

Time Translation Symmetry Breaking in an Isolated Spin-Orbit-Coupled Fluid of Light

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We study the interplay between intrinsic spin-orbit coupling and nonlinear photon-photon interactions in a nonparaxial, elliptically polarized fluid of light propagating in a bulk Kerr medium. We find that in situations where the nonlinear interactions induce birefringence, i.e., a polarization-dependent nonlinear refractive index, their interplay with spin-orbit coupling results in an interference between the two polarization components of the fluid traveling at different wave vectors, which entails the breaking of translation symmetry along the propagation direction. This phenomenon leads to a Floquet band structure in the Bogoliubov spectrum of the fluid, and to characteristic oscillations of its intensity correlations. We characterize these oscillations in detail and point out their exponential growth at large propagation distances, revealing the presence of parametric resonances.

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Spin-orbit coupling (SOC) in materials arises due to the interaction between the electron spin and its momentum, and lies at the heart of various phenomena and concepts such as spin Hall effects [1,2], topological insulators [3], and Majorana fermions [4]. In the context of quantum fluids, progress in the engineering of synthetic gauge fields has paved the way for intriguing phenomena resulting from the interplay between SOC and particle interactions in ultracold atoms [5,6]. In the ground states of Bose gases, e.g., this interplay yields stripe-superfluid or lattice phases [7–12]. In degenerate Fermi gases, on the other hand, SOC can significantly impact the celebrated BEC-BCS crossover [13] or lead to topological superfluids [14,15]. Beyond matter waves, SOC also exists in photonic systems [16,17]. This has been demonstrated, in particular, for exciton-polaritons in microcavities [18–23] or for photons tunneling in properly designed microstructures [24]. In those systems, effective photon-photon interactions also emerge due to the interaction between the underlying excitons, and their interplay with SOC has been investigated in numerous works [25–29].

An alternative optical platform where photon interactions can be realized are fluids of light in the propagating geometry [30]. Here the propagation of light through a nonlinear medium mimics, in the paraxial limit, the temporal evolution of a two-dimensional (2D) quantum fluid, the propagation axis playing the role of an effective time and the nonlinearity mediating the photon interactions. This analogy has been beautifully illustrated with measurements of the Bogoliubov dispersion [31,32], the dynamical formation of optical condensates [33,34], the spontaneous nucleation of vortices in a photonic lattice

[35], or the temporal dynamics of correlation functions following a quench [36,37]. Owing to the absence of cavity or underlying microstructure, fluids of light in the propagating geometry do not apparently seem to constitute a natural platform for achieving SOC. Nevertheless, recently a spin-orbit mechanism has been demonstrated in this system [38], based on the fundamental coupling between the polarization and the trajectory of optical fields subjected to a refractive-index gradient. Unlike linear setups, where the gradient is provided by the medium inhomogeneity [39–46], in nonlinear media SOC emerges in the presence of fairly strong spatial variations of the optical field itself, requiring it to deviate from its paraxial propagation regime. This induces a nonlinear index gradient, which couples to the optical spin via the polarization-trajectory coupling term of the wave equation [38].

In this Letter, we show that elliptically polarized fluids of light propagating in media displaying nonlinear birefringence exhibit a breaking of translation symmetry along the optical axis direction (i.e., the effective-time axis) due to SOC. In stark contrast with the case of linearly polarized fluids considered in Ref. [38], this leads to the emergence of a Floquet band structure in the excitation spectrum, analogously to what is observed in driven systems [47], but here in a purely *isolated* optical fluid. The breaking of translation symmetry also gives rise to peculiar oscillations in several physical quantities, in particular the intensity-correlation function of the fluid of light. By characterizing these oscillations in detail, we further point out their exponential growth at large propagation distances. This showcases the existence of parametric resonances, an original manifestation of the interplay between nonlinearity and SOC in fluid systems.

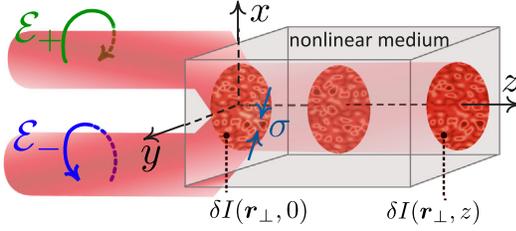


FIG. 1. We study the propagation of an optical field with two circularly polarized components \mathcal{E}_\pm in a nonlinear medium. Both \mathcal{E}_+ and \mathcal{E}_- display weak spatial intensity fluctuations $\delta I(\mathbf{r}_\perp, z = 0)$ (modeled by a speckle with correlation length σ), which couple to the optical spin via the nonlinear refractive index. Breaking of translation invariance along z is observed in the input-output correlation $\langle \delta I(\mathbf{r}_\perp, 0) \delta I(\mathbf{r}_\perp, z) \rangle$.

Our setup consists of a bulk nonlinear medium infinitely extended along the x and y axes and $z > 0$ (see Fig. 1). A monochromatic field $\mathcal{E}(\mathbf{r}, t) = \text{Re}[\mathcal{E}(\mathbf{r})e^{-i\omega_0 t}]$ propagates inside the material at frequency ω_0 . The components of the complex amplitude \mathcal{E} obey the nonlinear Helmholtz equation

$$\nabla^2 \mathcal{E} - \nabla(\nabla \cdot \mathcal{E}) + \frac{\omega_0^2}{c^2} [n_0^2 + 2n_0 \Delta n(\mathcal{E})] \mathcal{E} = 0, \quad (1)$$

with c the vacuum speed of light and n_0 the linear refractive index. The nonlinear refractive index $[\Delta n(\mathcal{E})]_{ij} = (n_{2,d} + n_{2,s})|\mathcal{E}|^2 \delta_{ij} - n_{2,s} \mathcal{E}_i^* \mathcal{E}_j$ ($i, j = x, y, z$) is a tensor featuring two independent Kerr indices $n_{2,d}$ and $n_{2,s}$. Equation (1) corresponds to the Euler-Lagrange equation for the action functional $\mathcal{S} = \int d^3 \mathbf{r} \mathcal{L}$, where [38]

$$\mathcal{L} = -\frac{1}{2\beta_0} (\nabla_i \mathcal{E}_j^* \nabla_i \mathcal{E}_j - \nabla_i \mathcal{E}_j^* \nabla_j \mathcal{E}_i - \beta_0^2 \mathcal{E}_i^* \mathcal{E}_i) - \frac{1}{2} [g_d \delta_{ij} \delta_{i'j'} + g_s (S_k)_{ij} (S_k)_{i'j'}] \mathcal{E}_i^* \mathcal{E}_j \mathcal{E}_{i'}^* \mathcal{E}_{j'}. \quad (2)$$

Here summation over repeated indices is implied, $\beta_0 = n_0 \omega_0 / c$ is the propagation constant, and $(S_k)_{ij} = -i \epsilon_{ijk}$ denotes the k^{th} spin-1 matrix. The simultaneous presence of a spin-independent ($g_d = -n_{2,d} \omega_0 / c$) and a spin-dependent ($g_s = -n_{2,s} \omega_0 / c$) nonlinear coupling, both assumed positive, is typical of isotropic systems described by multi-component fields, where the pairing in channels of different total spin can occur at different strengths. Besides optical nonlinear media [48,49], this behavior is observed in atomic spinor Bose-Einstein condensates [50] and in microcavity exciton polaritons [16,51–53], where the spin-dependent term is sometimes called self-induced Zeeman splitting. Note, however, that unlike polariton condensates [25–29], the fluid of light described by Eq. (1) is fully three-dimensional. In our setup, the mechanism of SOC of light originates from the term $\nabla(\nabla \cdot \mathcal{E}) \sim \nabla(\nabla \ln \Delta n \cdot \mathcal{E})$ in Eq. (1), which couples the

fluid polarization to its trajectory (via the nonlinear index gradient). In the corresponding Lagrangian formalism, Eq. (2), the SOC effects are encoded in the term $\propto \nabla_i \mathcal{E}_j^* \nabla_j \mathcal{E}_i$. While naturally present in Maxwell equations, the latter is discarded within the usual paraxial approximation [30]. In the following, we do *not* perform this approximation but work out the full Lagrangian [Eq. (2)].

Our aim is to determine the field amplitude inside the medium, given its transverse profile $\mathcal{E}[\mathbf{r}_\perp = (x, y), z = 0]$ at the air-medium interface. This can be regarded as the evolution problem of a 2D system with respect to the effective time z [30]. In the following, we assume that \mathcal{E} is the sum of a large homogeneous background and a small fluctuation, and treat the latter using Bogoliubov-Popov theory [54–57]. For that purpose, we write $\mathcal{E} = (\mathcal{E}_+, \mathcal{E}_z, \mathcal{E}_-)^T$ and employ the density-phase decomposition $\mathcal{E}_+ = \sqrt{I} \cos(\vartheta/2) e^{i(\Theta + \chi/2)}$, $\mathcal{E}_- = \sqrt{I} \sin(\vartheta/2) e^{i(\Theta - \chi/2)}$ of the field circular components $\mathcal{E}_\pm = \mp (\mathcal{E}_x \mp i \mathcal{E}_y) / \sqrt{2}$. Here I and ϑ quantify the total optical intensity of the transverse components and their relative weight, respectively, while Θ (χ) is their total (relative) phase. We then split the field into a background and a fluctuating contribution, writing $I = I_0 + \delta I$, $\vartheta = \vartheta_0 + \delta \vartheta$, and $\chi = \Delta k z + \delta \chi$, where $\Delta k = k_+ - k_-$. The wave numbers k_\pm of the two polarization components are imposed by Eq. (1):

$$k_\pm = \sqrt{\beta_0^2 - 2\beta_0(g_d \pm g_s \cos \vartheta_0) I_0}. \quad (3)$$

Notice that $\Delta k = k_+ - k_- \neq 0$ as soon as $g_s \neq 0$ and $\cos \vartheta_0 \neq 0$, i.e., when the background field is elliptically or circularly polarized. This defines the phenomenon of nonlinear circular birefringence, which will play a crucial role in the following.

Next, we insert the fluctuation variables into the Lagrangian [Eq. (2)] and determine the quadratic correction $\mathcal{S}^{(2)}$ to the background action. This is achieved by redefining $\mathcal{E}_z \rightarrow e^{i\Theta} \mathcal{E}_z$ and $\Theta \rightarrow \Theta + (k_+ + k_-)z/2$, and expanding Eq. (2) with respect to δI , $\delta \vartheta$, and $\delta \chi$. Note that the fluctuations of Θ are, in contrast, possibly large in two dimensions [56], but its derivatives remain small, and so does \mathcal{E}_z . In this procedure, $\mathcal{S}^{(2)}$ turns out to be independent of the z derivatives of \mathcal{E}_z , so that one can use the Euler-Lagrange equation $\delta \mathcal{S}^{(2)} / \delta \mathcal{E}_z^* = 0$ to eliminate \mathcal{E}_z [58]. The quadratic action $\mathcal{S}^{(2)} = \int dz \int d^2 \mathbf{q}_\perp / (2\pi)^2 \tilde{\mathcal{L}}^{(2)}$ can be written in terms of a single column vector $X = (\delta \tilde{I} / 2I_0, \delta \tilde{\vartheta} / 2, \tilde{\Theta}, \delta \tilde{\chi} / 2)^T$ for the Fourier variables with respect to \mathbf{r}_\perp , e.g., $\delta \tilde{I}(\mathbf{q}_\perp, z) = \int d^2 \mathbf{r}_\perp \delta I(\mathbf{r}_\perp, z) e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_\perp}$, with $\mathbf{q}_\perp = (q_\perp \cos \varphi_q, q_\perp \sin \varphi_q)$ the transverse momentum. We find that

$$\tilde{\mathcal{L}}^{(2)} = \dot{X}^\dagger \Lambda_2 \dot{X} + \dot{X}^\dagger \Lambda_1 X + X^\dagger \Lambda_1^T \dot{X} - X^\dagger \Lambda_0 X, \quad (4)$$

where $\dot{X} \equiv \partial_z X$. The 4×4 matrices $\Lambda_{0,1,2}$ are real π -periodic functions of the angular variable $\varphi(z) = \varphi_q + \Delta k z/2$ [58], which encodes a breaking of translation invariance along the effective time axis z , the central result of the Letter. This dependence stems from the SOC term in Eq. (2), and is completely absent in the paraxial framework. In our description, the paraxial approximation corresponds to taking \dot{X}/β_0 , $(q_\perp/\beta_0)^2$, and $g_{d,s}I_0/\beta_0$ small; the resulting expansion of $\tilde{\mathcal{L}}^{(2)}$ up to first order becomes $\varphi(z)$ independent and formally identical to the Bogoliubov Lagrangian of symmetric binary mixtures of atomic condensates [60,61].

In the same spirit as in Ref. [38], we define a Hamiltonian $\tilde{\mathcal{H}}^{(2)} = \Pi^T \dot{X} + \dot{X}^\dagger \Pi^* - \tilde{\mathcal{L}}^{(2)}$ depending on X and the conjugate momenta vector $\Pi = \partial \tilde{\mathcal{L}}^{(2)}/\partial \dot{X}^T$. The effective-time evolution of X and Π is governed by the Hamilton equations $\dot{X} = \partial \tilde{\mathcal{H}}^{(2)}/\partial \Pi^T$ and $\dot{\Pi} = -\partial \tilde{\mathcal{H}}^{(2)}/\partial X^T$, yielding the eight coupled equations

$$\begin{bmatrix} \dot{X} \\ \dot{\Pi}^* \end{bmatrix} = \begin{bmatrix} -\Lambda_2^{-1} \Lambda_1 & \Lambda_2^{-1} \\ -(\Lambda_1^T \Lambda_2^{-1} \Lambda_1 + \Lambda_0) & (\Lambda_2^{-1} \Lambda_1)^T \end{bmatrix} \begin{bmatrix} X \\ \Pi^* \end{bmatrix}. \quad (5)$$

In Ref. [38], Eq. (5) was solved in the $\Delta k = 0$ case (linearly polarized background field), where the matrix of coefficients is constant. Here on the contrary, we assume that the birefringence condition $\Delta k \neq 0$ is fulfilled. Hence, the coefficients of Eq. (5) oscillate in z with period $2\pi/\Delta k$. We stress that although this behavior is typical of Bogoliubov equations for periodically driven systems [62–66] (see also Ref. [67]), here it occurs in a purely *isolated* system due to the breaking of time-translation symmetry by the background. The underlying mechanism is the interplay between nonlinear birefringence and SOC. The latter is responsible for the presence, in Eq. (2), of interference terms between the two polarization components propagating at relative wave vector Δk . Note that this phenomenon is absent in linear birefringent media, where the fluctuation is always independent of the background field.

According to Floquet's theorem [68,69], the general solution of Eq. (5) has the form

$$\begin{bmatrix} X(\mathbf{q}_\perp, z) \\ \Pi^*(\mathbf{q}_\perp, z) \end{bmatrix} = \sum_\ell C_\ell(\mathbf{q}_\perp) \begin{bmatrix} X_{0,\ell}(\mathbf{q}_\perp, \varphi) \\ \Pi_{0,\ell}^*(\mathbf{q}_\perp, \varphi) \end{bmatrix} e^{-i\Omega_\ell(\mathbf{q}_\perp)z}. \quad (6)$$

Here the sum runs over eight independent solutions, labeled by ℓ and appearing with weight C_ℓ . These Floquet solutions are characterized by their eigenfunctions $X_{0,\ell}$ and $\Pi_{0,\ell}^*$ and the corresponding quasifrequencies, Ω_ℓ . Note that as is customary for Bogoliubov equations [70], for each solution with quasifrequency Ω_ℓ there exists another one with quasifrequency $-\Omega_\ell^*$ associated with the same physical oscillation, hence a total of four Bogoliubov modes. In the paraxial regime, in contrast, one has only a density (d) and a spin (s) mode, characterized by in- and out-of-phase intensity oscillations of the two polarization components, respectively, as discussed in the Supplemental Material [58].

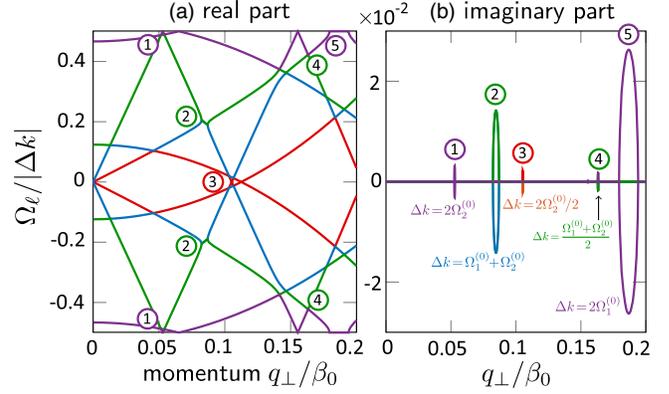


FIG. 2. (a) Real and (b) imaginary parts of the quasifrequency spectrum of an elliptically polarized fluid of light as functions of the transverse momentum. Each parametric resonance is assigned a number, which is used to identify the corresponding regions in the two plots. In (b) the corresponding resonance conditions are also provided. Here we have chosen the background polarization $\vartheta_0 = \pi/4$ and the nonlinear couplings $g_d I_0/\beta_0 = 0.2$, $g_s I_0/\beta_0 = 0.05$.

We first plot the real part of the quasifrequency spectrum in Fig. 2(a). Because the Ω_ℓ 's are defined modulo Δk , it is sufficient to take their real part in the first Brillouin zone, $-\Delta k/2 < \text{Re}\Omega_\ell \leq \Delta k/2$ [58]. In the $q_\perp \rightarrow 0$ limit, the spectrum in panel (a) displays the usual four phononic bands $\pm\Omega_{d,s}(q_\perp) \simeq \pm c_{d,s} q_\perp$, with two sound velocities $c_{d,s}$. Those correspond to the standard Bogoliubov modes in the paraxial regime, where $c_{d(s)}^2 = (g_d + g_s \pm \sqrt{g_d^2 + g_s^2 + 2g_d g_s \cos 2\vartheta_0}) I_0/2\beta_0$. On the contrary, the other four bands tend to a finite value and describe light propagating backward along z . At increasing q_\perp the various bands cross one another at several points, giving rise to an involved structure. In addition, some quasifrequencies—labeled by numbers in Fig. 2(a)—develop a finite imaginary part at certain values of q_\perp , see Fig. 2(b) (notice that complex quasifrequencies always occur in complex conjugate pairs). This important result reveals the presence of parametric resonances, which will be discussed in more details below.

A central feature of Eq. (6) is that the modes $X_{0,\ell}$ and $\Pi_{0,\ell}^*$ are π -periodic functions of $\varphi(z)$. Since $\varphi(z)$ contains a term linear in z , in general this leads to oscillations of specific observables in z . To illustrate this phenomenon and as an application of the above formalism, we now consider a concrete scenario where a fluid of light is initially prepared in the form of a (two-component) plane-wave background plus a small fluctuating field (see Fig. 1):

$$\begin{bmatrix} \mathcal{E}_+(\mathbf{r}_\perp, z=0) \\ \mathcal{E}_-(\mathbf{r}_\perp, z=0) \end{bmatrix} = \sqrt{I_0} \begin{bmatrix} \cos \frac{\vartheta_0}{2} + \epsilon \phi_+(\mathbf{r}_\perp) \\ \sin \frac{\vartheta_0}{2} + \epsilon \phi_-(\mathbf{r}_\perp) \end{bmatrix}, \quad (7)$$

where $0 < \epsilon \ll 1$, and ϕ_α ($\alpha = \pm$) is a two-component random complex speckle field of two-point correlation $\langle \phi_\alpha(\mathbf{r}_\perp) \phi_{\alpha'}^*(\mathbf{r}_\perp + \Delta\mathbf{r}_\perp) \rangle = \gamma(\Delta\mathbf{r}_\perp) \delta_{\alpha\alpha'}$, the brackets denoting statistical averaging. For definiteness we consider a Gaussian correlation $\gamma(\Delta\mathbf{r}_\perp) = \exp(-|\Delta\mathbf{r}_\perp|^2/4\sigma^2)$, with σ the correlation length. An initial state of the form of Eq. (7) was recently exploited experimentally [37] in order to realize an optical analog of the quench dynamics of thermal fluctuations in a 2D Bose gas, following Ref. [71]. The mode weights $C_\ell(\mathbf{q}_\perp)$ giving access to the state vector [Eq. (6)] at arbitrary z are easily deduced from the $z=0$ values of the fields [Eq. (7)] and the conjugate momenta, as shown in the Supplemental Material [58]. The latter are fixed requiring the vanishing of the weights of backward-propagating modes, as shown in the Supplemental Material [58]. The knowledge of the weights enables one to compute any statistical observable. One of the simplest is the function $g_2(z) \equiv \langle \delta I(\mathbf{r}_\perp, 0) \delta I(\mathbf{r}_\perp, z) \rangle$, which expresses how intensity fluctuations at a finite effective time $z > 0$ and at a given point in the transverse plane are correlated with their $z=0$ value (see Fig. 1). We find that [58]

$$\frac{g_2(z)}{\epsilon^2 I_0^2} = \sum_{\ell} \int_0^{\infty} \frac{q_\perp dq_\perp}{2\pi} \tilde{\gamma}(q_\perp) K_\ell(q_\perp, z) e^{-i\Omega_\ell(q_\perp)z}, \quad (8)$$

where $\tilde{\gamma}(q_\perp)$ is the speckle power spectrum. In the paraxial regime $q_\perp \rightarrow 0$ and $g_{d,s}I_0 \ll \beta_0$, the coefficients for the density and the spin modes $K_{d(s)} \simeq 1/2 \pm (g_d + g_s \cos 2\vartheta_0)/2\sqrt{g_d^2 + g_s^2 + 2g_d g_s \cos 2\vartheta_0}$ reduce to constants independent of q_\perp and z . g_2 in the paraxial limit is shown in Fig. 3 as a function of z (dashed curve), at fixed background polarization, nonlinearities, and correlation length. At $z=0$, $g_2(0)/\epsilon^2 I_0^2 = 2$, which corresponds to the Rayleigh law of the speckle field. At large z , the correlation is negative and approaches zero as $z \rightarrow +\infty$ following $g_2(z)/\epsilon^2 I_0^2 \simeq -(\sqrt{g_d I_0/\beta_0 z}/2\sigma)^{-2}$. In between these two asymptotic limits g_2 changes sign, in agreement with the sum rule $\int_0^{\infty} dz g_2(z) = 0$.

Beyond the paraxial limit, the emerging Floquet structure of the spectrum gives rise to periodic z oscillations of the coefficients K_ℓ . This manifests itself in oscillations of g_2 about its background value at large z , as seen in Fig. 3 (solid blue curve). The magnitude of these oscillations is directly controlled by two main physical parameters: first, the speckle correlation length σ , which controls by how much the fluid of light deviates from paraxiality, a necessary condition for the system to exhibit SOC, and second, the coupling strength g_s , which gives rise to nonlinear birefringence. These dependences are illustrated in the inset of Fig. 3: the oscillations' amplitude decreases (increases) with $\beta_0 \sigma$ ($g_s I_0/\beta_0$). In particular, at large σ the

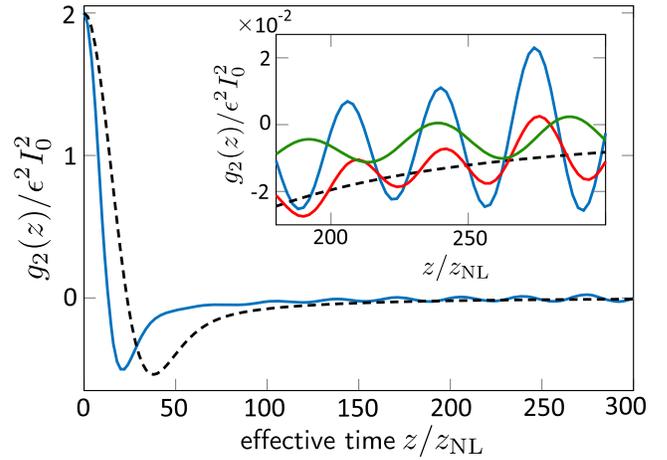


FIG. 3. Effective-time evolution of the intensity-intensity correlation function. In the main plot we compare the exact (solid blue curve) and paraxial (dashed curve) predictions for $\beta_0 \sigma = 15$ and the same parameters $\vartheta_0 = \pi/4$, $g_d I_0/\beta_0 = 0.2$, and $g_s I_0/\beta_0 = 0.05$ as in Fig. 2. z is measured in units of the nonlinear length $z_{\text{NL}} = 1/2g_d I_0$. In the inset we enlarge the large- z tail and include two additional curves showing the results for $\beta_0 \sigma = 22$ (red curve) and $g_s I_0/\beta_0 = 0.035$ (green curve), the other parameters being the same as above.

spectrum $\tilde{\gamma}(q_\perp)$ becomes very narrow, so that the integral [Eq. (8)] becomes dominated by the small- q_\perp modes, and $K_\ell(q_\perp, z)$ can be replaced by its constant, $q_\perp = 0$ value. The oscillation frequency, in contrast, is practically independent of σ . It is mainly governed by the nonlinear birefringence mismatch Δk of the fluid's two components. From a Fourier analysis of $g_2(z)$, we find that these oscillations are essentially harmonic at large enough σ , with a frequency $\simeq 0.8|\Delta k|$ [58].

A second remarkable consequence of the Floquet structure of the fluid's spectrum is a phenomenon of parametric resonance [72], analogous to that observed in periodically driven atomic [62–66] and photonic [73–77] systems. Parametric resonances are associated with the spontaneous emergence of complex frequencies in the Floquet spectrum, see Fig. 2(b). They can occur when the two lowest positive eigenvalues $\Omega_{1,2}^{(0)}$ of the time average of the matrix entering Eq. (5) fulfill $\Omega_a^{(0)}(q_\perp) + \Omega_b^{(0)}(q_\perp) \simeq n\Delta k$ for $a, b = 1, 2$ and integer $n > 0$ [58,65]. Parametric resonances induce an exponential growth of the population of the corresponding Bogoliubov modes at large z , resulting in a strong increase of the fringe amplitude in g_2 , as illustrated in Fig. 4(a). To explore more systematically the occurrence of these instabilities, we have analyzed their characteristic growth rate Γ for various interaction strengths $g_s I_0$ and background polarizations $\cos \vartheta_0$. To this aim, we studied the evolution of $g_2(z)$ within a fixed range of large values of z and fitted the data with an exponential law of the form $\exp(\Gamma z)$. The results are summarized by the diagram in Fig. 4(b). As

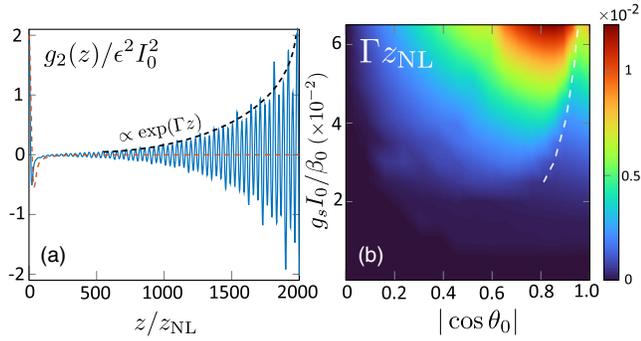


FIG. 4. (a) Evolution of $g_2(z)$ up to very large effective times z , revealing a phenomenon of parametric instability with a growth rate Γ . Parameters are the same as in Fig. 3. (b) Growth rate as a function of the background polarization, $\cos \theta_0$, and the spin-dependent nonlinear coupling, $g_s I_0/\beta_0$, at fixed $g_d I_0/\beta_0 = 0.2$ and $\beta_0 \sigma = 15$. The white dashed curve pinpoints a jump of Γ , corresponding to the appearance of an additional imaginary frequency participating in the dynamics.

expected, Γ vanishes when either $\cos \theta_0 \rightarrow 0$ (linearly polarized background) or $g_s \rightarrow 0$ (no spin-dependent nonlinearity). At large enough g_s , Fig. 4(b) also reveals a sharp increase of Γ as $|\cos \theta_0|$ is decreased from 1, pinpointed by the white dashed curve. This corresponds to the appearance of an additional complex frequency that starts to dominate the exponential growth in the chosen effective-time window, our fitting procedure yielding the largest Γ . Note that the exponential growth of g_2 is physical as long as $\langle (\delta I)^2 \rangle / I_0^2 \ll 1$ [and same for $\langle (\delta \vartheta)^2 \rangle$ and $\langle (\delta \chi)^2 \rangle$], corresponding to $z \ll \ln(\epsilon^{-2}) / \max_{\ell, q_\perp} [\text{Im} \Omega_\ell(q_\perp)]$. At larger z , nonlinear effects not captured by the Bogoliubov approximation eventually cause a saturation of the value of physical observables [62,65].

Several experimental studies of the polarization dependence of the nonlinear refractive index are available [78–87]. Typically one finds g_s/g_d of order 10^{-1} , not far from the value $g_s/g_d = 0.25$ chosen in this work. Taking this value and a nonlinear strength $g_d I_0/\beta_0 \sim 10^{-3}$ [37], we find that the oscillations of $g_2(z)$ are visible at propagation distances z of the order of a few tens of meters. Such a scale points toward optical fibers (see, e.g., Ref. [88]) as good candidates for the observation of the phenomena predicted in this Letter.

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