All Order Gravitational Waveforms from Scattering Amplitudes

Tim Adamo,[1,*] Andrea Cristofoli,[1,†] Anton Ilderton,[2,‡] and Sonja Klisch[1,§]
[1]*School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh, EH9 3FD, United Kingdom*
[2]*Higgs Centre, School of Physics and Astronomy, University of Edinburgh, Edinburgh, EH9 3FD, United Kingdom*

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Waveforms are classical observables associated with any radiative physical process. Using scattering amplitudes, these are usually computed in a weak-field regime to some finite order in the post-Newtonian or post-Minkowskian approximation. Here, we use strong-field amplitudes to compute the waveform produced in scattering of massive particles on gravitational plane waves, treated as exact nonlinear solutions of the vacuum Einstein equations. Notably, the waveform contains an infinite number of post-Minkowskian contributions, as well as tail effects. We also provide, and contrast with, analogous results in electromagnetism.

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The observation of gravitational waves has brought renewed importance to the study of general relativity and its observables. Surprisingly, scattering amplitudes—one of the key outputs of *quantum* field theory—are providing a new way to study *classical* general relativity; for reviews, see Refs. [1–3]. Starting from novel perspectives on [4–6], and a remarkable state-of-the-art calculation for [7], the conservative Hamiltonian of the gravitational two-body problem, a new program for providing higher-order post-Minkowskian (PM) approximations to gravitational observables has emerged based on the classical limit of scattering amplitudes. This has led to a variety of exciting new results for gravitational observables, e.g., Refs. [8–26], which build on many of the powerful structures in scattering amplitudes such as generalized unitarity and double copy, as well as techniques from effective field theory.

A key tool in this program has been the development of a formalism to systematize the extraction of classical physical observables from scattering amplitudes [27]. So far, all observables computed with this approach are valid for weak fields only: they are obtained from amplitudes at finite PM order, so truncate at a corresponding fixed order in the coupling [28–39]. This is in sharp contrast with other approaches to gravitational dynamics, such as the self-force paradigm [40–45], where perturbation theory is implemented around a curved background and the weak-field limit is not considered.

To address this gap, the amplitude-based approach can be generalized to curved backgrounds by means of strong-field scattering amplitudes and their classical limits [46]. This provides an alternative route to the computation of classical observables, as strong-field amplitudes encode a substantial amount of information about higher-order processes [47–54] and finite size effects [55–57] in trivial backgrounds, and can also admit remarkably compact formulas [58–60]. A key aspect is that even first-order perturbation theory around a curved background—which we refer to as "first postbackground," or 1PB, order—encodes *infinitely* many orders of the PM expansion. This is analogous to the relation between the PM and post-Newtonian expansions for bound orbits, where a fixed contribution of the former encodes infinitely many orders of the latter due to the virial theorem.

Here we show for the first time how classical observables encoding all order results can be extracted from scattering amplitudes. We derive expressions for the classical gravitational waveform emitted by a point particle scattering on a gravitational plane wave (an exact solution to the nonlinear Einstein equations), encoding all order contributions in the PM expansion when the flat spacetime limit is taken, as well as tail effects which usually enter at high order in the PM approximation. We also perform analogous calculations for charged particles scattering on electromagnetic plane waves. While our aim is not to study the phenomenology of electrodynamics, the waveforms do not seem to appear in an otherwise extensive literature [61–64], and it is revealing to compare and contrast with the gravitational case [65–68].

Note that plane waves are not just good models of gravitational waves, but also describe *any* spacetime in the neighborhood of a null geodesic [69]. This directly connects our results to the gravitational two-body problem: in the limit where one mass is negligible, the massless probe will experience the heavy body's metric as a plane wave. Indeed, plane wave or ultrarelativistic limits have been used to analyze gravitational self-force [70] and black hole quasinormal modes [71].

Asymptotic waveforms.—Let $|\Psi\rangle$ be a normalized superposition of free-particle (mass m) states,

$$|\Psi\rangle = \int d\Phi(p)\phi(p)e^{ip\cdot b/\hbar}|p\rangle, \tag{1}$$

where $d\Phi(p)$ is the Lorentz-invariant on-shell measure, the wave packet $\phi(p)$ has a well-defined classical limit (cf. Ref. [27]), and b_μ is the impact parameter. This state is evolved on an electromagnetic or gravitational plane wave background. In terms of the S matrix $\mathcal{S}$ on that background, the time-evolved state is simply $\mathcal{S}|\Psi\rangle$.

Our interest is in the classical gravitational or electromagnetic radiation emitted by a scalar particle as it scatters on these backgrounds, as measured by an asymptotic observer at future null infinity. The particular observable of interest is the waveform, encoded in the expectation value of the Maxwell and Riemann tensors, $\langle F_{\mu\nu}(x)\rangle$ and $\langle R_{\mu\nu\sigma\rho}(x)\rangle$. In coordinates $x^\mu = (t, \mathbf{x})$, approaching future null infinity corresponds to taking $r \equiv |\mathbf{x}| \to \infty$ while $u = t - r$ is held constant. Following Ref. [32], the waveform W is defined simply as the coefficient of the leading $1/r$ term in $\langle F\rangle$ or $\langle R\rangle$. It is a function of u and the two angular degrees of freedom encoded in the null vector $\hat{x}^\mu = (1, \hat{\mathbf{x}})$. Inserting complete sets of final states into the expectation value, and using the mode expansion of $F_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$, one easily obtains an expression for the waveform in terms of scattering amplitudes on the background. The leading contribution is at 1PB, meaning order e (the fundamental charge) in QED or order κ (the gravitational coupling) in gravity, but all orders in the background fields, and comes from interference between *tree-level* two-point and three-point amplitudes. Unlike in vacuum, two-point amplitudes on backgrounds are not trivial even at tree level, encoding, e.g., memory effects [46]. Defining the (theory-dependent) combination,

$$\alpha(k) = \int d\Phi(p')\langle\Psi|\mathcal{S}^\dagger|p'\rangle\langle p', k^\eta|\mathcal{S}|\Psi\rangle, \tag{2}$$

we arrive at, in QED and gravity, respectively,

$$\begin{aligned} W_{\mu\nu}(u,\hat{x}) &= -\frac{\hbar^{1/2}}{\pi}\mathrm{Re}\int_0^\infty \hat{d}\omega e^{-i\omega u}k_{[\mu}\varepsilon^{-\eta}_{\nu]}\alpha(k), \\ W_{\mu\nu\sigma\rho}(u,\hat{x}) &= -\frac{\kappa}{\pi\hbar^{1/2}}\mathrm{Im}\int_0^\infty \hat{d}\omega e^{-i\omega u}k_{[\mu}\varepsilon^{-\eta}_{\nu]}k_{[\sigma}\varepsilon^{-\eta}_{\rho]}\alpha(k), \end{aligned} \tag{3}$$

in which $k_\mu = \hbar\omega\hat{x}_\mu$ for ω a classical frequency (as will be useful later when taking the classical limit), $\varepsilon^\eta_\mu \equiv \varepsilon^\eta_\mu(k)$ is the photon polarization vector, and $\hat{d}x := dx/(2\pi)$. One can check that the combination of amplitudes in $\alpha(k)$ reproduces the radiation emitted due to geodesic motion, i.e., the first contribution of self-force effects [42].

Plane wave backgrounds.—Plane waves are highly symmetric vacuum solutions of the Einstein or Maxwell equations with two functional degrees of freedom. In gravity, they are described by metrics of the form [72]:

$$ds^2 = 2dx^+dx^- - dx^a dx^a - \kappa H_{ab}(x^-)x^a x^b (dx^-)^2, \tag{4}$$

where latin indices label the "transverse" directions $x^\perp = (x^1, x^2)$, while the 2×2 matrix $H_{ab}(x^-)$ is symmetric, traceless, and compactly supported on $x^-_i < x^- < x^-_f$ (ensuring the spacetime admits an S matrix [73]). The metric has a covariantly constant null Killing vector $n = \partial_+$ (or $n_\mu = \delta^-_\mu$) which will recur throughout. To ease notation, we absorb the gravitational coupling into the background, taking $\kappa H_{ab} \to H_{ab}$ from here on; as such, note that expressions below containing all orders in H implicitly contain all order PM contributions in κ.

Plane wave metrics have several associated geometric structures. First, there is a zweibein $E^a_i(x^-)$ and its inverse $E^{ia}(x^-)$, labeled by the index $i = 1, 2$, satisfying $\ddot{E}_{ia} = H_{ab}E^b_i$, $\dot{E}^a_{[i}E_{j]a} = 0$. The zweibein encodes gravitational (velocity) memory through the difference

$$\Delta E^i_a = E^i_a(x^- > x^-_f) - E^i_a(x^- < x^-_i), \tag{5}$$

which compares the relative transverse positions of two neighboring geodesics. The zweibein also defines a transverse metric $\gamma_{ij}(x^-) := E^a_{(i}E_{j)a}$ and deformation tensor $\sigma_{ab}(x^-) := \dot{E}^i_a E_{ib}$, the latter encoding the expansion and shear of the null geodesic congruence associated to Eq. (4). These definitions are completed by the initial condition $E^i_a(x^- < x^-_i) = \delta^i_a$, which yields $\gamma_{ij}(x^- < x^-_i) = \delta_{ij}$ and $\sigma_{ab}(x^- < x^-_i) = 0$.

Turning to electromagnetism, plane waves can be defined by the potential $A_\mu(x) = -x^b E_b(x^-)n_\mu$ in light front coordinates [given by the flat space part of Eq. (4)] and n_μ is as above. $E_b(x^-)$ is the two-component, compactly supported electric field. A useful associated quantity is

$$a_\perp(x^-) := \int_{-\infty}^{x^-} ds E_\perp(s), \tag{6}$$

such that $ea_\perp$ is the effective "work done" on a charge. The electromagnetic velocity memory effect is encoded in the constant $ea_\perp(x^- > x^-_f)$ [74]; this is the change in transverse momentum of a particle crossing the background from the asymptotic past to the future.

To simplify the presentation of our results, we make the assumption that velocity memory effects induced by our backgrounds are parametrically small, and thus negligible. (We relax this assumption in Supplemental Material B [75].) This means setting $a_b(x^- > x^-_f) = 0$ in electromagnetism, and $E^i_a(x^- > x^-_f) = \delta^i_a$ in gravity. The main simplification is that the tree-level two-point amplitudes reduce to $\langle p'|\mathcal{S}|\Psi\rangle \to e^{i\theta(p')}\phi(p')$, for a theory-dependent phase θ which can be absorbed by redefining u [76].

Electromagnetism.—We now construct the classical limit of the electromagnetic waveform $W_{\mu\nu}(u,\hat{x})$ from Eq. (3). Given our assumption of no memory, the only ingredient required is the three-point amplitude for a charged scalar, on an electromagnetic plane wave background, to emit a photon. Let the incoming (outgoing) scalar have momentum p_μ (p'_μ), and the emitted photon have momentum k_μ and helicity η. The amplitude is calculated by evaluating the cubic part of the action on the appropriate scattering states in a plane wave; see, e.g., Ref. [64]. The result is

$$\langle p', k^\eta|\mathcal{S}|\Psi\rangle = \int d\Phi(p)\phi(p)e^{ip\cdot b/\hbar}\hat{\delta}^3_{+,\perp}(p'+k-p)\mathcal{A}_3,$$
$$\mathcal{A}_3 = -\frac{2ie}{\hbar^{3/2}}\int_y \varepsilon^\eta\cdot P(y)\exp\left[\frac{i}{\hbar}\int_{-\infty}^y dz\frac{k\cdot P(z)}{p_+-k_+}\right], \tag{7}$$

where $\int_y := \int_{-\infty}^\infty dy$ and $\hat{\delta}(x) := 2\pi\delta(x)$. The "dressed" momentum $P_\mu(y)$ is the classical momentum of the particle in the background,

$$P_\mu(y) = p_\mu - ea_\mu(y) + n_\mu\frac{2ea(y)\cdot p - e^2a^2(y)}{2p_+}, \tag{8}$$

where $a_\mu(y) = \delta_\mu^\perp a_\perp(y)$, obeying $P^2(y) = m^2$. Only three components of overall momentum are conserved in $\mathcal{A}_3$ as the background breaks x^--translation symmetry.

Calculation of the waveform.—We assemble the QED waveform in Eq. (3) from Eq. (7), using the assumption of negligible memory effects. We perform the sum over photon helicities using the completeness relation in light front gauge. All gauge-dependent pieces vanish by antisymmetry or generate boundary terms which can be ignored [74], leaving only a contribution from $-\eta_{\mu\nu}$. An immediate simplification in the classical limit is that the δ function sets $p' = p$, and thus the wave packet appears as $|\phi(p)|^2$. This means that the impact parameter b drops out, and under the usual assumption that ϕ is sharply peaked around some classical momentum, we can integrate over p, localizing the integrand at the on-shell momentum of the incoming particle, which we continue to write as p for simplicity. This gives

$$W_{\mu\nu}(u,\hat{x}) = -\frac{ie}{4\pi^2p_+}\int_{y,\omega}\omega e^{-i\omega[u-\hat{x}\cdot X(y)]}\hat{x}_{[\mu}P_{\nu]}(y), \tag{9}$$

in which $X^\mu(y)$ is the classical particle orbit, obeying $X'_\mu(y) = P_\mu(y)/p_+$. Performing the frequency integral yields a very compact final expression for the classical waveform:

$$W_{\mu\nu}(u,\hat{x}) = \frac{e}{2\pi}\int_y \delta(u-\hat{x}\cdot X(y))\frac{d}{dy}\frac{\hat{x}_{[\mu}P_{\nu]}(y)}{\hat{x}\cdot P(y)}$$
$$= \frac{e}{2\pi}\sum_{\text{sols}}\frac{p_+}{\hat{x}\cdot P}\frac{d}{dx}\frac{\hat{x}_{[\mu}P_{\nu]}}{\hat{x}\cdot P}, \tag{10}$$

where the sum runs over all solutions of the δ-function constraint. It can be checked that this matches the result obtained directly from classical electrodynamics; see Supplemental Material A [75].

Properties of the waveform.—First observe that, due to the derivative, the waveform is vanishing in the absence of acceleration. Indeed, the final integration by parts, performed as part of the evaluation of the frequency integral, corresponds to removing Coulomb field contributions from the asymptotic waveform, i.e., restricting to the radiation field which is of interest [78].

Next, observe from Eq. (8) that the dressed momentum P, hence the orbit X, is quadratic in the coupling e: it follows immediately that the waveform contains terms of *all* orders in e. This is both explicit, due to the presence of P in the denominator, and implicit, in that one must solve the δ-function constraint. This requires inverting $\hat{x}\cdot X(y)$ which will introduce arbitrary *nonpolynomial* dependence on the coupling. (Even for the simple but unphysical choice of a "box" electric field, solving the constraint means solving a cubic equation.) In general, there will be *multiple* solutions to the constraint, meaning that the waveform at any given $(u, \hat{x}_\mu)$ is sourced at several points on the orbit.

We examine $W_{\mu\nu}$ by choosing a specific plane wave profile and other kinematic data; Fig. 1 illustrates the rich structure found in the classical waveform for a "Sauter pulse" defined by $ea_1 = m\xi\,\mathrm{sech}^2(\nu y^-)$ and $a_2 = 0$ for strength ξ and frequency ν. Furthermore, the all orders property of the waveform can be made explicit in the case of an impulsive plane wave, for which all integrals can be performed; see Supplemental Material B [75]. Alternatively, we can expand in powers of e, recovering the first perturbative contribution to our waveform, coming from Compton scattering in vacuum [32]; see Supplemental Material C [75].

For any plane wave, we can consider the waveform aligned with the direction of the background: $\hat{x}_\mu = \sqrt{2}n_\mu$ (the factor results from conventions). Parametrizing $\hat{x}^\mu$ by azimuthal and polar angles ϕ and θ, respectively, alignment with the background corresponds to $\theta = 0$. At this collinear point the argument of the δ function is simply $u - \sqrt{2}y$, and thus has a single point of support. Most of the structure in the waveform vanishes due to contraction or commutation with n_μ, and one finds

$$W_{\mu\nu}|_{\theta=0} = -\frac{e^2}{4\pi}\frac{F_{\mu\nu}\left(\frac{u}{\sqrt{2}}\right)}{p_+\sqrt{2}}, \tag{11}$$

a result we will later contrast with gravity. If we consider any other point on the celestial sphere, the waveform has a far richer structure, though—see again Fig. 1.

Gravity.—We now require the tree-level three-point amplitude for a massive scalar emitting a graviton, on the gravitational plane wave background. Let the on-shell incoming (outgoing) momentum for the scalar be p_μ (p'_μ), but let k_μ now be the emitted graviton momentum.

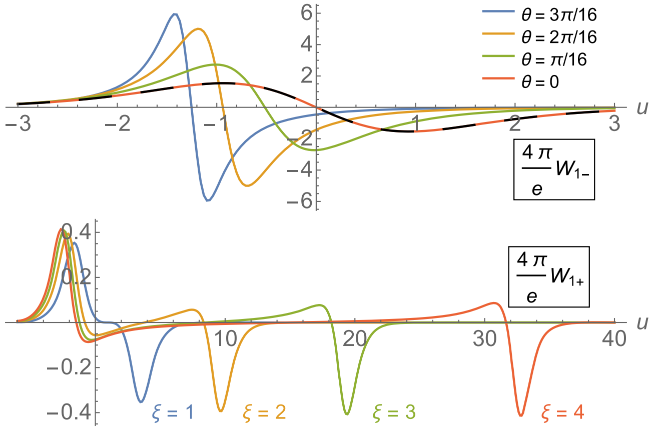

FIG. 1. Two examples of the waveform $W_{\mu\nu}(u,\hat{x})$ for a particle at rest struck by the wave $ea_1 = m\xi\text{sech}^2(\nu x^-)$ and $a_2 = 0$, for strength ξ and frequency ν. We work in units where $\nu = 1$. Top: $W_{1-}(u,\hat{x})$ as a function of u for various θ. We have fixed $\xi = 2$ and $\phi = 0$. At $\theta = 0$ (red and black dashed curve), the waveform is a multiple of the driving field $F_{\mu\nu}$ as in Eq. (11), but is very different for larger angles. Bottom: $W_{1+}(u,\hat{x})$ at fixed angles $\theta = \pi$, $\phi = 0$, showing the dependence of the waveform on the strength ξ of the background.

In contrast to QED, all particles are dressed in gravity: in scattering calculations, any particle of asymptotic momentum l_μ and mass m has the dressed momentum [79,80],

$$L_\mu(y)dy^\mu = l_+ dy^+ + (l_i E^i_a + l_+\sigma_{ab}y^b)dy^a + \left(\frac{m^2}{2l_+} + \gamma^{ij}\frac{l_i l_j}{2l_+} + \frac{l_+}{2}\dot{\sigma}_{bc}y^b y^c + l_i \dot{E}^i_b y^b\right)dy^-, \tag{12}$$

which obeys $g^{\mu\nu}L_\mu(y)L_\nu(y) = m^2$. Note that, in contrast to the dressed momentum Eq. (8) in QED, the gravitational dressing depends on the perpendicular coordinates y^a. The outgoing graviton polarization also becomes dressed by the background; it is conveniently expressed in terms of a projector acting on the free polarization:

$$\mathcal{E}^\eta_{\mu\nu}(k;y) = \mathbb{P}_{\mu\nu\sigma\rho}(k;y)\varepsilon^{\sigma\rho}_\eta := \left[\mathbb{P}_{\mu\rho}(k;y)\mathbb{P}_{\nu\sigma}(k;y) - \frac{i\hbar}{k_+}n_\mu n_\nu \delta^a_\rho \delta^b_\sigma \sigma_{ab}(y)\right]\varepsilon^{\sigma\rho}_\eta, \tag{13}$$

where $\mathbb{P}_{\mu\nu}(k;y) = g_{\mu\nu}(y) - 2K_{(\mu}(y)n_{\nu)}/k_+$ contains the dressed momentum $K_\mu(y)$ of the graviton. With these ingredients and the simplification of negligible memory, we can write down the required amplitude [80]:

$$\mathcal{A}_3 = -\frac{2i\kappa}{\hbar^{3/2}}\int_y \frac{\exp[i\mathcal{V}(y)]}{\sqrt{|E(y)|}}\mathcal{E}^\eta_{\mu\nu}(k;y)P^\mu(y)P'^\nu(y), \tag{14}$$

where the first line of Eq. (7) still holds; the exponent is

$$\mathcal{V}(y) := \frac{1}{\hbar}\int_{-\infty}^{y^-} dz \frac{P_\mu(z)K_\nu(z)g^{\mu\nu}(z)}{p_+ - k_+}, \tag{15}$$

and $|E(y)|$ is the zweibein determinant. It can be checked that all contractions between dressed momenta and polarizations appearing are independent of the transverse coordinates, even though their constituents are not. Hence the integrand in Eq. (14) is a function of only y^-, and is (trivially) evaluated on the classical particle orbit parametrized by y^-.

The calculation proceeds as in QED; we assemble the waveform Eq. (3) from the three-point amplitude Eq. (14). Similarly to the QED case, we can restrict the sum over graviton polarizations to physical degrees of freedom. To obtain the classical limit of the waveform, we inspect powers of $\hbar$ in the amplitude Eq. (14) and in the definition Eq. (3); we again find that all prefactors of $\hbar$ cancel, and the classical limit is obtained by setting $\hbar = 0$ everywhere else. This again allows the wave packet to be integrated out, arriving at

$$W_{\mu\nu\sigma\rho}(u,\hat{x}) = \frac{\kappa^2}{\pi}\int_0^\infty \hat{d}\omega\omega^2 e^{-i\omega u}\hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \frac{e^{i\omega\bar{\mathcal{V}}(y)}}{\sqrt{|E(y)|}} \times \left[\eta_{\nu]\gamma}\eta_{\rho]\delta} - \frac{1}{2}\eta_{\nu]\rho]}\eta_{\gamma\delta}\right]\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y) \times P_\alpha(y)P_\beta(y) + \text{c.c.}, \tag{16}$$

in which the reduced exponent $\bar{\mathcal{V}}$ is

$$\frac{1}{2p_+}\int_{-\infty}^y dz \frac{m^2}{p_+}\hat{x}^+ + \gamma^{ij}(z)\left(\frac{p_+}{\hat{x}_+}\hat{x}_i\hat{x}_j + \frac{\hat{x}_+}{p_+}p_i p_j - 2p_i\hat{x}_j\right) = \hat{x}\cdot X(y), \tag{17}$$

for $X(y)$ the classical particle orbit and $\bar{\mathbb{P}}_{\mu\nu\sigma\rho}(\hat{x},y) := \mathbb{P}_{\mu\nu\sigma\rho}(k,y)|_{\bar{k}=\omega\hat{x}}$ evaluated on that orbit. Now, the tracelike term in Eq. (16) arising from the polarization sum can be simplified by first observing that

$$\eta_{\gamma\delta}\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y)P_\alpha(y)P_\beta(y) = m^2 + \frac{2ip_+^2}{\omega\hat{x}_+}\left[i\partial_-\bar{\mathcal{V}} - \frac{1}{2}\sigma^a_a\right](y).$$

It can be checked that the term in brackets is exactly the y^- derivative of the entire integrand in Eq. (16), and hence gives a boundary term which can be dropped, leaving only the mass term.

It remains to perform the ω integral. However, in contrast to QED, the projector $\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y)$ contains terms with different scaling in ω. We highlight this by defining

$$T^0_{\nu\rho}(\hat{x},y) := \frac{\mathbb{P}_{\nu\alpha}(\hat{x},y)\mathbb{P}_{\rho\beta}(\hat{x},y)P^\alpha(y)P^\beta(y) - \frac{1}{2}\eta_{\nu\rho}m^2}{\sqrt{|E(y)|}},$$
$$T^1_{\nu\rho}(\hat{x},y) := \frac{\delta^a_\nu\delta^b_\rho\sigma_{ab}(y)}{\hat{x}_+\sqrt{|E(y)|}}p_+^2, \tag{18}$$

such that the integrand scales in the frequency as $\sim\omega^2 T^0 - i\omega T^1$. Combining the presented term in Eq. (16) with its complex conjugate and trading explicit ω factors for y^-–derivatives gives our final result for the waveform:

$$W_{\mu\nu\sigma\rho}(u,\hat{x}) = -\frac{\kappa^2}{\pi}\hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \delta(u-\bar{\mathcal{V}}(y)) \times [\mathcal{D}^2 T^0_{\rho]\nu]}(\hat{x},y) - \mathcal{D}T^1_{\rho]\nu]}(\hat{x},y)], \quad (19)$$

in which the derivative $\mathcal{D}$ acts as

$$\mathcal{D}f(y) \coloneqq \frac{d}{dy}\left(\frac{f(y)}{\partial_-\bar{\mathcal{V}}(y)}\right). \quad (20)$$

Again, for confirmation of this result via classical general relativity calculations, see Supplemental Material A [75].

Properties of the waveform.—Some insight into the gravitational waveform is provided by observing from Eq. (17) that $\bar{\mathcal{V}}$ is determined by the 0PB classical orbit $X^\mu(y)$ of a particle crossing the plane wave spacetime. The orbit itself goes like the integral of the transverse metric $\gamma^{ij} = E^{(i|a|}E^{j)}_a$. Reinstating explicit dependence on the gravitational coupling by taking $H_{ab} \to \kappa H_{ab}$, it is clear that the integral of γ^{ij} will contain terms which are at least linear in κ. Since Eq. (19) contains terms which go like $\bar{\mathcal{V}}^{-1}$, as well as an integral localized in terms of $\bar{\mathcal{V}}$, it follows that the waveform will contain terms of *all orders* in the background and hence in κ. To connect to the PM construction of the waveform we expand in κ, showing in Supplemental Material C [75] that the leading contribution comes from gravitational Compton scattering.

While the nonlinearity of general relativity makes it harder to evaluate the waveform analytically for test plane wave profiles, progress can be made in the impulsive case where $\kappa H_{ab}(x^-) = \delta(x^-)\kappa\,\mathrm{diag}(\lambda,-\lambda)$. This is demonstrated in Supplemental Material B: the resulting waveform is explicitly all orders in $\kappa\lambda$. See also [81].

The structure of Eq. (19) indicates the presence of *tail* effects in the gravitational waveform. This follows from the fact that the two terms in the waveform descend directly from those in the polarization tensor Eq. (13). The background dressing of this polarization is directly related to the failure of the Huygens principle for gravitational perturbations in plane wave spacetimes: initial data localized on a light cone spread outside of the light cone as it evolves [79,82,83]. These effects are present in both the T^0 and T^1 terms of the 1PB waveform, with the T^1 contribution being pure tail; by comparison, in the PM expansion of the two-body problem tail effects only emerge at fourth order (see, e.g., Ref. [17]).

These tail effects are a consequence of the inherent nonlinearity of gravity compared to electromagnetism, and this leads to another interesting feature of the gravitational waveform which is not present in QED. Consider the case, as in Eq. (11), where the direction of observation $\hat{x}^\mu$ aligns with the wave direction n^μ, corresponding to azimuthal angle $\theta = 0$. The plane wave metric is not asymptotically flat in precisely this (and only this) direction [84], so we approach it with caution. For any $\theta \neq 0$, the gravitational waveform is well defined, but in the limit $\theta \to 0$, it is divergent. To see this, one expands $\hat{x}_\mu$ for small θ, i.e., $\hat{x}_j = \sin\theta\{\cos\phi, \sin\phi\} \sim \theta$, and

$$\hat{x}_+ = \frac{1-\cos\theta}{\sqrt{2}} \sim \theta^2, \qquad \hat{x}_- = \frac{1+\cos\theta}{\sqrt{2}} \sim 1.$$

With this, it is simplest to pick components of W, and to focus on the pure tail term which contains the deformation tensor σ. The contribution of this term to W_{-a-b} is

$$\frac{\kappa^2 p_+^2 \hat{x}_-\hat{x}_-}{\pi\hat{x}_+}\int_y \delta(u-\bar{\mathcal{V}}(y))\mathcal{D}\frac{\sigma_{ab}(y)}{\sqrt{|E(y)|}} \sim \frac{1}{\theta^2}, \quad (21)$$

in which the $1/\hat{x}_+$ term generates the divergence (while $\bar{\mathcal{V}}$ and $\partial_-\bar{\mathcal{V}}$ remain finite in the limit $\theta \to 0$). The divergence reflects the fact that it is not possible to “scatter” gravitons in the n_μ direction, in which the background is not asymptotically flat; the interaction between the emitted radiation and the background never switches off. This is in contrast to QED, where the photon and background do not interact, and the waveform remains finite, cf. Eq. (11). (Indeed the distinction with QED is visible at the entirely perturbative level of the scalar-graviton Compton amplitude, which is singular at forward scattering [85].) The angular divergence would have physical consequences; it will enter, via the Riemann tensor, into the geodesic deviation equation for a null congruence at the next order of the PB expansion. The divergence will thus emerge as a physical singularity describing a region of spacetime in which null geodesics become infinitely separated. It would be interesting to investigate this.

Conclusions.—We have derived the gravitational waveform emitted by a massive particle when it scatters off a gravitational plane wave background, a solution to the fully nonlinear Einstein equations. Analogous formulas have been presented for the electromagnetic case. In contrast to existing results, these waveforms are manifestly *all orders* in the coupling, and exhibit a rich structure including tail effects that usually enter at higher order in the PM expansion. Our results underline the power of using strong-field amplitudes to study classical physics [46]. In future work we aim to go to higher orders in the PB expansion, including higher points and loops. There is no conceptual obstacle to doing so, and we expect this to provide easier access to observables of interest in classical gravity. It would also be interesting to consider other physically relevant strong backgrounds, like black holes or beams of gravitational radiation, and to analyze our results for specific profiles arising as plane wave limits of these backgrounds.

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*t.adamo@ed.ac.uk
†acristof@exseed.ed.ac.uk
‡anton.ilderton@ed.ac.uk
§s.klisch@ed.ac.uk

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