One Fixed Point Can Hide Another One: Nonperturbative Behavior of the Tetracritical Fixed Point of O(N) Models at Large N

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We show that at $N = \infty$ and below its upper critical dimension, $d < d_{\rm up}$, the critical and tetracritical behaviors of the O(N) models are associated with the same renormalization group fixed point (FP) potential. Only their derivatives make them different with the subtleties that taking their $N \to \infty$ limit and deriving them do not commute and that two relevant eigenperturbations show singularities. This invalidates both the ϵ—and the $1/N$—expansions. We also show how the Bardeen-Moshe-Bander line of tetracritical FPs at $N = \infty$ and $d = d_{\rm up}$ can be understood from a finite-N analysis.

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Field theories sometimes exhibit nonperturbative features such as confinement [1], presence of bound states [2], or exotic excitations [3], fixed points (FPs) of the renormalization group (RG) flows that are nonperturbative as in the Kardar-Parisi-Zhang equation [4], divergence of the perturbative RG flow at a finite RG scale [5], and the presence of a cusp in the FP potential as in the random field Ising model [6], to cite but a few. Very often, these nonperturbative effects are assumed either to occur in rather complicated theories such as gauge and string theories or in highly nontrivial statistical models.

O(N) models, which are the simplest scalar field theories, are often implicitly considered to be immune to these complex phenomena. Perturbative methods are therefore assumed to work almost all the time for these models, the exception to the rule being the Bardeen-Moshe-Bander (BMB) phenomenon [7], related to the existence of a line of tricritical FPs at $N = \infty$ and $d = 3$, which requires nonperturbative FPs to be fully understood from a large-N analysis [8]. From this viewpoint, the enormous success of the $\epsilon = 4 - d$ expansion for the perturbative calculation of the critical exponents associated with the Wilson-Fisher (WF) FP [9] could let us believe that the critical physics of the O(N) models is fully understood for any N and d, especially since it is corroborated by the $1/N$ and $\epsilon = d - 2$ expansions [9].

Our goal in this Letter is to show instead that although the critical physics of the O(N) models, described by the WF FP, is fully under perturbative control at both finite and infinite N, the tetracritical physics of these models at $N = \infty$—and probably of infinitely many multicritical behaviors—is not. We show below (i) that at $N = \infty$, it is also associated with the WF FP, which is unexpected, and (ii) that it nonetheless shows nonperturbative features that are beyond the reach of the standard implementation of both the large-N and ϵ- expansions. We show in particular a very intriguing phenomenon related to the large-N limit of the tetracritical FP of the O(N) models: from the second order, the derivatives of the $N = \infty$ tetracritical FP potential, that is, of the WF FP potential, are not identical to the limit of the derivatives of the finite-N tetracritical FP potentials when $N \to \infty$. This turns out to be crucial for understanding the large-N limit of tetracritical phenomena and shows that this limit is much less trivial than what is usually said [9–11].

The perturbative tetracritical FP corresponds to the massless $(\boldsymbol{\varphi}^2)^4$ theory, the upper critical dimension of which is $d_{\rm up} = 8/3$. It is found in perturbation theory in $\epsilon = 8/3 - d$ for all $N \geq 1$, and it is 3 times infrared unstable [12]. Calling $\lambda/(384N^3)$ the coupling in front of the dimensionless $(\boldsymbol{\varphi}^2)^4$ term, the large-N perturbative flow equation for λ reads as [13]

$$\partial_t \lambda = -3\epsilon\lambda + \frac{9\lambda^2}{4N} + \mathrm{O}(N^{-2}). \tag{1}$$

From Eq. (1), we find that at leading order in N, the nontrivial FP solution is $\tilde{\lambda} = 4\epsilon N/3$ from which follows that perturbation theory does not allow for a control of the large-N limit of the tetracritical FP at fixed ϵ. Only the double limit $N \to \infty$ and $\epsilon \to 0$ such that the product ϵN remains finite can possibly be under control. We come back on this point in the following.

Let us recall that in generic dimensions $d < 4$, the only nontrivial FP found in the standard large-N analysis of the O(N) models is the WF FP [14]. Thus, no tetracritical FP is found at $N = \infty$ and $d < 8/3$ which is paradoxical considering that it is perturbatively found for all $N < \infty$ and $\epsilon > 0$.

We show below that the solution to the paradox above lies in the field dependence of the tetracritical FP potential whereas it cannot be obtained from its field expansion and in particular from $\tilde{\lambda}$. The recourse to functional RG methods is therefore mandatory.

The best way to implement functional RG is to consider Wilson's RG, as it is inherently functional [15]. We recall below the take-away philosophy of the modern version of Wilson's RG known as the nonperturbative—or functional—renormalization group (NPRG).

NPRG is based on the idea of integrating fluctuations step by step [16]. It is implemented on the Gibbs free energy Γ [17–23] of a model defined by a Hamiltonian (or Euclidean action) H and a partition function $\mathcal{Z}$. To this model is associated a one-parameter family of models with Hamiltonians $H_k = H + \Delta H_k$ and partition functions $\mathcal{Z}_k$, where k is a momentum scale. In H_k, ΔH_k is chosen such that only the rapid fluctuations in the original model, those with wave numbers $|q| > k$, are summed over in the partition function $\mathcal{Z}_k$. Thus, the slow modes ($|q| < k$) need to be decoupled in $\mathcal{Z}_k$, and this is achieved by giving them a mass of order k, that is, by taking for ΔH_k a quadratic (masslike) term, which is nonvanishing only for the slow modes:

$$\mathcal{Z}_k[\boldsymbol{J}] = \int D\boldsymbol{\varphi}_i \exp(-H[\boldsymbol{\varphi}] - \Delta H_k[\boldsymbol{\varphi}] + \boldsymbol{J}\cdot\boldsymbol{\varphi}) \quad (2)$$

with $\Delta H_k[\boldsymbol{\varphi}] = \frac{1}{2}\int_q R_k(q^2)\varphi_i(q)\varphi_i(-q)$, where, for instance, $R_k(q^2) = (k^2 - q^2)\theta(k^2 - q^2)$ and $\boldsymbol{J}\cdot\boldsymbol{\varphi} = \int_x J_i(x)\,\varphi_i(x)$. The k-dependent Gibbs free energy $\Gamma_k[\boldsymbol{\phi}]$ is defined as the (slightly modified) Legendre transform of $\log \mathcal{Z}_k[\boldsymbol{J}]$:

$$\Gamma_k[\boldsymbol{\phi}] + \log \mathcal{Z}_k[\boldsymbol{J}] = \boldsymbol{J}\cdot\boldsymbol{\phi} - \frac{1}{2}\int_q R_k(q^2)\phi_i(q)\phi_i(-q) \quad (3)$$

with $\int_q = \int d^dq/(2\pi)^d$. With the choice of regulator function R_k above, $\Gamma_k[\phi]$ interpolates between the Hamiltonian H when k is of order of the ultraviolet cutoff Λ of the theory: $\Gamma_\Lambda \sim H$, and the Gibbs free energy Γ of the original model when $k = 0$: $\Gamma_{k=0} = \Gamma$. The exact RG flow equation of Γ_k gives the evolution of Γ_k with k between these two limiting cases. It is known as the Wetterich equation. It reads as [18]

$$\partial_t\Gamma_k[\boldsymbol{\phi}] = \frac{1}{2}\mathrm{Tr}\big(\partial_t R_k(q^2)\{\Gamma_k^{(2)}[q,-q;\boldsymbol{\phi}] + R_k(q)\}^{-1}\big), \quad (4)$$

where $t = \log(k/\Lambda)$, Tr stands for an integral over q and a trace over group indices, and $\Gamma_k^{(2)}[q,-q;\boldsymbol{\phi}]$ is the matrix of the Fourier transforms of $\delta^2\Gamma_k/\delta\phi_i(x)\delta\phi_j(y)$.

In most cases, Eq. (4) cannot be solved exactly, and approximations are mandatory. The best known approximation consists of expanding Γ_k in powers of the derivatives of ϕ_i and to truncate the expansion at a given finite order [24–32]. The approximation at lowest order is dubbed the local potential approximation (LPA). For the O(N) model it consists of approximating Γ_k by

$$\Gamma_k[\boldsymbol{\phi}] = \int_x \left(\frac{1}{2}(\nabla\phi_i)^2 + U_k(\phi)\right) \quad (5)$$

where $\phi = \sqrt{\phi_i\phi_i}$. Fixed points are found only for dimensionless quantities and the standard large-N limit by rescaling the field and the potential by factors $N^{-1/2}$ and N^{-1} respectively. Thus, we define the dimensionless and rescaled field $\bar{\phi}$ and potential $\bar{U}_k$ as $\bar{\phi} = v_d^{-(1/2)}k^{(2-d)/2}N^{-1/2}\phi$ and $\bar{U}_k(\bar{\phi}) = v_d^{-1}k^{-d}N^{-1}U_k(\phi)$ with $v_d^{-1} = 2^{d-1}d\pi^{d/2}\Gamma(d/2)$. The LPA flow of $\bar{U}_k$ then reads as

$$\partial_t\bar{U}_k(\bar{\phi}) = -d\bar{U}_k(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\bar{U}_k'(\bar{\phi}) + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}_k'(\bar{\phi})} + \frac{1}{N}\frac{1}{1 + \bar{U}_k''(\bar{\phi})} \quad (6)$$

with $\partial_t = k\partial_k$. The standard large-N limit of the LPA flow equation above is obtained by (i) replacing the factor $1 - 1/N$ by 1 and (ii) dropping the last term in Eq. (6) because it is assumed to be subleading [33]. As a consequence of the two steps above, the explicit dependence in N in Eq. (6) disappears in the large-N limit.

The crucial point of the large-N limit is that assuming point (ii) above, the resulting LPA flow equation on $\bar{U}_k$ can be shown to be *exact* in the limit $N \to \infty$ [34]. Under this assumption, all FPs of the O(N) models have been found exactly at $N = \infty$ [14,33–36]. The result is the following: In a generic dimension $d < 4$ there is only one non-Gaussian FP at $N = \infty$ which is the usual WF FP. The exceptions to the rule above are the BMB lines of FPs [7,14,37–39] existing in dimensions $d = 2 + 2/p$ with p an integer larger than 1.

We now show that the procedure described above is too restrictive to study the large-N limit of the tetracritical FPs. As said above, the standard large-N analysis consists of neglecting the last term in Eq. (6). However, this term is negligible only if $[1 + \bar{U}_k''(\bar{\phi})]^{-1}$ does not counterbalance at large N its $1/N$ prefactor for some finite values of $\bar{\phi}$. We now show that because of singularities in the third derivative of $\bar{U}_k(\bar{\phi})$, the contribution of the last term in Eq. (6) cannot be neglected in the FP equation of $\bar{U}_k''(\bar{\phi})$ obtained by differentiating Eq. (6) twice [see discussion below Eq. (8) for more detail]. This turns out to be sufficient to invalidate the standard large-N limit in the tetracritical case.

We have numerically solved Eq. (6) and have found for several values of N and $d < 8/3$ the perturbative tetracritical FP that we call $T_3(N, d)$. As expected, T_3 bifurcates from the Gaussian FP in $d = 8/3^-$. We have followed it

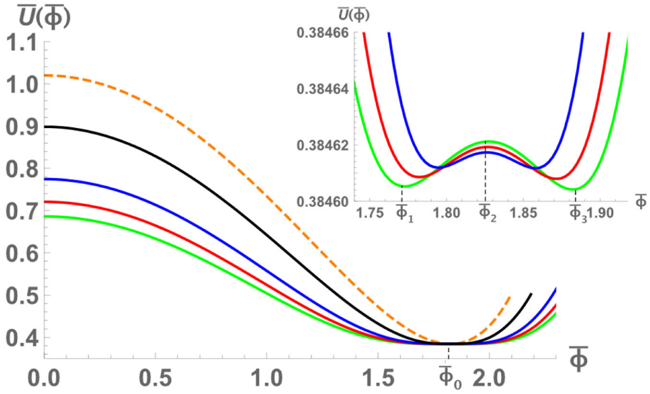

FIG. 1. $d = 2.6$: $\bar{U}(\bar{\phi})$ for the T_3 FP of Eq. (6). Green, red, blue and black curves correspond to $N = 1500$, 2250, 4500 and 42000. The orange dashed curve corresponds to the WF FP at $N = \infty$. Inset: Close view of $\bar{U}(\bar{\phi})$ around $\bar{\phi}_i$.

down to $d = 2.6$; see Figs. 1 and 3 of the Supplemental Material [40]. The FP potential of T_3, (i) shows as expected two maxima, one of which being located at $\bar{\phi} = 0$ and another one at $\bar{\phi}_2 > 0$, and two minima at $\bar{\phi}_1$ and $\bar{\phi}_3$ such that $\bar{\phi}_3 > \bar{\phi}_2 > \bar{\phi}_1 > 0$ (see Fig. 1); (ii) can be continuously followed up to arbitrarily large values of N at fixed $d < 8/3$; and (iii) has its three extrema $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3$ approaching each other when N is increased at fixed d. These extrema tend to a common value $\bar{\phi}_0$ when $N \to \infty$ which is the minimum of the FP potential; see Figs. 1 and 4 of the Supplemental Material [40]. Point (ii) above is paradoxical because it seems to contradict the standard large-N approach where only the WF FP is found in a generic dimension $d < 8/3$ at $N = \infty$. We now show that the WF FP potential at $N = \infty$ is in fact the limit when $N \to \infty$ of the potential of T_3 for $d < 8/3$. This solves the above paradox because it explains why on one hand there exists a nontrivial tetracritical FP at $N = \infty$ and $d < 8/3$ and on the other hand there is no other nontrivial and smooth solution of Eq. (6) at $N = \infty$ than the WF FP potential. However, this creates a new paradox since obviously the critical and tetracritical universal behaviors cannot be the same since the two FPs do not have the same number of unstable eigendirections. We now explain in detail this new paradox.

We can see in Fig. 1 that the FP potentials found in $d = 2.6$ for large values of N are extremely flat in the region $\bar{\phi} \in [\bar{\phi}_1, \bar{\phi}_3]$ because the three extrema are very close and the height of the barrier between the two minima very small. We have numerically found that the height of the barrier scales as N^{-1} and the distance between the two minima as $N^{-1/2}$ so that the curvatures $\bar{U}''(\bar{\phi}_i)$ at the three extrema approach constant values as $N \to \infty$; see Fig. 4 of the Supplemental Material [40]. This suggests that $\bar{U}''(\bar{\phi})$ while being well behaved everywhere but between the three extrema, changes very rapidly within a boundary layer around $\bar{\phi}_0$ of typical width $N^{-1/2}$, making divergent $\bar{U}'''(\bar{\phi}_0)$ when $N \to \infty$.

It is not common in physics to encounter this kind of situation where a sequence of functions $f_n(x)$ tends to a smooth function $f_\infty(x)$ whereas from a certain order p, their derivatives $f_n^{(p)}(x)$ do not tend to $f_\infty^{(p)}(x)$. However, a simple toy model explains trivially how this can occur. Consider the sequence of functions $f_n(x) = n^{-1}\sin(n^2 x)$. Obviously, $f_\infty(x) \equiv 0$ which implies that $f'_\infty(x) \equiv 0$ whereas $\lim_{n\to\infty} f'_n(0) = \infty$.

In our case, at fixed $d < 8/3$, the limit of the T_3 potentials when $N \to \infty$ is a nontrivial and well-defined function that therefore must be the WF FP potential. We have checked that it is indeed the limit of T_3 when $N \to \infty$; see Fig. 1. The difference between the critical and tetracritical behaviors is therefore not visible on the potentials themselves but only on their derivatives as we now show.

Let us study the boundary layer around $\bar{\phi}_0$. It is convenient for what follows to change variables. Following Ref. [41], we define $V(\mu) = U(\phi) + (\phi - \Phi)^2/2$ with $\mu = \Phi^2$ and $\phi - \Phi = -2\Phi V'(\mu)$. As above, it is convenient to rescale μ and $V(\mu)$: $\bar{\mu} = \mu/N$, $\bar{V} = V/N$. In terms of these quantities, the FP equation for $\bar{V}(\bar{\mu})$ reads as

$$0 = 1 - d\bar{V} + (d-2)\bar{\mu}\bar{V}' + 4\bar{\mu}\bar{V}'^2 - 2\bar{V}' - \frac{4}{N}\bar{\mu}\bar{V}''. \qquad (7)$$

Equation (7) has two remarkable features: (i) it is much simpler than Eq. (6) because the nonlinearity comes only from the $(\bar{V}')^2$ term, and (ii) it is the LPA equation obtained from the Wilson-Polchinski (WP) version of the NPRG [15,42,43]. Thus, $\bar{V}(\bar{\mu})$ is related to the potential $\bar{U}(\bar{\phi})$ of the Wetterich version of the RG by the Legendre transform of Eq. (3). The standard large-N analysis performed in this version of the NPRG consists here again of neglecting the last term in Eq. (7) because it is suppressed by a $1/N$ factor. Under the assumption that this term is indeed negligible, the resulting equation can be solved exactly in the large-N limit [14,35]. However, at large N, it is clear in Eq. (7) that we have to deal with singular perturbation theory since the small parameter used in the $1/N$ expansion is in front of the term of highest derivative, that is, $\bar{V}''$. In this case, it is well known that at large N a boundary layer can exist for a particular value of $\bar{\mu}$ that becomes a singularity at $N = \infty$, making this term non-negligible [44].

The value of $\bar{\mu}$ corresponding to $\bar{\phi}_0$ is called $\bar{\mu}_0$ and is the minimum of $\bar{V}(\bar{\mu})$ at $N = \infty$. We find for $\bar{V}(\bar{\mu})$ the same features about its three extrema $\bar{\mu}_i$ as for $\bar{U}(\bar{\phi})$ at $\bar{\phi}_i$: The three extrema $\bar{\mu}_i$ approach each other and to $\bar{\mu}_0$ as $N \to \infty$. The distances between them scale as $N^{-1/2}$ and the curvatures $\bar{V}''(\bar{\mu}_i)$ as N^0. Taking into account the scaling around $\bar{\mu}_0$ inside the boundary layer, we introduce another scaled variable $\tilde{\mu} = N^{1/2}(\bar{\mu} - \bar{\mu}_0)$. Since at $N = \infty$, $\bar{V}'(\bar{\mu})$ vanishes at $\bar{\mu} = \bar{\mu}_0$, $\bar{V}(\bar{\mu}_0)$ should approach $1/d$ at leading

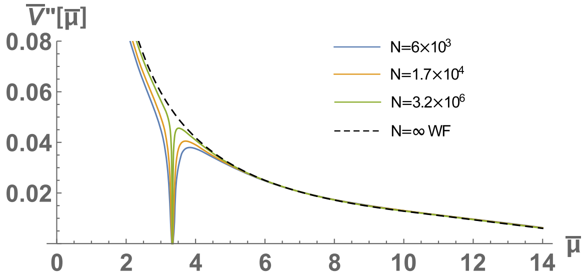

FIG. 2. Second derivative of the WF and T_3 FP potentials for different values of N in $d = 2.6$.

order in $N^{-1/2}$. We therefore define a scaled boundary layer by $\tilde{V}_N(\tilde{\mu}) = N[\bar{V}(\bar{\mu}_0 + N^{-1/2}\tilde{\mu}) - 1/d)]$ which implies $\tilde{V}''_N(\tilde{\mu}) = \bar{V}''(\bar{\mu}_0 + N^{-1/2}\tilde{\mu})$. We plot $\tilde{V}''_N(\tilde{\mu})$ for several values of N in Fig. 5 of the Supplemental Material [40].

By substituting $\tilde{V}_N(\tilde{\mu})$ with its value in Eq. (7) and solving it at order $O(N^{-1/2})$, we find that $\bar{\mu}_0 = 2/(d-2)$. At order $O(N^{-1})$, Eq. (7) becomes

$$-\frac{8\tilde{V}''_\infty(\tilde{\mu})}{d-2} + \frac{8\tilde{V}'_\infty(\tilde{\mu})^2}{d-2} + (d-2)\tilde{\mu}\tilde{V}'_\infty(\tilde{\mu}) - d\tilde{V}_\infty(\tilde{\mu}) = 0, \tag{8}$$

which is clearly invariant under $\tilde{\mu} \to -\tilde{\mu}$ from which it follows that $\tilde{V}'_\infty(0) = 0$. However this term is indispensable to describe the boundary layer of $\bar{U}''(\bar{\phi})$ or $\bar{V}''(\bar{\mu})$. At $\tilde{\mu} = \infty$, $\tilde{V}''_\infty(\tilde{\mu})$ should tend to a finite value that matches with $\bar{V}''(\mu)$ at $\bar{\mu}_0^+$. This implies that the solution of Eq. (8) should be quadratic when $\tilde{\mu} \to \infty$. Substituting $\tilde{V}_\infty(\tilde{\mu})$ with $\tilde{V}''_\infty(\tilde{\mu} = \infty)\tilde{\mu}^2/2$ in Eq. (8) and balancing the leading terms as $\tilde{\mu} \to \infty$, we find that $\tilde{V}''_\infty(\tilde{\mu} = \infty) = -(d-2)(d-4)/16$. We note that $\tilde{V}''_\infty(\tilde{\mu} = \infty)$ does not vanish in the range $2 < d < 8/3$ where we analyze T_3, which validates the above argument at leading order. Imposing the two boundary conditions found above at $\tilde{\mu} = 0$ and $\tilde{\mu} = \infty$ selects a unique and globally defined solution $\tilde{V}''_\infty(\tilde{\mu})$ of Eq. (8) shown in Fig. 5 of the Supplemental Material [40]. We find $\bar{V}''(\bar{\mu}_0^+) = \bar{V}''_{\rm WF}(\bar{\mu}_0) = \tilde{V}''_\infty(\tilde{\mu} = \infty)$ which proves the matching at $N = \infty$ between the boundary layer and the potential outside of the layer; see Fig. 2. We have shown in Fig. 6 of the Supplemental Material [40] the boundary layer for $\bar{U}''(\bar{\phi})$ analogous to that of $\bar{V}''(\bar{\mu})$. Note that the first term in Eq. (8) comes from the last term in Eq. (6) or Eq. (7), which is formally proportional to N^{-1} and neglected in the usual large-N analysis. However this term is indispensable to describe the boundary layer of $\bar{U}''(\bar{\phi})$ or $\bar{V}''(\bar{\mu})$. To conclude, we have proven that for $d < 8/3$, a boundary layer develops at large N for the second derivative of the T_3 potential that becomes a singularity when $N \to \infty$. What remains to be understood is its physical relevance.

At first sight, what we have obtained for T_3 looks paradoxical because we could think that its potential being identical to the WF potential at $N = \infty$, the linearized flow around these two FPs should also be identical and thus the same for all critical exponents. We now show that this naive argument is wrong.

We have computed in $d < 8/3$ the relevant eigenvalues of the RG flow around T_3 and WF at finite and large N and as expected we have found three for T_3 and one for WF. When $N \to \infty$, one of the three eigenvalues at T_3 tends as expected to $d-2$ which is the relevant eigenvalue ν^{-1} of the critical WF FP at $N = \infty$ [9,14]. The nontrivial point is that the two other relevant eigenvalues at T_3 have a well-defined limit when $N \to \infty$ although they do not play any role for the critical behavior of the $O(N = \infty)$ model. The solution to this paradox is that they are associated with eigenperturbations that become singular when $N \to \infty$. That these two eigenperturbations become singular is clear for one of them, called $\delta\bar{V}_2$, in Fig. 9 of the Supplemental Material [40] As for the other one, $\delta\bar{V}_1$, its slope at $\bar{\mu}_0$ diverges as $N^{1/3}$ which implies that at $N = \infty$, it becomes discontinuous at $\bar{\mu}_0$; see Figs. 9 and 10 of the Supplemental Material [40]. For ordinary second order phase transitions, these eigenperturbations are excluded which explains that the associated relevant eigenvalues do not play any role. This solves all the paradoxes associated with the tetracritical FPs at $N = \infty$ and $d < 8/3$.

What remains to be studied is the particular case $N = \infty$ and $d = 8/3$ where a line, called the BMB line, of smooth tetracritical FPs shows up. It is obtained in the WP version of the RG by integrating Eq. (7) in which the last term, proportional to $1/N$, has been discarded. It is given by the following implicit expression [14]:

$$\bar{\mu}_\pm = \frac{C}{\bar{V}'(1-2\bar{V}')}\left(\frac{\pm 2\bar{V}'}{1-2\bar{V}'}\right)^{4/3} + 2f(4\bar{V}'), \tag{9}$$

where $f(x)$, which is analytic for $x < 2$, is given by

$$f(x) = \frac{3}{2-x} + \frac{4x}{(2-x)^{7/3}} \int_0^1 dz \left(\frac{2-xz}{z}\right)^{1/3} \tag{10}$$

and $\bar{\mu}_\pm$ correspond to the two branches $\bar{\mu} > 3$ and $\bar{\mu} < 3$, respectively. The derivative of the potential $\bar{V}'$ is positive (negative) on the former (latter) branch and C is a non-negative integration constant. $\bar{V}(\bar{\mu})$ is analytic at $\bar{\mu} = \bar{\mu}_0 = 3$ and $\bar{V}'(\bar{\mu} = 3) = 0$. In Fig. 7 of the Supplemental Material [40] different $\bar{V}'(\bar{\mu})$ corresponding to different FPs of the BMB line are shown. All FPs along the BMB line share the same critical exponents, that is, the exponents of the Gaussian FP which is itself tetracritical. Notice that the WF FP, which corresponds to $C = 0$, is the end point of this line and deserves special attention. We come back on this point in the following.

From Eq. (1), we have seen that $\tilde{\lambda}$ remains constant at leading order in $1/N$ along the hyperbola of constant ϵN of the (d, N) plane. This suggests that when the double limits $d \to 8/3$ and $N \to \infty$ are taken at fixed $\alpha = \epsilon N$, T_3 converges in $d = 8/3$ to one of the FPs of the BMB line. We have analytically and numerically checked this and have derived analytically the relation between α and C: $\alpha = 162/C^3$; see the Supplemental Material [40] and Fig. 11.

Two extreme cases are worth studying. First, the Gaussian FP corresponds to the limit $N \to \infty$ at fixed dimension $d = 8/3$, that is, at $\alpha = 0$. It corresponds to $C = \infty$ in Eq. (9). Second, $\alpha = \infty$, which implies $C = 0$, corresponds to taking the limit $\epsilon \to 0$ at fixed $N = \infty$, that is, to following the WF FP at $N = \infty$ up to $d = 8/3$. However, at finite ϵ and $N = \infty$, we know from the analysis above that the last term in Eq. (7) cannot be neglected. Consistently, the same occurs for the BMB line: the WF FP potential is indeed the end point of the BMB line obtained by taking the limit $C \to 0$ in Eq. (9), but the derivatives of this potential can only be studied by retaining the last term in Eq. (7). Here again, this explains why the T_3 FP in the $C \to 0$ limit is 3 times as unstable and not only once unstable.

To conclude, we have solved the paradox of the apparent absence of a nontrivial tetracritical FP at $N = \infty$ and $d < 8/3$ by showing that this FP does exist but is nothing else than the WF FP up to the subtlety that the derivatives of the tetracritical FP potential are not the derivatives of the WF FP potential. This makes the large-N limit of the O(N) model much less trivial than is usually advocated at least for multicritical phenomena. The fact that the tetracritical FP has two more unstable infrared directions than the WF FP is related to this subtle point because they are associated with singular eigenperturbations, a possibility which is usually not considered. We conjecture that what has been found above at large N and for $d \leq 8/3$ is valid for all multicritical points with an odd number of eigendirections below or at their upper critical dimension because the BMB lines for all of them terminate at the WF FP [14], a fact that in itself is almost enough to imply everything else. Let us finally point out that what we have found for the tetracritical FP is very different from what was found around $d = 3$ at large-N in the tricritical case which required the existence of new FPs to be fully understood at finite N [45–48]. We also conjecture that this phenomenon is not specific to the O(N) models but should rather be generic.

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