Complete Hierarchy for High-Dimensional Steering Certification

Carlos de Gois,[1] Martin Plávala,[1] René Schwonnek,[2] and Otfried Gühne[1]
[1]*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany*
[2]*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167, Hannover, Germany*

(Received 6 February 2023; accepted 8 June 2023; published 7 July 2023)

High-dimensional quantum steering can be seen as a test for the dimensionality of entanglement, where the devices at one side are not characterized. As such, it is an important component in quantum informational protocols that make use of high-dimensional entanglement. Although it has been recently observed experimentally, the phenomenon of high-dimensional steering is lacking a general certification procedure. We provide necessary and sufficient conditions to certify the entanglement dimension in a steering scenario. These conditions are stated in terms of a hierarchy of semidefinite programs, which can also be used to quantify the phenomenon using the steering dimension robustness. To demonstrate the practical viability of our method, we characterize the dimensionality of entanglement in steering scenarios prepared with maximally entangled states measured in mutually unbiased bases. Our methods give significantly stronger bounds on the noise robustness necessary to experimentally certify high-dimensional entanglement.

DOI: 10.1103/PhysRevLett.131.010201

Introduction.—Controlling increasingly higher-dimensional quantum systems is one of the keys that can unlock the advantage of quantum technologies over classical predecessors. Indeed, a central promise of quantum computing is an improved scaling of the required qubits in comparison to classical bits, with a similar perspective applying, e.g., for quantum metrology or the capacity of quantum communication channels. On the other side, employing high-dimensional systems can improve the noise robustness of quantum information protocols and experiments [1–8], and by this the central bottleneck for applications like quantum cryptography or a quantum network can be removed.

In any of the above applications, the genuine use of high-dimensionality demands the ability of creating and controlling entangled quantum states with a high Schmidt number. In operational terms, this number asks for the minimal local Hilbert space dimension k that two parties, Alice and Bob, have to possess for holding shares of an entangled state ρ_{AB}.

Certifying this number in the context of the one-side device-independent setting of a steering experiment [see Fig. 1(a)] is a task that recently gained a significant amount of attention [9–12]. Here we assume a situation in which only one party, say Bob, has the ability to fully characterise his local quantum system. The specifics of the other party, Alice, are kept hidden. The only way in which Alice can interact with her system is by applying black-box measurements, of which she can only control an input x and observe a corresponding output a.

Alice's task in this situation is to find suitable black-box measurements and an initial shared quantum state ρ that allows her to convince Bob that they hold a state with a Schmidt number at least k. This task can be seen as a fundamental building block for the verification of quantum hardware, since a successful Alice will in the same run also

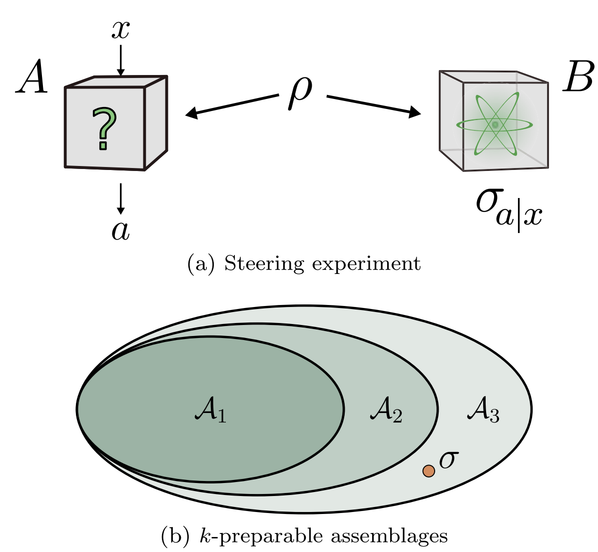

FIG. 1. In a steering experiment, Alice performs local measurements on a bipartite quantum state ρ shared between her and Bob. This results in a set of conditional states on Bob's side [a so-called state assemblage σ, see Eq. (1)], which can only be steerable if the shared state ρ were entangled. The central task in high-dimensional steering is to use an assemblage not only to witness entanglement in ρ, but to also quantify it. We say an assemblage is a k-preparable assemblage (and write that $\sigma \in \mathcal{A}_k$) if it can be prepared from a state ρ with Schmidt number smaller or equal to k.

prove her ability to control and nontrivially manipulate quantum states on a k-dimensional Hilbert space.

Despite its fundamental importance, the certification of high-dimensional steering can currently only be done for a very restricted set of cases. The usual tools employed here are linear steering witnesses, which typically come with the drawback that they are always designed around a given set of scenarios and resources that one aims to characterize. They usually only perform well when the corresponding black-box setting is applied, but tend to fail otherwise. For example, albeit in Ref. [9] tight witnesses are derived, they are limited to pairs of measurements. Their methods were later extended to more settings, but the obtained witnesses are quite weak [10,11].

In this Letter, we provide a general and complete hierarchy of semidefinite programs for high-dimensional steering certification. Our approach differs from the previous literature [9–11] in two important aspects. First, our starting point is the description of high-dimensional steering assemblages, while the previous methods were focused on optimizing a specific witness. Most importantly, our hierarchy holds for any number of measurement outcomes and measurement settings and it is always complete, meaning that for a given state assemblage on Bob's side one can, in principle, exactly determine the dimension of the entanglement that the assemblage certifies. Although this completeness is only guaranteed if all levels are tested, we demonstrate its usefulness in practice by obtaining significantly improved bounds on the noise robustness of assemblages that certify entanglement dimension $k > 2$.

High-dimensional steering.—In a steering experiment, Alice attempts to "steer" Bob's system by performing local measurements on her share of a quantum state ρ. Her measurements are uncharacterized, and her only input to the experiment is a label $x \in \{1, \ldots, X\}$, indicating a measurement choice. We denote her measurements by $\mathcal{M}_x = \{M_{a|x}\}$, where $M_{a|x} \geq 0$ are the effects and $\sum_a M_{a|x} = \mathbf{1}$. After Alice measures and announces her outcome to Bob, Bob's state ends up being

$$\sigma_{a|x} = \mathrm{tr}_A[\rho(M_{a|x} \otimes \mathbf{1}_B)]. \tag{1}$$

The collection of subnormalized states $\sigma = \{\sigma_{a|x}\}$ is called an assemblage. Since Alice cannot communicate with Bob, we require the assemblages to be nonsignaling: $\sum_a \sigma_{a|x} = \sum_a \sigma_{a|x'}$, for any choice of measurements x and x'. An assemblage is said to demonstrate steering if it cannot be explained by means of a local hidden state (LHS) model [13]

$$\sigma_{a|x} = \int_\Lambda d\lambda p(\lambda) p(a|x, \lambda) \sigma_\lambda, \tag{2}$$

where the states $\sigma_\lambda \in \mathcal{B}(H_B)$ are local to Bob, and λ is a latent, classical variable correlating Alice's and Bob's devices. Whenever the shared state ρ is separable, that is, whenever it can be written as $\rho = \sum_\lambda p(\lambda) \rho_\lambda^A \otimes \rho_\lambda^B$ for some local states ρ_λ^A and ρ_λ^B, all assemblages σ prepared from ρ admit a LHS model. Therefore, any steerable σ certifies entanglement of the shared state.

Entanglement, though, comes in many forms. In particular, it can be quantified [14], but steerability does not provide insights into how entangled ρ is. An object of recent discussion is whether, and how, can we use steering experiments to not only certify, but also quantify entanglement. The Schmidt number [15] is a popular quantifier in this and other correlation scenarios because it naturally leads to a notion of entanglement dimension (but other concepts also exist [16]). So, for example, if a bipartite state of local dimension d has Schmidt number 3, one can say it is only entangled in 3 of its d degrees of freedom and at the same time also certify that $d > 2$.

To properly define the Schmidt number, recall that a pure bipartite state $|\psi\rangle$ has Schmidt rank k if its Schmidt decomposition has k terms, $|\psi\rangle = \sum_{i=1}^k \nu_i |i, i\rangle$. Extending this definition to mixed states, we say that a state ρ has Schmidt number $sn(\rho) = k$ if (i) for any decomposition $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, at least one of the vectors $|\psi_i\rangle$ has Schmidt rank at least k, and (ii) there exists a decomposition of ρ with all vectors $|\psi_i\rangle$ having Schmidt rank at most k. We define the sets $\mathcal{S}_k = \{\rho | sn(\rho) \leq k\}$, and notice that they are convex, their extremal points are pure states, and $\mathcal{S}_{k-1} \subset \mathcal{S}_k$ for all $k \geq 2$, where $\mathcal{S}_1$ is the set of separable states.

Building on top of it, we say that an assemblage σ is k preparable if it can be prepared by local measurements on a state $\rho \in \mathcal{S}_k$ and we use the notation $sn(\sigma) \leq k$ to mean so, i.e., $sn(\sigma) \leq k$ if and only if $\sigma_{a|x} = \mathrm{tr}_A[\rho(M_{a|x} \otimes \mathbf{1}_B)]$ for some $\rho \in \mathcal{S}_k$. Furthermore, we define $\mathcal{A}_k = \{\sigma | sn(\sigma) \leq k\}$ as the set of all assemblages that can be prepared with states in $\mathcal{S}_k$. The sets $\mathcal{A}_k$ exhibit a nested structure $\mathcal{A}_{k-1} \subset \mathcal{A}_k$, for any $k \geq 2$ [9].

Naturally, the central question in high-dimensional steering is how to characterize these sets. In practice, given an assemblage, we want to be able to certify that it is not k preparable.

Main result.—Suppose $\sigma = \{\sigma_{a|x}\}$ is an assemblage in $\mathcal{A}_k$. Then, it can be obtained by means of a state ρ with Schmidt number k. Any such state can be seen as coming from a separable operator in an extended space, where an entangling projection was made. More precisely, we extend Alice's (A) and Bob's (B) subsystems with k-dimensional auxiliary spaces A' and B' and define the projection $\Pi_k = \mathbf{1} \otimes \sum_{i=1}^k |i, i\rangle_{A'B'}$. We call Π_k an entangling projection since it can be seen as an unnormalized maximally entangled state $|\phi_d^+\rangle = (1/\sqrt{d}) \sum_{i=1}^d |i, i\rangle$. Then, any pure state $|\psi\rangle = \sum_{i=1}^k \eta_i |i_A, i_B\rangle$ can be expressed as $|\psi\rangle = \Pi_k^\dagger(|i'_{AA'}, i'_{B'B}\rangle)$ if we set

$$|i'_{AA'}\rangle = \sum_{i=1}^{k} |i_A, i_{A'}\rangle, \qquad |i'_{B'B}\rangle = \sum_{i=1}^{k} \eta_i |i_{B'}, i_B\rangle. \quad (3)$$

Consequently, for mixed states,

$$\rho = \Pi_k^\dagger \underbrace{\left(\sum_i p_i \rho^i_{AA'} \otimes \rho^i_{B'B}\right)}_{\Omega} \Pi_k. \quad (4)$$

Therefore, any state ρ with Schmidt number k can be associated with an operator Ω which is separable w.r.t. the $AA'/B'B$ partition, at the expense of an entangling projection Π_k between A' and B' (Fig. 2). This fact has already been noted [17] and used to derive a Schmidt number certification hierarchy for entanglement [18].

With this in mind, when ρ has Schmidt number k we can recast the r.h.s. of Eq. (1) as

$$\begin{aligned} &\Pi_k^\dagger \left[\sum_i p_i \mathrm{tr}_A(\rho^i_{AA'}[M_{a|x} \otimes \mathbf{1}_{A'}]) \otimes \rho^i_{B'B}\right] \Pi_k \\ &= \Pi_k^\dagger \underbrace{\left[\sum_i p_i \tau^{i,a,x}_{A'} \otimes \rho^i_{B'B}\right]}_{\Omega_{a|x}} \Pi_k. \end{aligned} \quad (5)$$

Here, the set of $\tau^{i,a,x}_{A'}$ operators has the structure of an assemblage and, in particular, the nonsignaling condition $\sum_{i,a} \tau^{i,a,x}_{A'} = \sum_{i,a} \tau^{i,a,x'}_{A'}$, $\forall\ x \neq x'$ holds.

By this construction, an assemblage $\{\sigma_{a|x}\}$ is k preparable if and only if there are operators $\Omega_{a|x} \in \mathcal{B}(H_k \otimes H_k \otimes H_B)$ such that $\sigma_{a|x} = \Pi_k^\dagger \Omega_{a|x} \Pi_k$, where $\Omega_{a|x} = \sum_i p_i \tau^{i,a,x}_{A'} \otimes \rho^i_{B'B}$ and $\{\tau^{i,a,x}_{A'}\}$ is nonsignaling.

Although this provides an insight into the structure of high-dimensional steering assemblages, it does not evidence a way of determining the existence of the $\Omega_{a|x}$. As it turns out, this can be solved by means of a complete hierarchy of semidefinite tests, which can be seen as a generalization of the symmetric extensions criterion for entanglement certification [19].

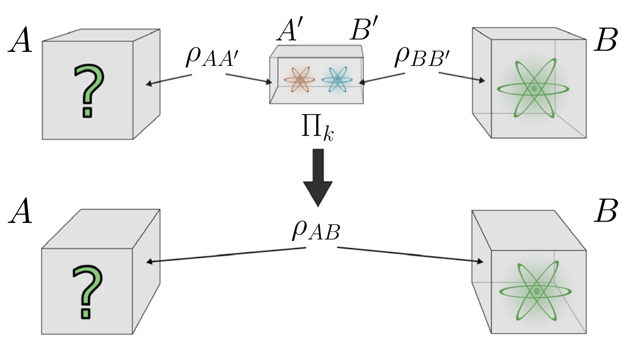

FIG. 2. Any state ρ with Schmidt number k acting on $\mathcal{B}(H_A \otimes H_B)$ is equivalent to some separable operator $\sum_i p_i \rho^i_{AA'} \otimes \rho^i_{B'B}$ in an extended Hilbert space, which is then projected with $\Pi_k = \mathbf{1} \otimes \sum_{i=1}^k |i,i\rangle_{A'B'}$ according to Eq. (4).

Given $N \geq 1$, $\Omega_{a|x}$ is said to have a nonsignaling symmetric extension of order N if there exists an operator $\Xi_{a|x} \in \mathcal{B}(H_{A'} \otimes H_{B'B}^{\otimes N})$ such that

$$\mathrm{tr}_{(B'B)_i:\ i\neq m}(\Xi_{a|x}) = \Omega_{a|x}, \quad \forall\ m \in \{1, \ldots, N\}, \quad (6a)$$

$$\sum_a \Xi_{a|x} = \sum_a \Xi_{a|x'}, \quad \forall\ x, x'. \quad (6b)$$

Here, the first condition ensures that when all but one of the systems in $B'B$ are traced out, $\Omega_{a|x}$ is recovered, while the second condition enforces nonsignaling in the extensions.

With the following theorem, we link the existence of symmetric extensions with that of the $\Omega_{a|x}$ operators described in Eq. (5).

Theorem 1.—An assemblage $\{\sigma_{a|x}\}$ is k preparable if and only if there are corresponding operators $\Omega_{a|x} \in \mathcal{B}(H_{A'} \otimes H_{B'B})$ such that $H_{A'}$, $H_{B'}$ are of dimension k and $\Omega_{a|x}$ has a nonsignaling symmetric extension [Eq. (6)] to any order N.

Proof.—The "only if" condition, essential for the applications described ahead, is easily seen to be satisfied by taking $\Xi_{a|x} = \sum_i p_i \tau^{i,a,x}_{A'} \otimes (\rho^i_{B'B})^{\otimes N}$, where we, without loss of generality, assumed that $\mathrm{tr}(\rho^i_{B'B}) = 1$. We postpone the other direction of the proof to the Supplemental Material [20]. ■

Theorem 1 tells us that, to test if an assemblage $\{\sigma_{a|x}\}$ is not k preparable, we can check whether there exists a nonsignaling symmetric extension of $\Omega_{a|x}$ to some order N. Thus, for some given assemblage $\{\sigma_{a|x}\}$, Schmidt number k, and hierarchy level N, we must search for extensions $\{\Xi_{a|x}\}$ that satisfy the constraints in Eq. (6) with $\sigma_{a|x} = \Pi_k^\dagger \Omega_{a|x} \Pi_k$.

This can be done with semidefinite programming. If the program is unfeasible (i.e., the symmetric extension does not exist), then $\{\sigma_{a|x}\}$ is not k preparable. Otherwise, the test is inconclusive and we can proceed to test a higher N. This hierarchy is complete in the sense that unfeasibility will eventually occur for all assemblages which are not k preparable, but never for k-preparable ones.

Certifying high-dimensional steering with maximally entangled states and MUB measurements.—Consider, as an example, the assemblage arising from a set of X d-dimensional mutually unbiased bases (MUB) measurements [21] acting on the maximally entangled state $|\phi_d^+\rangle = (1/\sqrt{d}) \sum_{i=1}^d |i, i\rangle$,

$$\sigma = \{\{\mathrm{tr}_A[|\phi_d^+\rangle\langle\phi_d^+| \cdot (M_{a|x} \otimes \mathbf{1}_B)]\}_{a=1}^d\}_{x=1}^X. \quad (7)$$

By mixing σ with the white noise assemblage $\sigma_{\mathbf{1}}$ (the one with elements $\mathbf{1}/d^2$) at a visibility η, we get a new assemblage with elements $\sigma_{a|x}(\eta) = \eta\sigma_{a|x} + (1-\eta)\mathbf{1}/d^2$.

Since σ_1 is always a feasible point in the program described above, one may rewrite it as a maximization on η. Any solution $\eta^* < 1$ to this maximization problem certifies that the assemblage $\{\sigma_{a|x}\}$ does not have nonsignaling symmetric extensions and therefore that $sn(\sigma) > k$.

However, even for reasonable values of the putative Schmidt number k and hierarchy level N, the size of the program increases superexponentially. To mitigate this blow up, one can make use of the symmetries in the problem.

First, observe that if the extension is of the form $\Xi_{a|x} = \sum_i p_i \tau_{A'}^{i,a,x} \otimes (\rho_{B'B}^i)^{\otimes N}$ then we can, without loss of generality, assume that $\rho_{B'B}^i$ are pure states and thus $\Xi_{a|x}$ acts on $S = H_{A'} \otimes \mathrm{Sym}_N(B'B)$, where $\mathrm{Sym}_N(B'B)$ is the symmetric subspace of $\mathcal{B}(H_{B'B}^{\otimes N})$. This space is of dimension $d_S = d_{A'} \times \binom{kd_B+N-1}{kd_B-1}$ [22], considerably reducing the number of variables in our problem. To put this observation to use, notice that, from the projector

$$\Pi_S = \mathbf{1}_{A'} \otimes \frac{1}{n!} \sum_{\pi \in S_n} P_\pi, \tag{8}$$

where S_N is the N elements permutation group, one may construct a matrix P_S whose d_S rows span the symmetric subspace. With that in hand, we can start the whole ordeal with a $d_S \times d_S$ matrix $\Xi'_{a|x}$ and substitute $\Xi_{a|x}$ for $P_S^\dagger \Xi'_{a|x} P_S$ in all constraints. More than that, due to the permutational symmetry, all partial traces need only be evaluated on one of the $B'B$ subsystems. Therefore, we not only reduce the dimension on the optimization variables, but also decrease the number of constraints.

Figure 3 shows numerical results obtained with an implementation of the method hereby described. We additionally made use of positive partial transpose constraints—which are known to accelerate convergence in the symmetric extensions hierarchy [23]—and explicitly enforced that $\mathrm{tr}(\Xi_{a|x}) = k\,\mathrm{tr}(\sigma_{a|x})$, which also leads to better results.

The results shown in Fig. 3 were computed for the lowest hierarchy level ($N = 2$), and even though Theorem 1 only guarantees convergence in the limit $N \to \infty$, they are already of practical significance. Remarkably, this lowest level already provide better bounds than the known values for high-dimensional steering witnesses with $X > 2$ measurements [[10], Table III], except for the $d = 5$, $X = 3$ case. In comparison to the previous results for $X = 2$ measurements [9], level $N = 2$ of our hierarchy provides slightly worse visibilities, but moving to level $N = 3$ makes up for it. For example, while they find a visibility of 0.886 (0.838) for dimension 3 (4), our method improves it to 0.844 (0.811).

Although most of these results [with the exception of $(d = 5, X = 5)$] were computed in a standard computer, increasing the desired Schmidt number k or the dimension d rapidly leads to instances that are too large to compute

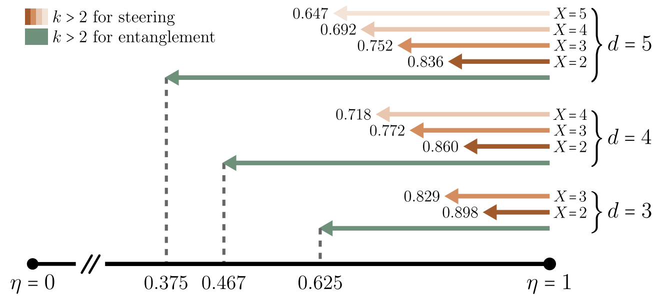

FIG. 3. Schmidt number (k) certification for assemblages prepared with maximally entangled states and X measurements on MUBs. The assemblages were mixed with the white noise assemblage and an upper bound on the minimum visibility such that 2-preparability can be falsified was computed. All results were computed with two copies of the $B'B$ system (i.e., $N = 2$). Increasing N can lead to smaller bounds. For instance, in the $d = 3$, $X = 3$ case, by setting $N = 3$ we were able to certify that $k > 2$ for all $\eta \geq 0.789$. For comparison, we also show analogous bounds for the entanglement scenario, obtained through a similar procedure: To certify the Schmidt number in an entanglement scenario, the standard symmetric extensions hierarchy can be directly applied to the observation illustrated in Fig. 2 (see also [18]).

with standard methods. For particular applications, however, it is possible to significantly reduce the computational cost by, for instance, exploiting symmetries of the assemblages. One example is described in the Supplemental Material [20]. It can be used when the measurements in the assemblage preparation have unitary symmetry (as is the case for MUBs), and effectively reduces the number of variables we must consider by a factor of X (the number of measurements).

Robustness of high-dimensional steering.—The optimal value η^*, as described above, can be interpreted as the distance between the assemblage and the set $\mathcal{A}_k$, measured along the line segment connecting σ to the white noise assemblage. A particularly interesting alternative choice of noise model to be considered is that of all k-preparable assemblages. Borrowing ideas from the entanglement and steering robustnesses [24,25], we define the k-preparability robustness w.r.t. noise model $\mathcal{N}$ as the real number $R_{\mathcal{N}}(\sigma, k) = t^*$ resulting from the following program, in the limit $N \to \infty$.

$$\min\ t \tag{9a}$$

s.t.

$$\frac{\sigma_{a|x} + t\pi_{a|x}}{1 + t} = \sigma^k_{a|x} \tag{9b}$$

$$\Pi_k^\dagger(\mathrm{tr}_{(BB')_i:\ i\neq m}[\Xi_{a|x}])\Pi_k = \sigma^k_{a|x}, \tag{9c}$$

$$\sum_a \Xi_{a|x} = \sum_a \Xi_{a|x'}, \quad \forall\ x, x' \tag{9d}$$

Since the minimization ranges over $\Xi_{a,x}$, t and the noise operators $\pi_{a|x} \in \mathcal{N}$, this is not a semidefinite program. Nevertheless, when $\mathcal{N}$ is any set defined by a linear matrix inequality (LMI), it can be turned into one, as shown in the Supplemental Material [20].

We have already shown that the set of Schmidt number k assemblages ($\mathcal{A}_k$) can be approximated by means of a similar hierarchy of semidefinite programs. Thus, any choice N for the hierarchy level will provide an upper bound $R^N_{\mathcal{N}}(\sigma, k) \geq R_{\mathcal{N}}(\sigma, k)$. By taking $\mathcal{N} = \mathcal{A}_k$, we can interpret $R_{\mathcal{A}_k}(\sigma, k)$ as the minimal distance between the assemblage σ and the set of Schmidt number k assemblages.

Conclusions.—We have provided a complete set of criteria for the certification of high-dimensional steering. The criteria are formulated as a sequence of semidefinite programs, such that any kind of high-dimensional steering will be detected at one point of the sequence. Moreover, we have demonstrated the practical implementation of our method and it has turned out that it improves existing results significantly.

In the context of known results in quantum steering, our results have further consequences. To start with, steering is known to be in one-to-one correspondence with measurement incompatibility [26,27], a connection which was recently extended to high-dimensional steering and k-simulatability of measurements [12], in the sense defined in [28]. Thus our results also solve the problem of quantifying measurement incompatibility in terms of a dimension. On a more practical side, it is also possible to use our method to optimize high-dimensional steering witnesses (or its dual to obtain them), which might be a fruitful way of constructing tighter witnesses than possible with the current approaches [7,10].

Although our hierarchy for Schmidt number certification can be stated by means of semidefinite programs, it leads to a formidable computational problem. Ultimately, the size of the problem is dictated by the number of measurements and outcomes, the tentative Schmidt number k and the hierarchy level N. In particular, the Hilbert space dimension grows exponentially with k and N. A natural further step is thus to find cheaper relaxations or to adapt the formulation to specific problems by making use of further symmetries, as done in Refs. [29–32].

Another possible extension comes from the key elements in our proposal, which are the lifting of the state ρ by means of the ancillary spaces $H_{A'}$ and $H_{B'}$ [cf. Eq. (4)], and the reformulation of the symmetric extension criterion to the operators $\Omega_{a|x}$. In light of recent extensions of the symmetric extensions hierarchy for general cones [2], it should be possible to extend the ideas herein presented to Bell nonlocality and other correlation scenarios.

Many thanks to T. Cope, H. C. Nguyen, and R. Uola for useful discussions, and to the developers and maintainers of JULIA [33], SCS [34], and JUMP [35]. We acknowledge support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation, Projects No. 447948357 and No. 440958198), the Sino-German Center for Research Promotion (Project M-0294), the ERC (Consolidator Grant 683107/TempoQ), and the German Ministry of Education and Research (Project QuKuK, BMBF Grant No. 16KIS1618K). C. G. acknowledges support from the House of Young Talents of the University of Siegen. M. P. acknowledges support from the Alexander von Humboldt Foundation. R. S. acknowledges financial support by the BMBF project ATIQ and the Quantum Valley Lower Saxony. The OMNI cluster of the University of Siegen was used for the computations.

[1] T. Vértesi, S. Pironio, and N. Brunner, Phys. Rev. Lett. **104**, 060401 (2010).
[2] G. Aubrun, A. Müller-Hermes, and M. Plávala, arXiv:2206.11805.
[3] S. Ecker, F. Bouchard, L. Bulla, F. Brandt, O. Kohout, F. Steinlechner, R. Fickler, M. Malik, Y. Guryanova, R. Ursin, and M. Huber, Phys. Rev. X **9**, 041042 (2019).
[4] F. Zhu, M. Tyler, N. H. Valencia, M. Malik, and J. Leach, AVS Quantum Sci. **3**, 011401 (2021).
[5] P. Skrzypczyk and D. Cavalcanti, Phys. Rev. A **92**, 022354 (2015).
[6] M. Marciniak, A. Rutkowski, Z. Yin, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **115**, 170401 (2015).
[7] R. Qu, Y. Wang, M. An, F. Wang, Q. Quan, H. Li, H. Gao, F. Li, and P. Zhang, Phys. Rev. Lett. **128**, 240402 (2022).
[8] N. Miklin, A. Chaturvedi, M. Bourennane, M. Pawłowski, and A. Cabello, Phys. Rev. Lett. **129**, 230403 (2022).
[9] S. Designolle, V. Srivastav, R. Uola, N. H. Valencia, W. McCutcheon, M. Malik, and N. Brunner, Phys. Rev. Lett. **126**, 200404 (2021).
[10] S. Designolle, Phys. Rev. A **105**, 032430 (2022).
[11] R. Qu, Y. Wang, X. Zhang, S. Ru, F. Wang, H. Gao, F. Li, and P. Zhang, Optica **9**, 473 (2022).
[12] B. D. Jones, R. Uola, T. Cope, M. Ioannou, S. Designolle, P. Sekatski, and N. Brunner, arXiv:2207.04080.
[13] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[14] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[15] B. M. Terhal and P. Horodecki, Phys. Rev. A **61**, 040301(R) (2000).
[16] T. Kraft, C. Ritz, N. Brunner, M. Huber, and O. Gühne, Phys. Rev. Lett. **120**, 060502 (2018).
[17] F. Hulpke, D. Bruß, M. Lewenstein, and A. Sanpera, Quantum Inf. Comput. **4**, 207 (2004).
[18] M. Weilenmann, B. Dive, D. Trillo, E. A. Aguilar, and M. Navascués, Phys. Rev. Lett. **124**, 200502 (2020).
[19] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A **69**, 022308 (2004).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.010201 for proof for the main result, discussion on measurement symmetries and on steering robustness.

[21] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, Int. J. Quantum. Inform. **08**, 535 (2010).
[22] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).
[23] M. Navascués, M. Owari, and M. B. Plenio, Phys. Rev. A **80**, 052306 (2009).
[24] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
[25] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).
[26] M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014).
[27] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, Phys. Rev. Lett. **115**, 230402 (2015).
[28] M. Ioannou, P. Sekatski, S. Designolle, B. D. Jones, R. Uola, and N. Brunner, Phys. Rev. Lett. **129**, 190401 (2022).
[29] H. C. Nguyen, S. Designolle, M. Barakat, and O. Gühne, arXiv:2003.12553.
[30] E. A. Aguilar, J. J. Borkała, P. Mironowicz, and M. Pawłowski, Phys. Rev. Lett. **121**, 050501 (2018).
[31] A. Tavakoli, D. Rosset, and M.-O. Renou, Phys. Rev. Lett. **122**, 070501 (2019).
[32] M. Ioannou and D. Rosset, arXiv:2112.10803.
[33] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, SIAM Rev. **59**, 65 (2017).
[34] B. O'Donoghue, E. Chu, N. Parikh, and S. Boyd, J. Optim. Theory Appl. **169**, 1042 (2016).
[35] I. Dunning, J. Huchette, and M. Lubin, SIAM Rev. **59**, 295 (2017).