Hierarchies of Frequentist Bounds for Quantum Metrology: From Cramér-Rao to Barankin

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We derive lower bounds on the variance of estimators in quantum metrology by choosing test observables that define constraints on the unbiasedness of the estimator. The quantum bounds are obtained by analytical optimization over all possible quantum measurements and estimators that satisfy the given constraints. We obtain hierarchies of increasingly tight bounds that include the quantum Cramér-Rao bound at the lowest order. In the opposite limit, the quantum Barankin bound is the variance of the locally best unbiased estimator in quantum metrology. Our results reveal generalizations of the quantum Fisher information that are able to avoid regularity conditions and identify threshold behavior in quantum measurements with mixed states, caused by finite data.

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Introduction.—Quantum metrology provides the theory foundation for identifying precision limits and developing quantum-enhanced strategies in quantum measurements [1–6]. Today, quantum sensing and metrology represent some of the most advanced quantum technologies with applications ranging from gravitational wave detection [7] to atomic clocks and interferometers [6]. The cornerstone of frequentist quantum metrology, the quantum Cramér-Rao bound (QCRB), identifies a lower bound on the variance of an estimator for a parameter encoded in a quantum state [8]. The inverse of the QCRB is a measure for the quantum state's sensitivity under small variations of a parameter, known as the quantum Fisher information (QFI). Nowadays, the QFI is used, far beyond its original purpose in quantum metrology, as an extremely powerful and versatile tool in quantum information theory. For instance, it efficiently detects multipartite entanglement [9,10], Einstein-Podolsky-Rosen steering [11], quantum phase transitions [12–14], non-Markovian open-system evolutions [15], and allows us to sharpen uncertainty relations [16,17], quantum speed limits [18], and the quantum Zeno effect [19].

The QCRB can be understood as the natural quantization of the classical CRB by analytical optimization over all possible quantum measurements [8]. As such, the QCRB inherits many properties from its classical counterpart, including its shortcomings. For instance, the CRB is undefined for estimation problems that do not satisfy certain regularity conditions. Furthermore, saturation of the CRB typically only occurs in the asymptotic limit of many repeated measurements, i.e., when the signal-to-noise ratio is large. Consequently, the QCRB provides only very limited information about the achievable precision in few-shot scenarios. To realistically identify limits on the variance of unbiased estimators, tighter bounds are required that generalize the QCRB for low signal-to-noise ratio. In quantum metrology, the unattainability of the QCRB has been pointed out using Bayesian approaches that involve some form of prior information [20–22]. This so-called threshold behavior has been studied in classical information theory within the frequentist paradigm, to which the CRB belongs, through comparison with tighter bounds [23–27].

Generalizations of the CRB appeared already in its contemporary literature, most notably the families of bounds by Bhattacharyya [28] and Barankin [29]. Both families consist of hierarchies of bounds that impose increasingly demanding conditions on the unbiasedness of an estimator. The Bhattacharyya bounds compare the estimator to the identity function in a neighborhood of the unknown parameter's true value by means of a Taylor-like expansion involving higher-order derivatives. The Barankin bounds impose unbiasedness of the estimator at an increasing number of n test points within the range of possible values for the unknown parameter. In the limit $n \to \infty$, the Barankin bounds identify the variance of the locally best unbiased estimator [29]. By avoiding derivatives altogether, the Barankin bounds circumvent the regularity conditions of the CRB and, more generally, the Bhattacharyya bounds. The Abel bounds [30] combine the ideas of Bhattacharyya and Barankin and involve a combination of test points and higher-order derivatives. In all cases, these approaches produce hierarchies of bounds that are tighter than the CRB, which in turn is recovered at the respective lowest order.

In this Letter, we derive a hierarchy of increasingly tight bounds for quantum parameter estimation that include the QCRB as a special case at the lowest order. Like the QCRB, all generalized quantum bounds are obtained from classical frequentist bounds by an analytical optimization over all possible quantum measurements. Each bound is determined by a choice of test observables that reflect generalized unbiasedness conditions on the estimator. In the limit of pure states, the hierarchy collapses and all bounds reproduce the QCRB. As relevant applications, we show how our approach naturally quantizes the bounds by Bhattacharyya [28], Barankin [29], Hammersley [31], Chapman and Robbins [32], and Abel [30]. We provide explicit expressions for the bounds and the quantum information functions, i.e., generalizations of the QFI based on the Bures metric, that are defined by their inverses. We illustrate our results with an analysis of threshold behavior in quantum phase estimation with a qubit.

Hierarchy of quantum bounds.—We consider an estimator $\theta_{\mathrm{est}}(x)$ for θ that is a function of the obtained measurement results x. For a fixed measurement setting, described by the positive operator-valued measure (POVM) $E_x \geq 0$, with $\sum_x E_x = \mathbb{I}$, the probability to obtain the result x when the parameter of interest takes on the value θ is given by $p(x|\theta) = \mathrm{Tr}\{E_x \rho(\theta)\}$, where $\rho(\theta)$ is the quantum state of the system. In the following, we describe the state of the measurement apparatus with a generic (single-copy) density matrix $\rho(\theta)$. The m-shot scenario is included in this description by replacing $\rho(\theta)$ with $\rho(\theta)^{\otimes m}$.

The quantity of central interest is the variance of the estimator,

$$(\Delta\theta_{\mathrm{est}})^2 = \sum_x p(x|\theta)[\theta_{\mathrm{est}}(x) - \langle\theta_{\mathrm{est}}\rangle_\theta]^2, \tag{1}$$

with the average value $\langle\theta_{\mathrm{est}}\rangle_\theta = \sum_x p(x|\theta)\theta_{\mathrm{est}}(x)$. We are interested in the locally best estimators, i.e., those that minimize the variance when the unknown parameter takes on the value θ. We consider estimators that are unbiased over the range $\Theta \subset \mathbb{R}$ of possible parameter values, i.e., $\langle\theta_{\mathrm{est}}\rangle_{\theta_0} = \theta_0$ for all $\theta_0 \in \Theta$. The starting point for our derivation of lower bounds on (1) is the choice of a family of test observables $\boldsymbol{G} = (G_1, \ldots, G_n)^\top$. Each G_k is a Hermitian operator on the same Hilbert space as $\rho(\theta)$. Typically, G_k will be a linear function of $\rho(\theta_k)$ where $\theta_k \in \Theta$. In the following, we derive generalized quantum bounds for any choice of the G_k, and further below, we show how suitable choices for the G_k lead to a hierarchy of bounds that generalize the QCRB. The number n of different test observables G_k determines the order of the bound. Each G_k allows us to impose an unbiasedness constraint λ_k on the estimator θ_{est}.

In combination with a measurement of the POVM element E_x, the test observables give rise to the classical functions $g_k(x) = \mathrm{Tr}\{E_x G_k\}$. We first derive classical bounds that hold for any choice of the test functions $\boldsymbol{g}(x) = [g_1(x), \ldots, g_n(x)]^\top$ and in a second step optimize these bounds over all choices of POVMs. The bounds are based on linear combinations of the test functions of the form $\boldsymbol{a}^\top \boldsymbol{g}(x) = \sum_{k=1}^n a_k g_k(x)$, where $\boldsymbol{a} = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$ is a vector of coefficients. We find that $(\Delta\theta_{\mathrm{est}})^2 \geq (\Delta\theta_{\mathrm{est}})^2_C$, with [33]

$$(\Delta\theta_{\mathrm{est}})^2_C = \sup_{\boldsymbol{a}} \frac{(\boldsymbol{a}^\top \boldsymbol{\lambda})^2}{\boldsymbol{a}^\top C \boldsymbol{a}} = \boldsymbol{\lambda}^\top C^{-1} \boldsymbol{\lambda}, \tag{2}$$

where the second equality holds when C is invertible. We have introduced the bias conditions $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$ with

$$\lambda_k = \sum_{x \in X_+} g_k(x)(\theta_{\mathrm{est}}(x) - \langle\theta_{\mathrm{est}}\rangle_\theta), \tag{3}$$

and C is a real, symmetric $n \times n$ classical information matrix with elements

$$C_{kl} = \sum_{x \in X_+} \frac{g_k(x) g_l(x)}{p(x|\theta)}. \tag{4}$$

The sums are restricted to those events $x \in X_+$ that occur with nonzero probability $p(x|\theta) > 0$.

To obtain the corresponding quantum bounds, we now minimize the right-hand side of Eq. (2) over all POVMs $\{E_x\}$. We assume that the λ_k are fixed properties of the estimator that characterize the bias. Thus, each test observable G_k imposes another constraint (3) on the unbiasedness of θ_{est}. The analytical optimization over all POVMs yields $(\Delta\theta_{\mathrm{est}})^2 \geq (\Delta\theta_{\mathrm{est}})^2_C \geq (\Delta\theta_{\mathrm{est}})^2_Q$ with [33]

$$(\Delta\theta_{\mathrm{est}})^2_Q = \min_{\{E_x\}} (\Delta\theta_{\mathrm{est}})^2_C = \sup_{\boldsymbol{a}} \frac{(\boldsymbol{a}^\top \boldsymbol{\lambda})^2}{\boldsymbol{a}^\top Q \boldsymbol{a}} = \boldsymbol{\lambda}^\top Q^{-1} \boldsymbol{\lambda}, \tag{5}$$

where, again, the final equality in Eq. (5) holds for Q invertible, and we obtain the real, symmetric $n \times n$ quantum information matrix Q with elements

$$Q_{kl} = \mathrm{Tr}\{G_k \Omega_{\rho(\theta)}(G_l)\}. \tag{6}$$

The operator $\Omega_A(X)$ is defined by the property

$$X = \frac{A\Omega_A(X) + \Omega_A(X)A}{2}. \tag{7}$$

Intuitively, $\Omega_A(X)$ describes a symmetric "division" of X by the operator A. Furthermore, the bound (5) is saturated by an optimal projective measurement whose elements E_x are the projectors onto the eigenstates of $\Omega_{\rho(\theta)}(\boldsymbol{a}^\top \boldsymbol{G})$. The optimal coefficient vector $\boldsymbol{a}$ achieves the supremum in Eq. (5) and is given by $\boldsymbol{a} = \alpha Q^{-1} \boldsymbol{\lambda}$ when Q^{-1} exists, with $\alpha \in \mathbb{R}$ a normalization constant.

The bound (5) and definition (7) apply to test observables that satisfy $\Pi_\perp G_k \Pi_\perp = 0$ for all k, where $\Pi_\perp$ is the projector onto the eigenspace of $\rho(\theta)$ with eigenvalue 0. This condition can always be satisfied by adding $\epsilon\mathbb{I}$ to $\rho(\theta)$ with $\epsilon \ll 1$. A discussion of technical details and saturation conditions is provided in Ref. [33].

To illustrate our approach, consider the simple scenario of a single test observable ($n = 1$) chosen as $G_1 = \partial\rho(\theta)/\partial\theta$. It is easy to see that the bound (2) yields the classical CRB with bias condition $\lambda_1 = (\partial/\partial\theta)\langle\theta_{\text{est}}\rangle_\theta$ and the Fisher information C. The associated quantum bound (5) is the well-known QCRB, i.e., Q yields the quantum Fisher information $F_Q[\rho(\theta)]$. More general choices of the test observables reveal a rich variety of quantum bounds that provide more realistic estimates of the smallest achievable variance of unbiased estimators than the QCRB. The result (5) also includes special cases that are discussed in Ref. [34].

Quantum Barankin bounds.—Consider the family of test observables $G_k = \rho(\theta_k)$, representing the density matrix at different values of the parameter θ within the range of possible parameters, called test points $\theta_k \in \Theta$. We are interested in estimators that are unbiased at the true value θ and at each of these test points, $\langle\theta_{\text{est}}\rangle_{\theta_k} = \theta_k$, which in (3) yield $\lambda_k = \theta_k - \theta$.

Classical bounds for estimators that are unbiased at n arbitrary test points within the range Θ were first derived by Barankin in 1949 [29]. The Barankin bound (BaB) of order n is recovered from Eq. (2) for the choice $\boldsymbol{G} = [\rho(\theta_1), \ldots, \rho(\theta_n)]^\top$, after a maximization over the choice of test points λ_k, and reads $(\Delta\theta_{\text{est}})^2_{C_{\text{Ba}}} = \sup_{\boldsymbol{\lambda}} \boldsymbol{\lambda}^\top C_{\text{Ba}}^{-1} \boldsymbol{\lambda}$, with the Barankin information matrix

$$(C_{\text{Ba}})_{kl} = \sum_{x\in X_+} p(x|\theta) L(x|\theta + \lambda_k, \theta) L(x|\theta + \lambda_l, \theta), \quad (8)$$

and $L(x|\theta + \lambda_k, \theta) = p(x|\theta + \lambda_k)/p(x|\theta)$ is the likelihood ratio. Barankin further showed that an additional optimization of all BaBs over the number n of test points (or equivalently $n \to \infty$) yields the variance of the locally best unbiased estimator, which is unique [29]. The Barankin bound is typically hard to determine but has the advantage of avoiding the regularity conditions of the CRB. Efficient approximations of the Barankin bound based on small n have proven to be useful to highlight the limitations of the CRB at small signal-to-noise ratio [23–25,27].

We identify the quantum Barankin bounds (QBaBs) by optimizing over all possible POVMs, which according to Eq. (5) yields $\min_{\{E_x\}} (\Delta\theta_{\text{est}})^2_{C_{\text{Ba}}} = \sup_{\boldsymbol{\lambda}} \boldsymbol{\lambda}^\top Q_{\text{Ba}}^{-1} \boldsymbol{\lambda}$ with the quantum Barankin information (QBaI) matrix

$$(Q_{\text{Ba}})_{kl} = \text{Tr}\{\rho(\theta + \lambda_k)\Omega_{\rho(\theta)}(\rho(\theta + \lambda_l))\}. \quad (9)$$

The QBaBs can be improved at no additional computational cost by adding $\rho(\theta_0) = \rho(\theta)$ to the set $\boldsymbol{G}$ [23], with the corresponding bias constraint $\lambda_0 = 0$, leading to [33]

$$(\Delta\theta_{\text{est}})^2_{Q_{\text{Ba}}} = \sup_{\boldsymbol{\lambda}} \boldsymbol{\lambda}^\top (Q_{\text{Ba}} - \boldsymbol{e}\boldsymbol{e}^\top)^{-1} \boldsymbol{\lambda}, \quad (10)$$

where $\boldsymbol{e} = (1, \ldots, 1)^\top \in \mathbb{R}^n$.

At its lowest order, $n = 1$, the classical BaB reduces to the Hammersley-Chapman-Robbins bound (HCRB) [31,32], which reads

$$(\Delta\theta_{\text{est}})^2_{C_{\text{HCRB}}} = \sup_\lambda \frac{\lambda^2}{\chi^2[p(\cdot|\theta + \lambda), p(\cdot|\theta)]}. \quad (11)$$

Here, $\chi^2[p(\cdot|\theta+\lambda), p(\cdot|\theta)] = \sum_{x\in X_+} p(x|\theta+\lambda)^2/p(x|\theta) - 1$ is the χ^2 divergence of $p(\cdot|\theta + \lambda)$ with respect to $p(\cdot|\theta)$. The quantum Hammersley-Chapman-Robbins bound (QHCRB), $(\Delta\theta_{\text{est}})^2_{Q_{\text{HCRB}}} = \min_{\{E_x\}} (\Delta\theta_{\text{est}})^2_{C_{\text{HCRB}}}$, i.e., the QBaB (10) at $n = 1$ reads

$$(\Delta\theta_{\text{est}})^2_{Q_{\text{HCRB}}} = \sup_\lambda \frac{\lambda^2}{\text{Tr}\{\rho(\theta + \lambda)\Omega_{\rho(\theta)}[\rho(\theta + \lambda)]\} - 1}. \quad (12)$$

The denominator indeed coincides with the quantum χ^2 divergence $\chi^2_Q[\rho(\theta+\lambda), \rho(\theta)] = \max_{\{E_x\}} \chi^2[p(\cdot|\theta+\lambda), p(\cdot|\theta)] = \text{Tr}\{\rho(\theta+\lambda)\Omega_{\rho(\theta)}(\rho(\theta+\lambda))\} - 1$ [35].

Already at this lowest order, the QBaBs generalize the QCRB: If $\rho(\theta)$ is differentiable, we recover the QCRB from the right-hand side of Eq. (12) by replacing the supremum with the limit $\lambda \to 0$ [33,35]. Generally, however, the QHCRB is tighter, $(\Delta\theta_{\text{est}})^2_{Q_{\text{HCRB}}} \geq (\Delta\theta_{\text{est}})^2_{Q_{\text{CRB}}} = F_Q[\rho(\theta)]^{-1}$, and applicable to a wider range of problems.

Quantum Bhattacharyya bounds.—Let us consider the test observables $G_k = \partial^k\rho(\theta)/\partial\theta^k$, assuming that these derivatives exist. The bias conditions (3) here identify $\lambda_k = (\partial^k/\partial\theta^k)\langle\theta_{\text{est}}\rangle_\theta$ for $k > 0$ as fixed properties of the estimator. Unbiased estimators satisfy $\lambda_1 = 1$ and $\lambda_k = 0$ for all $k > 1$.

Choosing the family of test functions $\boldsymbol{G} = \{\partial\rho(\theta)/\partial\theta, \ldots, \partial^n\rho(\theta)/\partial\theta^n\}^\top$ with corresponding bias conditions (3) $\boldsymbol{\lambda} = (1, 0, \ldots, 0)^\top$ leads in Eq. (2) to the Bhattacharyya bound (BhB) of order n [28], $(\Delta\theta_{\text{est}})^2_{C_{\text{Bh}}} = \boldsymbol{\lambda}^\top C_{\text{Bh}}^{-1} \boldsymbol{\lambda}$, with the Bhattacharyya information matrix

$$(C_{\text{Bh}})_{kl} = \sum_{x\in X_+} \frac{1}{p(x|\theta)} \left(\frac{\partial^k p(x|\theta)}{\partial\theta^k}\right) \left(\frac{\partial^l p(x|\theta)}{\partial\theta^l}\right). \quad (13)$$

Conceptually, the BhBs account for the unbiasedness of θ_{est} in an increasingly large region around the true value θ by adding constraints on the higher-order terms of a Taylor expansion.

The quantum Bhattacharyya bounds (QBhBs) $(\Delta\theta_{\text{est}})^2_{Q_{\text{Bh}}} = \min_{\{E_x\}} (\Delta\theta_{\text{est}})^2_{C_{\text{Bh}}} = \boldsymbol{\lambda}^\top Q_{\text{Bh}}^{-1} \boldsymbol{\lambda}$ follow from Eq. (5) with the quantum Bhattacharyya information (QBhI) matrix

$$(Q_{\text{Bh}})_{kl} = \text{Tr}\left\{\frac{\partial^k\rho(\theta)}{\partial\theta^k} \Omega_{\rho(\theta)} \left(\frac{\partial^l\rho(\theta)}{\partial\theta^l}\right)\right\}. \quad (14)$$

It is easy to see that at $n = 1$, the QBhB coincides with the QCRB, while for higher orders, it is generally tighter.

Compared to the QBaBs, the QBhBs require stronger regularity conditions, since all higher-order derivatives must exist. In contrast, the QBhBs avoid the optimization over the parameters $\boldsymbol{\lambda}$ of the QBaBs, which can be computationally expensive. Since higher-order derivatives can be recovered from the differences of infinitesimally separated test points, the QBaBs include the QBhBs as special cases [29].

Quantum Abel bounds.—The bounds of Barankin and Bhattacharyya can be combined into hybrid bounds by considering a combination of test observables that contain both $\rho(\theta+\lambda_k)$ and $\partial^l\rho(\theta)/\partial\theta^l$. Such bounds were first discussed by Abel in 1993 [30]. We obtain the classical Abel bounds of order (r,s), $(\Delta\theta_{\rm est})^2_{C_{\rm A}}$, as well as the corresponding quantum Abel bounds (QABs) by considering the family of $r+s$ test observables $\boldsymbol{G}=\{\rho(\theta+\lambda_1),\ldots,\rho(\theta+\lambda_r),\partial\rho(\theta)/\partial\theta,\ldots,\partial^s\rho(\theta)/\partial\theta^s\}^\top$ in Eqs. (2) and (5), respectively. The QABs read [33]

$$(\Delta\theta_{\rm est})^2_{Q_{\rm A}}=\min_{\{E_x\}}(\Delta\theta_{\rm est})^2_{C_{\rm A}}=\sup_{\lambda_1,\ldots,\lambda_r}\boldsymbol{\lambda}^\top(Q_{\rm A}-\boldsymbol{f}\boldsymbol{f}^\top)^{-1}\boldsymbol{\lambda},\qquad(15)$$

where $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_r,1,0,\ldots,0)\in\mathbb{R}^{r+s}$ combines the unbiasedness conditions of the Barankin and Bhattacharyya bounds and $\boldsymbol{f}=\boldsymbol{e}\oplus\boldsymbol{0}\in\mathbb{R}^{r+s}$, where $\boldsymbol{0}\in\mathbb{R}^s$ is the zero vector. The $(r+s)\times(r+s)$ quantum Abel information (QAI) matrix

$$Q_{\rm A}=\begin{pmatrix}Q_{\rm Ba} & Q_{\rm H}\\ Q_{\rm H}^\top & Q_{\rm Bh}\end{pmatrix}\qquad(16)$$

contains the $r\times r$ QBaI matrix $Q_{\rm Ba}$ (9) and the $s\times s$ QBhI matrix $Q_{\rm Bh}$ (14), as well as the $r\times s$ hybrid matrix

$$(Q_{\rm H})_{kl}=\mathrm{Tr}\left\{\rho(\theta+\lambda_k)\Omega_{\rho(\theta)}\left(\frac{\partial^l\rho(\theta)}{\partial\theta^l}\right)\right\}.\qquad(17)$$

Even more general bounds can be obtained by including higher-order derivatives also at each of the test points $\theta+\lambda_k$, see, e.g., [27].

Quantum phase estimation with a qubit.—Let us now illustrate these bounds by applying them to the relevant example of quantum phase estimation. Our goal is to estimate the phase θ of a single qubit $\rho=\frac{1}{2}(\mathbb{I}+\boldsymbol{r}^\top\boldsymbol{\sigma})$ with Bloch vector $\boldsymbol{r}=(0,r,0)^\top$, which is imprinted by the rotation $U(\theta)=e^{-i\sigma_z\theta/2}$ as $\rho(\theta)=U(\theta)\rho U(\theta)^\dagger$. To study threshold behavior, we analytically determine and compare the bounds QCRB $=$ QAB$_{(0,1)}$, QHCRB $=$ QAB$_{(1,0)}$ and the hybrid bound QAB$_{(1,1)}$ for m independent copies of the same qubit, $\rho(\theta)^{\otimes m}$ [33]. The results are shown in Fig. 1 for up to $m=7$ qubits with entropy $S(\rho)=-\mathrm{Tr}\{\rho\ln\rho\}=0.6$, corresponding to $r=|\boldsymbol{r}|\approx0.42$. For better comparison, all bounds are normalized with respect to the QCRB, i.e., the plot shows $(\Delta\theta_{\rm est})^2_Q/(\Delta\theta_{\rm est})^2_{Q_{\rm CRB}}$ with $(\Delta\theta_{\rm est})^2_{Q_{\rm CRB}}=1/(mr^2)$. We note that the QCRB is overly optimistic for estimators that are unbiased in the range $\Theta=(\pi,\pi]$ when m is small. Both bounds QHCRB and QAB$_{(1,1)}$ reveal threshold behavior: They identify larger values on the lowest possible variance of an unbiased estimator, but after sufficiently many measurements, they approach the QCRB. Since the bounds at small m are determined by the edge of Θ, a smaller range Θ allows for smaller variances of unbiased estimators. Furthermore, the threshold behavior is more pronounced as the entropy of the qubit grows and it disappears in the limit of pure states [33].

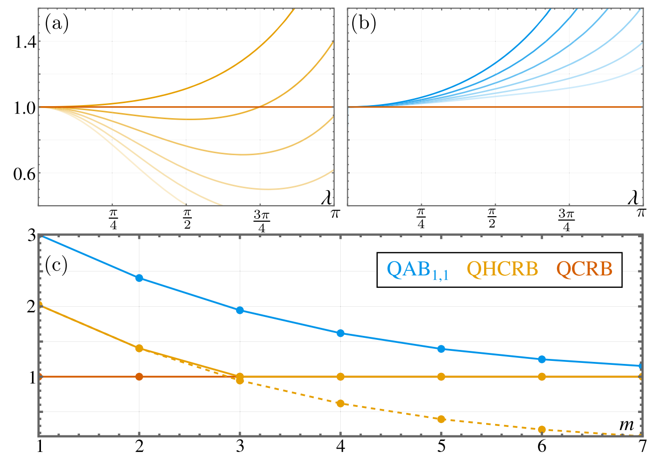

FIG. 1. Bounds on the variance of an unbiased estimator in the interval $\Theta=(-\pi,\pi]$ for m independent measurements on a noisy qubit with entropy $S(\rho)=0.6$. The plots show $(\Delta\theta_{\rm est})^2_{Q_{\rm A}}/(\Delta\theta_{\rm est})^2_{\rm QCRB}$ for the QABs of order (1,1), $(1,0)=$ QHCRB and $(0,1)=$ QCRB. The QAB bounds (1,0) and (1,1) are obtained by maximizing over the single free parameter λ in (15) and the dependence is shown in (a) and (b), respectively. Lighter colors indicate larger values of $m=1,\ldots,7$. Below the threshold value $m=3$, the maximum appears at the edge of the parameter range Θ at $\lambda_{\rm max}=\pi$ and the QHCRB is larger than the QCRB (c). For $m\geq3$, the edge value (dashed orange line) is smaller than the one obtained at $\lambda_{\rm max}\to0$, see also (a). In this case, the QHCRB coincides with the QCRB. The hybrid QAB$_{(1,1)}$ always takes on its maximum at $\lambda_{\rm max}=\pi$ and converges to the QCRB from above in the limit of large m.

The constant value of $m(\Delta\theta_{\rm est})^2_{Q_{\rm CRB}}$ is due to the additivity of the QFI: This reflects the asymptotic limit, where each additional measurement adds as much information as the previous one. This is not the case at low data, as is shown by the more general bounds that do not satisfy additivity. As a consequence, saturation of these bounds in the m-shot scenario typically requires joint measurements on all m copies.

General properties.—The quantum information matrix Q is closely related to the Bures metric [36], which stands out as the smallest among the family of metrics that contract under quantum channels [37,38]. When the G_k are linear functions of $\rho(\cdot)$, the quantum information function $I_{\boldsymbol{a},\boldsymbol{\lambda}}[\rho(\cdot)]=(\boldsymbol{a}^\top Q\boldsymbol{a})/(\boldsymbol{a}^\top\boldsymbol{\lambda})^2=\mathrm{Tr}\{(\boldsymbol{a}^\top\boldsymbol{G})\Omega_{\rho(\theta)}(\boldsymbol{a}^\top\boldsymbol{G})\}/(\boldsymbol{a}^\top\boldsymbol{\lambda})^2$ is convex in the quantum state $\rho(\cdot)$ as a consequence of the joint convexity of the Bures inner product [33,38–40]. The bounds (5) are furthermore bounded from below by the QCRB and therefore subject to the same separability limits that were derived using the QFI [9,10]. We show in Ref. [33] that in the limit of pure states, all QABs converge to the QCRB. We further provide explicit expressions for Q from the spectral decomposition of $\rho(\theta)$, from matrix vectorization techniques, and from the Bloch vector in the case of two-level systems. In the following we present an equivalent formulation of these bounds as optimization problems with constraints.

Locally best unbiased quantum parameter estimation.—The bounds (5) are tight in the sense that they identify the minimum variance at θ by optimization over all measurements and over all estimators that satisfy the unbiasedness constraints determined by $\boldsymbol{G}$ and $\boldsymbol{\lambda}$. In other words, the bound (5) is the solution to [33]

$$\min_{\{E_x\}} \min_{\theta_{\mathrm{est}}} (\Delta\theta_{\mathrm{est}})^2,$$
$$\text{s.t.} \sum_{x\in X_+} \mathrm{Tr}\{E_x \boldsymbol{G}\}(\theta_{\mathrm{est}}(x) - \langle\theta_{\mathrm{est}}\rangle_\theta) = \boldsymbol{\lambda}. \qquad (18)$$

Moreover, the estimator that achieves the second minimum is unique. For estimators that are unbiased throughout Θ, the locally best variance is identified by the quantum Barankin bound in the limit $n \to \infty$, which demands unbiasedness at all points in Θ. All other bounds discussed here are approximations of this limit. This extends Barankin's result [29,41] to the quantum realm by optimization over all measurements.

Conclusions.—We derived hierarchies of generalized quantum bounds on the variance of unbiased estimators in quantum metrology from unbiasedness constraints. The bounds converge towards the QCRB from above in two limits: When large amounts of data, i.e., many copies of the state, are available, and when the state becomes pure. For few-shot measurements with mixed states, the more general bounds reveal tighter constraints on the precision of unbiased quantum parameter estimation than the QCRB. We identify the optimal measurement observable and the estimator that achieves the smallest variance, given a set of unbiasedness constraints. Besides leading to important generalizations of the QFI and the QCRB, these bounds are useful to study threshold behavior in quantum measurements and to identify error bounds when regularity assumptions of the QCRB cannot be fulfilled.

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