Bootstrapping the String Kawai-Lewellen-Tye Kernel

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We show that double-copy maps for amplitudes in effective field theory are severely constrained at four points by self-consistency and locality at six points. The resulting double-copy kernel depends only on two parameters as well as a specific symmetric function in s, t, u and interpolates between the original Kawai-Lewellen-Tye (KLT) string double copy and the open and closed string period integrals. Amplitudes double copied with this map must obey either the string monodromy relations or the field theory Kleiss-Kuijf (KK) and Bern, Carrasco, and Johansson (BCJ) relations; there are no other options. Our construction elucidates the "single-valued projection" property of the Riemann zeta-function values for the four-point string theory double copy.

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Introduction.—The double copy is a map between observables in a variety of theories, including, famously, Yang-Mills (YM) theory to gravity. It originates in string theory as a relation between open and closed string amplitudes [1] and has, in its field theory incarnation, dramatically simplified higher-order amplitude computations. The double copy has a wide range of applications, such as derivations of compact formulas for n-point tree-level amplitudes [2–12] and the simplification of higher-loop calculations to examine, for example, the UV behavior of supersymmetric scattering amplitudes [13–26]. It is used in investigations of gravitational wave physics [27,28], studies of the interplay between quantum field theory and string theory [29–35], methods for finding exact solutions to classical equations of motion [36–44], computations of boundary observables in (anti)-de Sitter [45–57] explorations of amplitudes in the soft and Regge limit [58–61], and manifestation of symmetries in amplitudes [62–70]. Reviews of the double copy include Refs. [4,71–73].

In this Letter, we study the double copy in the context of d-dimensional effective field theory (EFT) with massless states and local higher-derivative operators. Reference [74] proposed an algorithm, the Kawai-Lewellen-Tye (KLT) double-copy bootstrap, to systematically construct higher-derivative corrections to the field theory double-copy map. When implemented at minimal rank (motivated by the absence of spurious poles) for four- and five-point amplitudes, Ref. [74] found that the bootstrap gives an EFT double-copy map that is more general than the string kernel.

We extend the minimal rank KLT bootstrap to six-point and find novel restrictions on the four- and five-point double-copy maps that leave very few parameters free. Specifically, we find that any EFT amplitude compatible with the four-point double-copy map must obey either the string monodromy relations or the field theory KK and BCJ relations [2,75–79]. It is surprising that string monodromy arises in an EFT context because it is a property of the world sheet description of string scattering [75–79].

Based on the low-energy expansion of the generalized double-copy map, we propose a closed-form expression for the four-point map: it takes a factorized form of one function that depends on two parameters (a "left" and a "right" choice of α') and a symmetric function U in s, t, u of a simple form. Hence, the most general double-copy map is a generalization of the string theory double copy such that a continuous interpolation is possible between the string kernel with $\alpha'_L = \alpha'_R \neq 0$, the open string period integrals ("Z theory") with either α'_L or α'_R zero, and the closed string period integrals with $\alpha'_L = \alpha'_R = 0$. In an EFT context, the double copy is effectively independent of the symmetric function U in the sense that U can be absorbed into the input amplitudes. In a string theory context, the symmetric function provides insight into the single-valued projection property of closed string amplitudes [80,81].

Review: KLT bootstrap.—The KLT double copy maps on-shell tree amplitudes, A_n^L and A_n^R, of two theories L and R to on-shell tree amplitudes in a third theory, denoted $L \otimes R$. Assuming all states are massless, we have

$$A_n^{L\otimes R} = \sum_{a\in B_L, b\in B_R} A_n^L[a] S_n[a|b] A_n^R[b], \quad (1)$$

where the sum is over two "basis choices" B_L and B_R of $(n-3)!$ color orderings among the $(n-1)!$ single-trace structures associated with a local or global color group of the L and R theories. The double-copy kernel S_n is a function of the n-point Mandelstam variables. In the original KLT construction of Ref. [1], the L and R amplitudes are open string tree amplitudes and S_n is the string kernel. In the low-energy limit $\alpha' \to 0$, the open (closed) string amplitudes reduce to YM (gravity) tree amplitudes and the string kernel becomes the field theory kernel.

A crucial property of Eq. (1) is that the double-copied amplitudes, $A_n^{L\otimes R}$, are independent of the choice of color orderings, $B_{L/R}$, in the sum. As a consequence, the double copy can be formulated as a multiplicative map of field theories, $\mathrm{FT}^{L\otimes R} = \mathrm{FT}^L \otimes \mathrm{FT}^R$, where the multiplication rule $\otimes$ is defined by the kernel S_n. In string theory, basis independence is ensured by the string monodromy relations satisfied by the open string tree amplitudes. In field theory, the KK and BCJ relations, denoted as KKBCJ relations, guarantee the needed basis independence.

The goal of the KLT double-copy bootstrap introduced in Ref. [74] is to determine the most general form of the double-copy kernel, S_n, along with the associated linear relations required by the L and R amplitudes such that the double copy is a map on the space of field theories. This requires that the output $A_n^{L\otimes R}$ is free of spurious poles and independent of the basis choices $B_{L/R}$ in Eq. (1). Reference [74] showed that these desired properties are linked to the existence of an identity element $\mathbb{1}$, i.e., a field theory whose tree amplitudes obey

$$\mathbb{1} \otimes \mathbb{1} = \mathbb{1}, \tag{2}$$

$$L \otimes \mathbb{1} = L, \qquad \mathbb{1} \otimes R = R. \tag{3}$$

This is the KLT algebra. The identity model associated with the field theory kernel is the cubic biadjoint scalar (BAS) model,

$$\mathcal{L}_{\mathrm{BAS}} = -\frac{1}{2}(\partial\phi^{aa'})^2 + \frac{1}{6} f^{abc}\tilde{f}^{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}, \tag{4}$$

for which Eq. (3) becomes the KKBCJ relations.

The identity model amplitudes of the string kernel were constructed in Ref. [82], and their low-energy α' expansion shows that the identity model is a BAS model plus a particular tower of higher-derivative terms with fixed coefficients controlled by α'.

The core proposal of Ref. [74] is to take the KLT algebra to be a fundamental property of the double copy. Specifically, Eq. (2) becomes a bootstrap equation for the kernel while the generalized KKBCJ relations arise from Eq. (2). As the candidate for the identity model, we consider BAS EFT: the cubic BAS in Eq. (4) plus all possible local higher-derivative terms.

Let $m_n[a|b]$ be the doubly color-ordered tree amplitudes of the general BAS EFT. The bootstrap equation $\mathbb{1} \otimes \mathbb{1} = \mathbb{1}$ says

$$m_n[c|d] = \sum_{a\in B_L, b\in B_R} m_n[c|a] S_n[a|b] m_n[b|d]. \tag{5}$$

The sums are over the $(n-3)!$ color orderings in the bases defined below Eq. (1). Since there is a total of $(n-1)!$ color orderings, we organize the amplitudes $m_n[a|b]$ into $(n-1)! \times (n-1)!$ matrices $\mathbf{m}_n$. If the color choices c and d in Eq. (5) run over $(n-3)!$ color orderings B_1 and B_2, then Eq. (5) becomes a matrix equation

$$\mathbf{m}_n^{B_1B_2} = \mathbf{m}_n^{B_1B_L}.\mathbf{S}_n^{B_LB_R}.\mathbf{m}_n^{B_RB_2}, \tag{6}$$

where the superscript indicates the choices of rows and columns for the $(n-3)! \times (n-3)!$ submatrices of $\mathbf{m}_n$. Choosing $B_1 = B_R$ and $B_2 = B_L$, it follows from Eq. (6) that

$$\mathbf{S}_n^{B_LB_R} = (\mathbf{m}_n^{B_RB_L})^{-1}. \tag{7}$$

This shows that the kernel is uniquely linked to the tree amplitudes of the identity model. Plugging Eq. (7) into Eq. (6) gives a nontrivial condition: the rank of the $(n-1)! \times (n-1)!$ matrix $\mathbf{m}_n$ must be $(n-3)!$. Generic BAS EFT tree amplitudes give $\mathbf{m}_n$ with full rank $(n-1)!$, so the "minimal rank" condition imposes nontrivial relations on the Wilson coefficients. This way $\mathbb{1} \otimes \mathbb{1} = \mathbb{1}$ becomes a bootstrap equation for the double-copy kernel [83]. We now summarize the results of Ref. [74] for four- and five-point.

Four-point: At four-point, all $3! \times 3!$ entries of $\mathbf{m}_n$ can be expressed in terms of just three functions [86]

$$f_1(s,t) = m_4[1234|1234], \qquad f_2(s,t) = m_4[1234|1243],$$
$$f_6(s,t) = m_4[1234|1432] \tag{8}$$

via cyclicity and momentum relabeling, and the minimal rank 1 condition is solved by

$$f_6(s,t) = f_1(s,t) = \frac{f_2(s,t)f_2(u,s)}{f_2(t,s)},$$
$$f_2(s,t)f_2(u,s)f_2(t,u) = f_2(t,s)f_2(u,t)f_2(s,u). \tag{9}$$

These equations ensure that f_1 is cyclic, $f_1(u,t) = f_1(s,t)$. These conditions are solved by the tree amplitudes of the BAS in Eq. (4) and by the inverse string kernel of Ref. [82], respectively,

$$f_2^{\mathrm{BAS}} = -\frac{1}{s}, \qquad f_2^{\mathrm{str}}(s,t) = -\frac{\pi\alpha'}{\sin(\pi\alpha' s)}. \tag{10}$$

Since on-shell local operators are in one-to-one correspondence with local on-shell matrix elements, the most general ansatz for the BAS EFT is given by higher-order polynomial terms in f_2 as

$$f_2(s,t) = -\frac{1}{s} + \sum_{k=0}^{N}\sum_{r=0}^{k} a_{k,r} s^r t^{k-r}. \qquad (11)$$

Here all possible local operators up to $2N$ derivatives are included. The Wilson coefficients $a_{2i,2i}$ must be set to zero for all $i = 0, 1, 2, \ldots$ to avoid unphysical poles in f_1 via Eq. (9). The four-point bootstrap equation, Eq. (9), is solved order by order in the Mandelstam variables and for the first few orders we find

k	Constraints
1	None
2	$a_{2,1} = a_{2,0}$
3	None
4	$a_{4,3} = a_{4,0} - a_{4,1} + a_{4,2}$
5	$a_{5,4} = a_{5,0} - a_{5,1} + a_{5,3} + a_{1,0}a_{1,1}(a_{1,0} - a_{1,1}) + a_{1,1}(a_{3,1} - a_{3,2}) - a_{1,0}(a_{3,0} - a_{3,2} + a_{3,3})$.

To match the inverse string kernel $f_2^{\rm str}$ in Eq. (10), set all $a_{k,r} = 0$ except for $a_{k,k}$ with k odd for which

$$a_{1,1} = -\frac{\pi^2\alpha'^2}{6}, \quad a_{3,3} = -\frac{7\pi^4\alpha'^4}{360}, \quad a_{5,5} = -\frac{31\pi^6\alpha'^6}{15120}, \ldots.$$

A closed-form expression for the inverse string kernel coefficients can be given in terms of the Bernoulli numbers.

Five-point: The five-point KLT bootstrap requires $\mathbf{m}_5$ to have rank 2, which significantly constrains the Wilson coefficients of the local five-point interactions. Since the BAS EFT amplitudes m_5 depend on m_4 through factorization, it is noteworthy that no additional constraints [as explicitly checked up to $O(s^8)$] arise on the four-point Wilson coefficients $a_{k,r}$ in the five-point analysis.

New results at six points.—New constraints on the four-point double-copy kernel arise from the six-point KLT bootstrap. The six-point analysis is done by first using cyclicity and momentum relabeling to parameterize the $(6-1)! = 120$ distinct amplitudes $m_6[123456|b]$ in terms of 24 "basis" amplitudes. All pole terms of these 24 basis amplitudes are then fixed by their factorization to the known four- and five-point amplitudes. Local six-point terms are included as all possible polynomial terms in a choice of the nine Mandelstam variables that are independent under six-point momentum conservation. Table I gives the parameter count at each order.

We impose minimal rank 6 on the 120×120 matrix of doubly color ordered six-point amplitudes of BAS EFT by setting the 7×7 minors to zero up to and including cubic order $\mathcal{O}(s^3)$. This constrains the four-point coefficients $a_{k,r}$ up to and including $\mathcal{O}(s^5)$ and the full results are summarized in Table II. In the section "Hybrid decomposition," we extend these results to higher order. The result is that at four-point, all parameters are fixed except

$$a_{1,0}, a_{1,1} \quad \text{and} \quad a_{2k,0} \quad \text{for } k = 1, 2, 3, \ldots. \qquad (12)$$

TABLE I. Number of free parameters at each order in Mandelstams in the BAS + EFT ansatz before and after the six-point bootstrap. The counting in the table refers to parameters left free after the four-point and five-point minimal rank bootstrap. For example, for $k = 4$, the three free parameters at four-point are $a_{4,0}$, $a_{4,1}$, $a_{4,2}$, because $a_{4,3}$ was fixed by the four-point bootstrap and we needed $a_{4,4} = 0$ to ensure locality of f_1. The six-point bootstrap condition fixes all $3 + 4 + 1080$ parameters except 1, namely, $a_{4,0}$.

k	Four-point $O(s^k)$	Five-point $O(s^{k-1})$	Six-point $O(s^{k-2})$	Free after six-point bootstrap	
1	2	0	0	2	$a_{1,0}$, $a_{1,1}$
2	1	0	24	1	$a_{2,0}$
3	4	0	216	0	
4	3	4	1080	1	$a_{4,0}$
5	5	10	3960	0	

The generalized KKBCJ relations arising from Eq. (3) can all be written in terms of ratios of f_2. For example one of the relations required by $L \otimes \mathbb{1} = L$ is

$$A_4^L[1234] = \frac{f_2(u,s)}{f_2(t,s)} A_4^L[1243]. \qquad (13)$$

The ratios of f_2 turn out to be independent of all $a_{2k,0}$. In fact, the generalized L-sector KKBCJ relations only

TABLE II. Combined results for the coefficients $a_{k,r}$ up to $\mathcal{O}(s^5)$ in the four-point amplitude f_2 in Eq. (11) after the KLT bootstrap at four-, five-, and six-point. Note that $a_{1,0}$, $a_{1,1}$, and $a_{2k,0}$ for $k = 1, 2, \ldots$ remain unfixed.

$a_{2,1} \quad a_{2,0}$
$a_{3,0} = \frac{2}{5}a_{1,0}(a_{1,0} - 2a_{1,1})$
$a_{3,1} = (1/10)a_{1,0}(a_{1,0} - 12a_{1,1})$
$a_{3,2} = \frac{1}{5}a_{1,0}(2a_{1,0} - 9a_{1,1})$
$a_{3,3} = -(7/10)a_{1,1}^2$
$a_{4,1} = -a_{1,0}a_{2,0} + 2a_{4,0}$
$a_{4,2} = -(a_{1,0} + a_{1,1})a_{2,0} + 2a_{4,0}$
$a_{4,3} = -a_{1,1}a_{2,0} + a_{4,0}$
$a_{5,0} = (8/35)a_{1,0}(a_{1,0}^2 - 3a_{1,0}a_{1,1} + 3a_{1,1}^2)$
$a_{5,1} = (2/35)a_{1,0}(3a_{1,0}^2 - 16a_{1,0}a_{1,1} + 30a_{1,1}^2) - (a_{2,0}^2/2)$
$a_{5,2} = (1/70)a_{1,0}(23a_{1,0}^2 - 104a_{1,0}a_{1,1} + 216a_{1,1}^2) - a_{2,0}^2$
$a_{5,3} = (1/70)a_{1,0}(12a_{1,0}^2 - 71a_{1,0}a_{1,1} + 204a_{1,1}^2) - (a_{2,0}^2/2)$
$a_{5,4} = (1/70)a_{1,0}(16a_{1,0}^2 - 76a_{1,0}a_{1,1} + 153a_{1,1}^2)$
$a_{5,5} = (31/70)a_{1,1}^3$

depend on a single parameter, namely, $a_{1,1} - a_{1,0}$ whereas the relations from $\mathbb{1} \otimes R = R$ only depend on $a_{1,1}$. Writing

$$a_{1,1} = -\frac{\pi^2 \alpha_R'^2}{6}, \qquad a_{1,0} = \frac{\pi^2}{6}(\alpha_L'^2 - \alpha_R'^2), \tag{14}$$

our results for the L and R generalized KKBCJ relations can be identified precisely as the low-energy expansion of the string monodromy relations with separate L or R choices of α'. Depending on whether $\alpha'_{L,R}$ are zero or not means that that generalized four-point KKBCJ relations are then either the field theory KKBCJ relations or the string monodromy relations.

We now connect the results of the KLT bootstrap to known special cases.

Inverse string kernel: The inverse string kernel has monodromy relations with the same α' for the two color orderings, i.e., $\alpha'_L = \alpha'_R$. By Eq. (14), this choice requires $a_{1,0} = 0$. Table II then shows that almost all coefficients vanish except the $a_{2k,0}$'s and $a_{k,k}$ for k odd, which directly give the string kernel coefficients. Thus,

$$\alpha' \equiv \alpha'_L = \alpha'_R \quad \text{and} \quad a_{2k,0} = 0 \tag{15}$$

matches the inverse string kernel.

Hybrid models GF (generalized Z theory): Reference [87] studied BAS EFTs with tree amplitudes that satisfy the field theory KKBCJ relations on the second color ordering. These relations are imposed as $(n-1)! - (n-3)!$ null vector conditions on the matrix $\mathbf{m}_n$. Hence, it has rank $(n-3)!$ and must be in the same class as the inverse double-copy kernels studied in this Letter. We call these models hybrid models and denote them by GF to indicate that they are general (i.e., no imposed constraints) on the first color-structure and obey field theory KKBCJ relations on the second color structure. It was shown in Ref. [87] that at four-point the generalized KKBCJ relations of the first color structure (i.e., G) are the string monodromy relations with $a_{1,0}$ simply a choice of the scale of α'.

We obtain the GF models of Ref. [87] by setting $a_{1,1} = a_{1,0}$, i.e., $\alpha'_L = 0$. Additionally choosing

$$a_{2k,0} = -\zeta(2k+1)\alpha'^{2k+1}, \tag{16}$$

where $\alpha' = \alpha'_R$ and $\zeta(p)$ is the Riemann Zeta function, we match the four-point amplitudes of Z theory in Ref. [88]. It is useful to note that the open string tree amplitudes can be obtained as the double copy

$$\text{open string tree} = Z \otimes_{\text{FF}} \text{YM}, \tag{17}$$

where $\otimes_{\text{FF}}$ indicates the field theory kernel with no higher-derivative corrections.

Closed string J integrals: The closed string tree amplitudes can be written in terms of period integrals, the "J integrals," as a field theory kernel double copy [32,89–92]

$$\text{closed string tree} = \text{YM} \otimes_{\text{FF}} J \otimes_{\text{FF}} \text{YM}. \tag{18}$$

The J-integral amplitudes are a special case of the BAS EFT amplitudes, namely, those that obey the field theory KKBCJ relations on both color structures, i.e., $\alpha'_L = \alpha'_R = 0$. We obtain the J-integral amplitudes by additionally choosing $a_{2k,0} = -2\zeta(2k+1)\alpha'^{2k+1}$.

Hybrid decomposition.—The process of solving the vanishing conditions for 7×7 minors of $\mathbf{m}_6$ becomes increasingly difficult at higher orders in the derivative expansion. However, we can go to much higher orders using the hybrid decomposition conjecture which posits that the most general BAS EFT (denoted $\mathrm{G\tilde{G}}$) (i.e., the tree amplitude matrix $\mathbf{m}_n$ has minimal rank $(n-3)!$) can be obtained by a field theory double copy of two hybrid models:

$$\mathrm{G\tilde{G}} = \mathrm{GF} \otimes_{\text{FF}} \mathrm{F\tilde{G}}. \tag{19}$$

Here GF is the hybrid model described above and $\mathrm{F\tilde{G}}$ is the hybrid model with first and second color orders interchanged. GF and $\mathrm{F\tilde{G}}$ have independent coefficients, $a^{\text{GF}}_{k,r}$ and $a^{\mathrm{F\tilde{G}}}_{k,r}$. The subscript FF on the product in Eq. (19) emphasizes that the double copy is done with the standard field theory kernel.

At four-, five-, and six-points, one can directly test Eq. (19) using the results for $\mathrm{G\tilde{G}}$ presented in the section "New results at six points" and those for GF in Ref. [87]. Further evidence will be presented in Ref. [93]. The key point here is that it is much easier to solve the linear field theory KKBCJ constraints for GF than it is to impose vanishing 7×7 determinant conditions for $\mathrm{G\tilde{G}}$. The hybrid model GF was solved to order 18 in the Mandelstam expansion at four-point with constraints from six-point KKBCJ relations in Ref. [87].

A closed-form expression for f_2^{GF} was proposed in Ref. [87] and using it with the hybrid conjecture Eq. (19), we find

$$f_2^{\mathrm{G\tilde{G}}}(s,t) = f_2^{(0)}(s,t)U(s,t,u), \tag{20}$$

where

$$f_2^{(0)} = -\frac{\pi}{s}\sqrt{\frac{\alpha'_R \alpha'_L s^2 \sin(\pi\alpha'_R t)\sin(\pi\alpha'_L u)}{\sin(\pi\alpha'_R s)\sin(\pi\alpha'_R u)\sin(\pi\alpha'_L s)\sin(\pi\alpha'_L t)}},$$

$$\log U = \sum_{k=1}^{\infty} \frac{a_{2k,0}}{2k+1}(s^{2k+1} + t^{2k+1} + u^{2k+1}). \tag{21}$$

Here, $a_{2k,0} = a^{\text{GF}}_{2k,0} + a^{\mathrm{F\tilde{G}}}_{2k,0}$, and $\alpha'_{\text{R/L}}$ are the α''s associated with the four-point string monodromy relations of the GF and F$\tilde{\rm G}$ models, respectively. Using the definitions in Eq. (14), the low-energy expansion of Eqs. (20) and (21) matches the results in Table II. When $\alpha'_R = \alpha'_L$ and $a_{2k,0} = 0$ we recover the inverse string kernel $f^{\rm str}$ in Eq. (10) from Eqs. (20) and (21).

When $\alpha'_L \neq \alpha'_R$, the closed form expression in Eq. (21) has unphysical poles, so it should be understood only in the low-energy expansion. This may pinpoint the string kernel with $\alpha'_L = \alpha'_R$.

Additional constraints on the four-point coefficients $a_{k,r}$ from a higher-point bootstrap are unlikely. Conditions on $a_{2k,0}$ are impossible because Z theory has $a_{2k,0}$ fixed to $\zeta(2k+1)$ and there are (conjecturally) no polynomial conditions that can relate these odd-argument transcendental numbers [81,88,94,95].

Implications for the KLT double copy.—Monodromy relations: When the closed-form expression in Eqs. (20) and (21) are plugged in to the generalized KKBCJ relations, e.g., Eq. (13), we find the string monodromy relations,

$$\sin(\pi\alpha'_L u) A_4^L[1234] = \sin(\pi\alpha'_L t) A_4^L[1243]. \quad (22)$$

In particular, U drops out from the ratios of f_2 which is why there is no dependence on the $a_{2k,0}$'s. Moreover, by the hybrid decomposition construction, the monodromy relations from the hybrid models are inherited by G$\tilde{\rm G}$ in (19). This means that the generalized KKBCJ relations of the minimal double-copy kernel at four-point must be either the string monodromy relations ($\alpha_{\rm L/R}$ nonzero) or the field theory KKBCJ relations ($\alpha_{\rm L/R}$ zero). It is interesting that such stringy properties arise from the basic assumptions of the KLT algebra.

Kernel equivalence: Even though U in Eq. (20) does not enter the generalized KKBCJ conditions, such as in Eq. (13), it does contribute to the final double result. However, any contribution from the U in the kernel can be absorbed into the input amplitudes. Concretely, if we use U to redefine the Wilson coefficients of the higher-derivative terms of the L or R sector amplitudes, e.g., $A_4^{\rm R}[b] \to UA_4^{\rm R}[b]$ for all color orders b and simultaneously rescale $m_4[a|b] \to m_4[a|b]U$, the result of the double copy is unchanged:

$$\begin{aligned} A_4^{L\otimes R} &= A_4^L[a] S_4[a|b] A_4^R[b] \to A_4^L[a](S_4[a|b]U^{-1})(UA_4^R[b]) \\ &= A_4^{L\otimes R}. \end{aligned} \quad (23)$$

This means that f_2 and $f_2 U$ are functionally equivalent in the low energy expansion.

Single-valued projection: With the closed string tree amplitudes being the double copy of the open string using the string kernel, denoted $\otimes_{\alpha'\alpha'}$, we have from Eq. (17) the identity

$$\text{closed string tree} = \text{YM} \otimes_{\rm FF} Z^T \otimes_{\alpha'\alpha'} Z \otimes_{\rm FF} \text{YM}, \quad (24)$$

where Z^T is the model whose tree amplitudes are $(\mathbf{m}_n^Z)^T$. Comparing with Eq. (18) gives [32,89–92]

$$J = Z^T \otimes_{\alpha'\alpha'} Z. \quad (25)$$

Now, Eq. (20) is valid for the string kernel, with $U = 1$ and for Z theory whose U_Z is given by (21) with coefficients (16). At four-point, the J integrals have $f_2^{(0)} = -1/s$, so all the α' dependence from the $f_2^{(0)}$'s of Z theory and the string kernel must cancel: this eliminates π. Only U_Z^2 remains, which has parameters $2a_{2k,0}^Z$ due to its exponentiated form in Eq. (21). Thus at four-point, Eqs. (20) and (21) reproduce the "single-valued projection" $J = {\rm SV}(Z)$ [80,81]:

$${\rm SV}: \zeta(\text{even}) \to 0, \qquad \zeta(\text{odd}) \to 2\zeta(\text{odd}). \quad (26)$$

This implies that the closed (super)string tree amplitude can be obtained as SV(open) $\otimes_{\rm FF}$ (S)YM [95,96].

The generalization to $\alpha'_L \neq \alpha'_R$ is straightforward. It would be interesting to understand if the double-copy bootstrap can similarly provide an explanation of the single-valued projection at a higher point.

KLT vs BCJ: Higher-derivative corrections were incorporated into the BCJ-representation [2] of the double copy in Refs. [97–101]. It appears that the BCJ construction can be truncated at finite order in the derivative expansion, in contrast to the KLT kernel in this work where locality at six-point requires an infinite tower of higher-derivative corrections.

BAS at leading order: We assumed the BAS EFT to have a nonvanishing cubic interaction term. This ensured that the usual field theory double copy was obtained from the low-energy limit of the generalized double copy. Relaxing this condition may result in a different form of the double copy.

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