Nucleation at Finite Temperature: A Gauge-Invariant Perturbative Framework

Johan Löfgren [ORCID],[1,*] Michael J. Ramsey-Musolf,[2,3,4,5,†] Philipp Schicho [ORCID],[6,‡] and Tuomas V. I. Tenkanen [ORCID][4,5,7,§]

[1]*Department of Physics and Astronomy, Uppsala University, Box 516, SE-751 20 Uppsala, Sweden*
[2]*Amherst Center for Fundamental Interactions, Department of Physics,*
University of Massachusetts, Amherst, Massachusetts 01003, USA
[3]*Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125 USA*
[4]*Tsung-Dao Lee Institute and School of Physics and Astronomy,*
Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China
[5]*Shanghai Key Laboratory for Particle Physics and Cosmology, Key Laboratory for Particle Astrophysics & Cosmology (MOE),*
Shanghai Jiao Tong University, Shanghai 200240, China
[6]*Department of Physics and Helsinki Institute of Physics, P.O. Box 64, FI-00014 University of Helsinki, Finland*
[7]*Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

We present a gauge-invariant framework for bubble nucleation in theories with radiative symmetry breaking at high temperature. As a procedure, this perturbative framework establishes a practical, gauge-invariant computation of the leading order nucleation rate, based on a consistent power counting in the high-temperature expansion. In model building and particle phenomenology, this framework has applications such as the computation of the bubble nucleation temperature and the rate for electroweak baryogenesis and gravitational wave signals from cosmic phase transitions.

Achieving a rigorous understanding of the thermal history of the electroweak (EW) symmetry breaking has been a long-standing challenge at the interface between particle physics and cosmology. The standard history of the standard model (SM) of particle physics contains no electroweak phase transition. Instead a smooth crossover between high and low temperature phases occurs [1,2]. A nonstandard history with a first-order phase transition is conceivable in theories beyond the SM (BSM) that are well motivated from the electroweak to TeV scale. BSM scenarios with extended scalar sectors can contain rich patterns of symmetry breaking [3] and potentially produce EW baryogenesis [4–6], and also rich phenomenology for high energy collider physics [7] and the production of primordial gravitational waves [8]. These phenomena have been actively studied for decades and recently sparked increased interest in next-generation gravitational wave detector experiments [9–11] such as LISA [12].

One key ingredient for both EW baryogenesis and stochastic gravitational wave background production is a first-order phase transition in the primordial plasma of particles that proceeds via nucleating bubbles of the low-temperature phase. The thermodynamics and bubble dynamics of such transitions can be reliably described nonperturbatively using lattice simulations [13–15]. However, exploring BSM scenarios and their thermodynamics invariably requires the use of perturbation theory, since fully comprehensive nonperturbative analyses of a multidimensional parameter space are computationally out of reach. A qualitative and quantitative assessment of the perturbative reliability requires robust theory computations when comparing to lattice data. In that respect, an unphysical gauge dependence of the bubble nucleation rate has plagued particle physics phenomenology applications of the thermal phase transition literature for decades (cf. Refs. [8,16] and references therein). While by now resolved at zero temperature [17–22], a similar, long-standing and open problem exists at finite temperature if the potential barrier between the phases is radiatively generated. Such a scenario in EW symmetry breaking is relevant for the initial of two transition steps [23] or a single transition step wherein new scalars both strengthen the barrier while reducing the magnitude of the effective Higgs boson self interaction [24]. It is also relevant for a phase transition in a dark sector [25].

We resolve this long-standing problem by providing a practical, gauge-invariant framework for thermal bubble nucleation, intended for model building and particle phenomenology applications. The reliability of the approach can be assessed by "benchmarking," against lattice analyses in the future. Such benchmarking is incompatible with the conventional approach since there the nucleation rate is gauge dependent and an ill-defined unphysical quantity.

The bubble nucleation rate per unit volume, Γ, has the semiclassical approximation [26–29]

$$\Gamma = Ae^{-\mathcal{B}}. \qquad (1)$$

Here, the exponent $\mathcal{B} = S_3/T$ with S_3 being the three-dimensional Euclidean effective action evaluated at the "bounce" solution that solves the classical Euclidean field equations [27,30]. The prefactor A is dimensionful and given by the characteristic mass scales of the theory, resulting from computing the functional determinants. The leading behavior of the rate Γ is encoded in the exponent $\mathcal{B}$.

The effective action is computable using the background field (ϕ) method. Therein, $V^{\text{eff}}(\phi, T)$ is the thermal effective potential describing the equilibrium free energy of the system, and $Z^{\frac{1}{2}}(\phi, T)$ is the field renormalization factor. Both $V^{\text{eff}}(\phi, T)$ and $Z(\phi, T)$ admit an expansion in the weak gauge coupling, denoted generically here as g. After the gradient expansion in powers of spatial derivatives ($\partial_i \phi$), the effective action reads as [31,32]

$$S_3 = \int \mathrm{d}^3 x \left[V^{\text{eff}}(\phi, T) + \frac{1}{2} Z(\phi, T)(\partial_i \phi)^2 + \dots \right], \quad (2)$$

where the ellipsis contains terms of additional ($\partial_i \phi$) powers. While the gradient expansion in general does not converge [15,33–35], one may obtain a self-consistent, gauge-invariant estimate of Γ by ensuring the three following conditions apply. (A) Fields that generate the barrier must be parametrically heavier than the nucleating field; in this case leading orders can be correctly described by a derivative expansion. (B) Temperatures are assumed to be in the vicinity of the critical temperature, T_c, such that a specific power counting in g applies, wherein the leading order potential has a radiatively generated barrier. (C) Bubbles are assumed to have a thick-wall which can arise after supercooling when the nucleation temperature, T_n, is sufficiently below T_c but still within the criteria of (B).

Condition (C) is particularly relevant for strong transitions interesting for cosmological applications [35]. Hence, we focus on a temperature regime where the first two leading terms of the effective action are enhanced by powers of g and presentable schematically as

$$S_3 = a_0 g^{-\frac{3}{2}} + a_1 g^{-\frac{1}{2}} + \Delta. \qquad (3)$$

Here $a_{0,1}$ are numerical coefficients that are computable in the gradient expansion at leading (LO) and next-to-leading orders (NLO), respectively. Δ encodes formally higher order corrections—that are $\mathcal{O}(1)$, up to potential logarithms of g [34]—that we presently do not compute. The bubble nucleation rate then reads as

$$\Gamma = A' e^{-(a_0 g^{-\frac{3}{2}} + a_1 g^{-\frac{1}{2}})}. \qquad (4)$$

We formally collected higher order effects encoded in Δ in Eq. (3), to a yet unspecified prefactor $A' \equiv Ae^{-\Delta}$. The expression [Eq. (4)] results from an expansion based on the chain of scale hierarchies at high temperature,

$$\pi T \gg gT \gg g^{\frac{3}{2}}T, \qquad (5)$$

where πT is the thermal scale of nonzero Matsubara modes, gT is the intermediate scale of zero modes that are thermally screened, and $g^{\frac{3}{2}}T$ is a characteristic scale of bubble nucleation, related to an effective mass of the nucleating field. For a systematic treatment of thermodynamics in the framework of three-dimensional, high temperature effective field theory (3D EFT) see Refs. [36,37] and [38,39] for relevant applications. This framework was recently extended [34] by presenting an approach for an effective description for bubble nucleation.

Here, we work at the leading high temperature expansion [40] that allows us to perform the computation without formally constructing a 3D EFT. Our goal is to compute LO and NLO terms in $\mathcal{B}$ and show their gauge invariance. We emphasize that this computation agrees with the generic EFT approach [34] at leading orders, and an accompanying article [35] revisits this computation in the EFT context, which allows one to pursue improvement in terms of higher order corrections.

As in earlier literature both at zero [17,41] and high [33] temperature, we use the Abelian Higgs model as a concrete illustration. The Euclidean Lagrangian density for the Abelian Higgs model is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \Phi)^* (D_\mu \Phi) + \mu^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2 + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}, \qquad (6)$$

with a U(1) gauge field B_μ and a complex scalar field Φ. The covariant derivative for complex Higgs reads as $D_\mu \Phi = \partial_\mu \Phi - ig B_\mu \Phi$, where g is the gauge coupling and the field strength tensor $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. We expand Φ in real fields,

$$\Phi = \frac{\phi}{\sqrt{2}} + \frac{1}{\sqrt{2}}(H + i\chi), \qquad (7)$$

where $\phi \equiv \phi(x)$ is a spatially dependent classical background field and H and χ are propagating quantum degrees of freedom. In perturbation theory the gauge can be fixed using R_ξ gauge [33,42]

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\xi} [-(\partial_\mu B_\mu + ig\xi(\tilde{\phi}^* \Phi - \Phi^* \tilde{\phi}))]^2, \qquad (8)$$

with the Fadeev-Popov ghost $(c, \bar{c})$ Lagrangian

$$\mathcal{L}_{FP} = \bar{c}(-\Box + \xi g^2 (\tilde{\phi}^* \Phi + \Phi^* \tilde{\phi}))c. \qquad (9)$$

Although one need not relate $\tilde{\phi}$ directly to ϕ, it is convenient to identify $\tilde{\phi} = \phi/\sqrt{2}$ to eliminate mixing between the gauge field and Goldstone mode χ.

Equilibrium thermodynamics are described by the effective potential, whose minima should be separated by a barrier for a first-order phase transition. While the Abelian Higgs model admits no tree-level barrier, the barrier arises radiatively through loop corrections. Similar to Refs. [17,33,40,43], we show that one-loop gauge field contributions can arise at the same order as the tree-level potential provided that model parameters assume the following power counting. First, we assume that

$$\phi \sim T, \qquad (10)$$

which sets the size of the background field to the characteristic mass scale of the problem, the temperature. Second, we relate the size of the scalar self-coupling and gauge coupling by [40]

$$\lambda \sim g^3. \qquad (11)$$

By assuming that the quartic coupling is sufficiently small compared with gauge coupling, the cubic term induced by the gauge field will contribute at leading order, as seen below. Third, we assume

$$\mu^2 \sim (gT)^2, \qquad (12)$$

which installs the high-temperature expansion of $\mu^2 \ll T^2$. Finally, the effective thermally corrected mass of the scalar at leading order is

$$\mu_{eff}^2 \equiv \mu^2 + (4\lambda + 3g^2)\frac{T^2}{12} \sim \mathcal{O}(g^{2+N}T^2). \qquad (13)$$

Here, the first and third terms are individually of $\mathcal{O}(g^2 T^2)$. However, for $\mu^2 < 0$, a cancellation between the first and remaining terms can render the overall sum in μ_{eff}^2 smaller—parametrized by $N > 0$—for some temperature range. Henceforth, we assume a temperature window where $N = 1$ and argue below that this occurs in the vicinity of the phase transition critical temperature [40,44].

The assumptions of Eqs. (10)–(13) induce the chain of thermal scale hierarchies in Eq. (5). Nonzero Matsubara modes have masses $\sim \pi T$. In the unbroken phase ($\phi = 0$) zero Matsubara modes have the following scales: Higgs and Goldstone fields have masses $\sim \mu_{eff} \sim g^{\frac{3}{2}}T$. The gauge field temporal mode, B_0, has a thermal mass $\bar{m}_{B_0} \sim gT$; while spatial gauge fields remain massless ($m_B = 0$) in the unbroken phase. In the broken phase ($\phi > 0$), spatial and temporal gauge bosons, Goldstone and ghost fields have masses $\sim g\phi \sim gT$, while the Higgs field that undergoes nucleation has a parametrically lighter mass $\sim g^{\frac{3}{2}}T$.

The full one-loop effective potential is found using the following background field dependent mass eigenvalues

$$\bar{m}_H^2 = \mu_{eff}^2 + 3\lambda\phi^2, \qquad (14)$$

$$\bar{m}_\chi^2 = \mu_{eff}^2 + \lambda\phi^2 + g^2\xi\phi^2, \qquad (15)$$

$$\bar{m}_{B_0}^2 = \frac{1}{3}g^2 T^2 + g^2\phi^2, \qquad (16)$$

$$m_c^2 = g^2\xi\phi^2, \qquad m_B^2 = g^2\phi^2, \qquad (17)$$

where masses for the zero Matsubara modes of the Higgs, Goldstone, and temporal gauge field B_0 include resummed thermal corrections, e.g., the T^2 term in the third line describes Debye screening. The mass of the longitudinal component of the gauge field equals the ghost mass. Neither spatial gauge fields, nor ghost fields, develop thermal masses. Based on the power counting in g, the leading contribution to the effective potential is of $\mathcal{O}(g^3 T^4)$ and reads as

$$V_{LO}^{eff} = \frac{1}{2}\mu_{eff}^2\phi^2 + \frac{1}{4}\lambda\phi^4 - \frac{g^3 T}{12\pi}$$
$$\times \left[2\phi^3 + \left(\frac{1}{3}T^2 + \phi^2\right)^{\frac{3}{2}}\right], \qquad (18)$$

the second line is the gauge field contribution with the first term therein results from the spatial vector field and the second term the Debye mass (16) of the temporal component. Near the phase transition, all terms in the potential should be approximately of the same order of magnitude which is assured by construction, given the assumed power counting in g in Eqs. (10)–(13). The leading-order potential of Eq. (18) at $\mathcal{O}(g^3 T^4)$ is gauge invariant.

The one-loop contribution to the effective potential from the longitudinal gauge field, Goldstone field, and ghost fields is

$$-\frac{T}{12\pi}(\bar{m}_\chi^3 - m_c^3) \sim \mathcal{O}(g^4 T^4). \qquad (19)$$

Note that ghosts contribute with a relative minus sign to other fields. These contributions give rise to an explicit ξ dependence, but these terms are of higher order compared with Eq. (18) due to a cancellation at leading order. Below we include the remaining $\mathcal{O}(g^4 T^4)$ terms at NLO in the effective potential which expands as

$$V^{eff} = V_{LO}^{eff} + V_{NLO}^{eff} + \mathcal{O}(g^{\frac{9}{2}}T^4), \qquad (20)$$

where

$$V_{\mathrm{NLO}}^{\mathrm{eff}} = \frac{1}{(4\pi)^2}\left\{ g^4 T^2 \phi^2 \left(-1 + \ln\left(\frac{4g^2\phi^2}{\Lambda^2}\right)\right) \right.$$
$$+ \sqrt{\xi}gT\phi(g^3 T\phi - 2\pi(\mu_{\mathrm{eff}}^2 + \lambda\phi^2))$$
$$+ g^4 T^2 \left[\frac{1}{2}\sqrt{\xi}\phi\sqrt{\frac{1}{3}T^2 + \phi^2}\right.$$
$$\left.\left. + \frac{1}{2}\phi^2 \left(-1 + \ln\left(\frac{4g^2(\frac{1}{3}T^2 + \phi^2)}{\Lambda^2}\right)\right)\right]\right\}. \quad (21)$$

The last term on the second line originates from the one-loop Goldstone-ghost contribution in Eq. (19), and the last two lines correspond to the two-loop contributions of the B_0 field. The remaining terms result from two-loop diagrams with spatial gauge fields, ghosts, and scalars (χ, H). The $\mathcal{O}(g^{\frac{9}{2}}T^4)$ term in Eq. (20) arises at one-loop order from the Higgs field.

Consistent treatment of the effective action [Eq. (2)] at NLO also requires including the one-loop Higgs field renormalization:

$$Z = 1 + Z_{\mathrm{NLO}} + \mathcal{O}(g^{\frac{3}{2}}), \quad (22)$$

where

$$Z_{\mathrm{NLO}}(\phi) = \frac{gT}{48\pi}\left[-\frac{22}{\phi} + \frac{\phi^2}{(\frac{1}{3}T^2 + \phi^2)^{\frac{3}{2}}}\right], \quad (23)$$

is detailed in Ref. [35]; cf. Refs. [33,45] for a similar computation. The first term is from the spatial gauge fields and ghosts while the second term is from the temporal gauge field component. Notably, at order $\mathcal{O}(g)$ the field renormalization is independent of the gauge-fixing parameter. Higher order terms at $\mathcal{O}(g^{\frac{3}{2}})$ arise from two-loop diagrams involving gauge, ghost, and Goldstone fields, and one-loop diagrams with internal Higgs legs. Importantly, the gauge dependence arising from $\bar{m}_\chi^2 - m_c^2$ is formally of higher order in our power counting. The corresponding contributions to S_3 are asymptotically suppressed, arising from the tail of the bounce solution wherein $\phi \ll T$.

In the semiclassical approximation, the background field extremizes the leading-order action and can be found from the equation of motion for the leading-order potential

$$\nabla^2 \phi_b(x) = \left.\frac{\partial V_{\mathrm{LO}}^{\mathrm{eff}}}{\partial \phi}\right|_{\phi=\phi_b}, \quad \begin{cases}\phi_b(\infty) &= 0\\ \phi_b'(0) &= 0\end{cases}, \quad (24)$$

where $\nabla^2 \equiv \partial_i \partial_i$ is the three-dimensional Laplacian operator, and ϕ_b is the "bounce solution" [46]. We expand the exponent of the nucleation rate as $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$:

$$\mathcal{B}_0 = \beta \int d^3x \left[V_{\mathrm{LO}}^{\mathrm{eff}}(\phi_b) + \frac{1}{2}(\partial_i \phi_b)^2\right], \quad (25)$$

$$\mathcal{B}_1 = \beta \int d^3x \left[V_{\mathrm{NLO}}^{\mathrm{eff}}(\phi_b) + \frac{1}{2}Z_{\mathrm{NLO}}(\phi_b)(\partial_i \phi_b)^2\right], \quad (26)$$

where $\beta \equiv 1/T$. The characteristic length scale R for nucleation is related to the typical bubble size, given by the inverse mass of the nucleating field $R \sim m_H^{-1} \sim g^{-\frac{3}{2}}T^{-1}$. This gives rise to the formal scaling $\int d^3x \sim g^{-\frac{9}{2}}T^{-3}$. Together with the power counting for the effective potential and field renormalization, this establishes the relative importance of the first two leading exponent terms of the nucleation rate: $\mathcal{B}_0 \sim g^{-\frac{3}{2}}$, $\mathcal{B}_1 \sim g^{-\frac{1}{2}}$ as already foreshadowed in Eqs. (3) and (4). Despite the $1/\phi$ behavior of $Z_{\mathrm{NLO}}(\phi_b)$, the contribution of the term $Z_{\mathrm{NLO}}(\phi_b)(\partial_i \phi_b)^2$ is finite also in the region of vanishing ϕ_b [35] (cf. also Refs. [34,47,48]). NLO corrections to the bounce solution, $\phi = \phi_b + \Delta\phi$, also induce corrections to the effective action and hence the tunneling rate. However, these are higher order contributions than we consider here:

$$S_{\mathrm{eff}}(\phi) = S_{\mathrm{eff}}^{\mathrm{LO}}(\phi) + S_{\mathrm{eff}}^{\mathrm{NLO}}(\phi) + \dots$$
$$= S_{\mathrm{eff}}^{\mathrm{LO}}(\phi_b) + [S_{\mathrm{eff}}^{\mathrm{NLO}}(\phi_b) + (\Delta\phi)\delta S_{\mathrm{eff}}^{\mathrm{LO}}(\phi_b)] + \dots$$
$$= \mathcal{B}_0 + \mathcal{B}_1 + \dots \quad (27)$$

In the second line, the terms inside the [ ] are $\mathcal{O}(g)$; $\delta S_{\mathrm{eff}}^{\mathrm{LO}}(\phi_b) = 0$ since ϕ_b extremizes the LO action. Thus, the corrections $\Delta\phi$ to the bounce solution first enter at higher orders, so we may neglect them here.

Because the bounce solution has to be solved numerically from Eq. (24), the exponents $\mathcal{B}_{0,1}$ are necessarily obtained by numerical integration. Nevertheless, their gauge independence can still be proven analytically. The gauge-fixing parameter is absent at leading order since Z and V^{eff} are both ξ independent at this order, implying gauge independence of both the bounce solution ϕ_b and exponent $\mathcal{B}_0$. The gauge invariance of $\mathcal{B}_1$ is not immediately obvious since $V_{\mathrm{NLO}}^{\mathrm{eff}}$ in Eq. (21) explicitly depends on ξ. To proceed, we utilize the Nielsen identities [42,49] in analogy to Ref. [17]. These identities have been discussed in the context of finite temperature in, e.g., Refs. [33,41,50,51]. The Nielsen identity (in d-dimensional Euclidean space)

$$\xi\frac{\partial S^{\mathrm{eff}}}{\partial \xi} = -\int d^d\mathbf{x} \frac{\delta S^{\mathrm{eff}}}{\delta\phi(x)}\mathcal{C}(x), \quad (28)$$

relates the variation of the effective action with the gauge parameter to the corresponding Nielsen functional

$$\mathcal{C}(x) = \frac{ig}{2}\int d^d\mathbf{y} \langle\chi(x)c(x)\bar{c}(y)$$
$$\times [\partial_i B_i(y) + \sqrt{2}g\xi\phi\chi(y)]\rangle, \quad (29)$$

which admits a derivative expansion [33]

$$\mathcal{C}(x) = C(\phi) + D(\phi)(\partial_\mu \phi)^2 - \partial_\mu(\tilde{D}(\phi)\partial_\mu \phi) + \mathcal{O}(\partial^4).$$
(30)

Together with the expansion of the effective action [Eq. (2)], this yields the Nielsen identities for the effective potential and field renormalization factor

$$\xi \frac{\partial}{\partial \xi} V^{\text{eff}} = -C \frac{\partial}{\partial \phi} V^{\text{eff}},$$
(31)

$$\xi \frac{\partial}{\partial \xi} Z = -C \frac{\partial}{\partial \phi} Z - 2Z \frac{\partial}{\partial \phi} C - 2D \frac{\partial}{\partial \phi} V^{\text{eff}} - 2\tilde{D} \frac{\partial^2}{\partial \phi^2} V^{\text{eff}}.$$
(32)

To employ these relations, we expand them in powers of g by first quoting the scaling of the Nielsen coefficients

$$C = C_{\text{LO}} + \mathcal{O}(g^{\frac{3}{2}}T), \qquad C_{\text{LO}} = T \frac{\sqrt{\xi}}{16\pi} g,$$
(33)

$$D = \mathcal{O}(g^{-1}T^{-3}), \qquad \tilde{D} = \mathcal{O}(g^{-1}T^{-2}).$$
(34)

These coefficients are computed at leading order in Refs. [33,35]. We do not need the explicit expressions of D, $\tilde{D}$ because the terms on the second line of Eq. (32) appear at $\mathcal{O}(g^2)$, and are hence suppressed relative to those on the first line at $\mathcal{O}(g)$. The leading order $C_{\text{LO}} \sim gT$ is independent of the scalar background field ϕ at finite temperature. An explicit counting in powers of g in the identities from Eqs. (31) and (32) yields

$$\xi \frac{\partial}{\partial \xi} V^{\text{eff}}_{\text{NLO}} = -C_{\text{LO}} \frac{\partial}{\partial \phi} V^{\text{eff}}_{\text{LO}},$$
(35)

$$\xi \frac{\partial}{\partial \xi} Z_{\text{NLO}} = -2 \frac{\partial}{\partial \phi} C_{\text{LO}},$$
(36)

at $\mathcal{O}(g^4 T^4)$ in the first and $\mathcal{O}(gT)$ in the second Nielsen identity. A combination of the explicit expressions [Eqs. (18), (21), (23), and (33)] readily verifies both identities. In particular, the equality [Eq. (36)] holds since the NLO field renormalization is ξ independent and at LO C is ϕ independent.

Using the above Nielsen identities, we demonstrate gauge independence of $\mathcal{B}_1$:

$$\begin{aligned}
\xi \frac{\partial}{\partial \xi} \mathcal{B}_1 &= \xi \frac{\partial}{\partial \xi} \beta \int d^3 x \left[V^{\text{eff}}_{\text{NLO}}(\phi_b) + \frac{1}{2} Z_{\text{NLO}}(\partial_\mu \phi_b)^2 \right] \\
&\overset{(A)}{=} \beta \int d^3 x \left[-C_{\text{LO}} \frac{\partial}{\partial \phi} V^{\text{eff}}_{\text{LO}}(\phi_b) \right] \\
&\overset{(B)}{=} -C_{\text{LO}} \beta \int d^3 x [\nabla^2 \phi_b] \\
&\overset{(C)}{=} -C_{\text{LO}} \beta \int d^2 S \cdot (\partial \phi_b) \\
&\overset{(D)}{=} 0.
\end{aligned}$$
(37)

Step (A) uses the Nielsen identity [Eq. (35)] and ξ independence of Z_{NLO}; (B) applies the equation of motion [Eq. (24)] and moves C_{LO} outside the integrand due to its ϕ independence; (C) applies Gauss's theorem; (D) follows from the asymptotic behavior of the bounce solution at the boundary,

$$\nabla^2 \phi_b \sim \mu^2_{\text{eff}} \phi_b, \qquad \phi_b(\infty) = 0,$$
(38)

$$\Rightarrow \phi_b(r) \sim c \frac{e^{-\mu_{\text{eff}} r}}{r} \quad \text{as } r \to \infty.$$
(39)

This completes our proof of gauge invariance of the leading exponential of the nucleation rate.

Figure 1 illustrates the quantitative impact of applying our framework as compared to the conventional gauge-dependent approach. The left panel of Fig. 1 gives the λ dependence of the strength of the transition, characterized by latent heat released in the transition L, scaled to T_c^4. The right panel of Fig. 1 shows the T dependence of S_3/T. We fix $g^2 = 0.1$ and $\mu^2 = -M^2/2$, where $M = 100$, and where M and T are in arbitrary units of mass. Furthermore, we define λ and g at an initial renormalization scale $\Lambda_0 = 100$ and run to a scale $\Lambda = T$ using one-loop renormalization group evolution [35].

The dashed (solid) line in Fig. 1 (left) corresponds to the gauge-invariant result for L/T_c^4 at LO (LO + NLO). For comparison, we also illustrate the conventional analysis based on direct minimization of the full one-loop effective potential (in the high-T expansion). The latter approach is gauge dependent, as illustrated by the light gray band in which the gauge-fixing parameter ranges over $\xi = [0, 4]$.

The right panel of Fig. 1 shows the T dependence of $S_3/T = \mathcal{B}_0 + \mathcal{B}_1$ for fixed $\lambda = 0.001$ as the black solid curve. We have obtained this result using FINDBOUNCE [52] to find the bounce solution ϕ_b of Eq. (24). For comparison, the light gray band significantly varies in S_3/T when varying $\xi = [0, 4]$, which illustrates the gauge dependence that is often encountered in the past literature (e.g., Refs. [8,53]). The latter result is based on computing Eq. (1) using Eq. (2), where the effective potential is directly computed at one-loop order in a fixed gauge. Therein, the bounce solution is found from the same gauge-dependent one-loop effective potential.
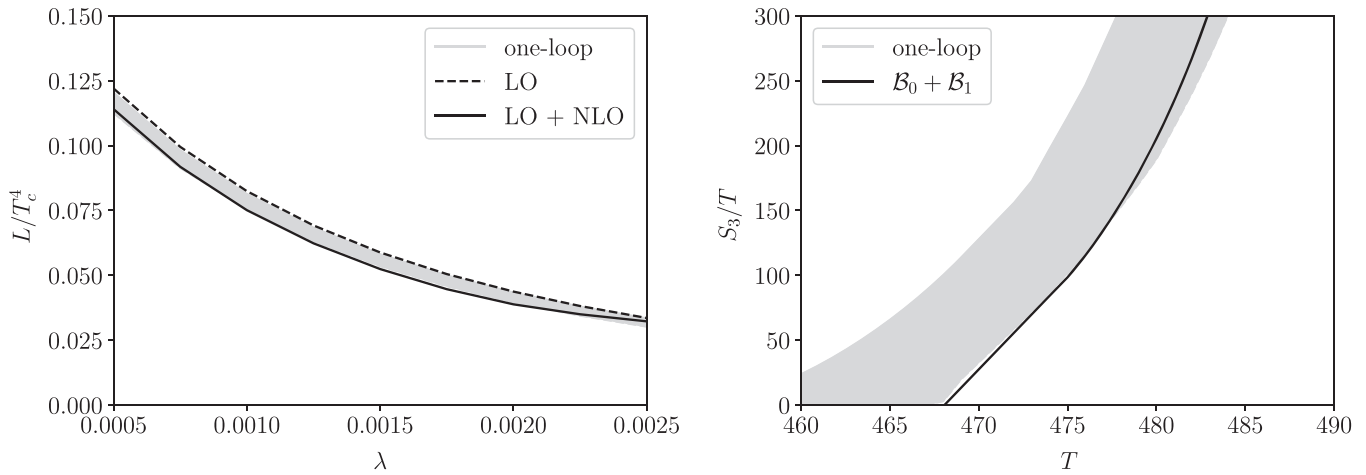
FIG. 1. An illustration of a gauge-independent, perturbative determination of the thermodynamics for a first-order phase transition, with $g^2 = 0.1$ and $M = 100$, where M and T are in arbitrary units of mass. Left: the dimensionless ratio L/T_c^4 as a function of λ at LO (black dashed) and NLO (black solid). Right: the action $S_3/T = \mathcal{B}_0 + \mathcal{B}_1$ as a function of temperature at $\lambda = 0.001$ (black solid). For comparison, both panels illustrate the conventional gauge-dependent determination, wherein the light gray band shows the variation of the gauge-fixing parameter in the range of $\xi = [0, 4]$.

This leftover gauge-dependence immediately signals an inconsistency, as the computation of a physical quantity should contain no ξ dependence.

By denoting the difference between the two approaches as $\Delta S_3 \equiv S_3^{\text{1-loop}} - (\mathcal{B}_0 + \mathcal{B}_1)$, ΔS_3 contains the ξ-dependent terms. Within the context of the power counting of Eqs. (11)–(13), these ξ-dependent terms are formally NNLO, i.e., $\mathcal{O}(g^0)$ and beyond, since they do not enter $\mathcal{B}_0 + \mathcal{B}_1$. Indeed, the organization of the perturbative expansion provided herein allows one to clearly identify the origin—as a function of g—of the ξ dependence in the "conventional" approach. As a framework for obtaining a ξ-independent computation to a given order in g, our framework also delineates the limits of validity of the perturbative series—also as a function of g—when implemented in combination with the derivative expansion.

For a discussion of other theoretical inconsistencies encountered in the "conventional" approach, such as the appearance of an imaginary part in S_3 and double counting of contributions from the nucleating field, see Refs. [34,54,55]. The renormalization scale dependence of the results in Fig. 1, that can be used to monitor the accuracy of perturbation theory (cf. Refs. [54,56]), is further discussed in Ref. [35].

The framework presented in this Letter provides a way to obtain a gauge-independent, perturbative estimate of the thermal nucleation rate in the presence of radiative barriers. While we have worked in the Abelian Higgs model along the lines of previous literature [17,33], our framework readily generalizes to more complicated gauge field theories, with radiatively generated barrier, and our practical approach can facilitate corresponding model-building phenomenological studies. Ultimately, one must assess the quantitative and qualitative reliability of perturbative nucleation rate computations through comparison with nonperturbative calculations (see Refs. [15,57]). A meaningful comparison requires a well-defined, gauge-invariant perturbative computation, which the framework presented herein provides.

[*] johan.lofgren@physics.uu.se
[†] mjrm@sjtu.edu.cn; mjrm@physics.umass.edu
[‡] philipp.schicho@helsinki.fi
[§] tuomas.tenkanen@su.se

[1] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Phys. Rev. Lett. **77**, 2887 (1996).
[2] F. Csikor, Z. Fodor, and J. Heitger, Phys. Rev. Lett. **82**, 21 (1999).
[3] S. Weinberg, Phys. Rev. D **9**, 3357 (1974).
[4] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, Phys. Lett. **155B**, 36 (1985).
[5] M. E. Shaposhnikov, Nucl. Phys. **B287**, 757 (1987).
[6] D. E. Morrissey and M. J. Ramsey-Musolf, New J. Phys. **14**, 125003 (2012).
[7] M. J. Ramsey-Musolf, J. High Energy Phys. 09 (2020) 179.
[8] C. Caprini et al., J. Cosmol. Astropart. Phys. 03 (2020) 024.

[9] S. Kawamura *et al.*, Classical Quantum Gravity **28**, 094011 (2011).

[10] G. M. Harry, P. Fritschel, D. A. Shaddock, W. Folkner, and E. S. Phinney, Classical Quantum Gravity **23**, 4887 (2006).

[11] W.-H. Ruan, Z.-K. Guo, R.-G. Cai, and Y.-Z. Zhang, Int. J. Mod. Phys. A **35**, 2050075 (2020).

[12] P. Amaro-Seoane *et al.* (LISA Collaboration), arXiv:1702 .00786.

[13] K. Farakos, K. Kajantie, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B442**, 317 (1995).

[14] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B466**, 189 (1996).

[15] G. D. Moore and K. Rummukainen, Phys. Rev. D **63**, 045002 (2001).

[16] C. Caprini *et al.*, J. Cosmol. Astropart. Phys. 04 (2016) 001.

[17] D. Metaxas and E. J. Weinberg, Phys. Rev. D **53**, 836 (1996).

[18] J. Baacke and K. Heitmann, Phys. Rev. D **60**, 105037 (1999).

[19] A. Andreassen, W. Frost, and M. D. Schwartz, Phys. Rev. D **91**, 016009 (2015).

[20] A. Andreassen, W. Frost, and M. D. Schwartz, Phys. Rev. Lett. **113**, 241801 (2014).

[21] B. Garbrecht and P. Millington, Phys. Rev. D **92**, 125022 (2015).

[22] M. Endo, T. Moroi, M. M. Nojiri, and Y. Shoji, Phys. Lett. B **771**, 281 (2017).

[23] L. Niemi, M. J. Ramsey-Musolf, T. V. I. Tenkanen, and D. J. Weir, Phys. Rev. Lett. **126**, 171802 (2021).

[24] J. O. Andersen, T. Gorda, A. Helset, L. Niemi, T. V. I. Tenkanen, A. Tranberg, A. Vuorinen, and D. J. Weir, Phys. Rev. Lett. **121**, 191802 (2018).

[25] D. Croon, V. Sanz, and G. White, J. High Energy Phys. 08 (2018) 203.

[26] J. S. Langer, Ann. Phys. (Berlin) **54**, 258 (1969).

[27] S. R. Coleman, Phys. Rev. D **15**, 2929 (1977); **16**, 1248(E) (1977).

[28] A. D. Linde, Phys. Lett. **96B**, 289 (1980).

[29] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).

[30] A. D. Linde, Nucl. Phys. **B216**, 421 (1983); **223B**, 544 (1983).

[31] J. Langer, Physica (Amsterdam) **73**, 61 (1974).

[32] I. Moss, D. Toms, and A. Wright, Phys. Rev. D **46**, 1671 (1992).

[33] M. Garny and T. Konstandin, J. High Energy Phys. 07 (2012) 189.

[34] O. Gould and J. Hirvonen, Phys. Rev. D **104**, 096015 (2021).

[35] J. Hirvonen, J. Löfgren, M. J. Ramsey-Musolf, P. Schicho, and T. V. I. Tenkanen, J. High Energy Phys. 07 (2022) 135.

[36] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B458**, 90 (1996).

[37] E. Braaten and A. Nieto, Phys. Rev. D **51**, 6990 (1995).

[38] J. Kripfganz, A. Laser, and M. G. Schmidt, Nucl. Phys. **B433**, 467 (1995).

[39] J. Kripfganz, A. Laser, and M. G. Schmidt, Z. Phys. C **73**, 353 (1997).

[40] P. B. Arnold and O. Espinosa, Phys. Rev. D **47**, 3546 (1993); **50**, 6662(E) (1994).

[41] S. Arunasalam and M. J. Ramsey-Musolf, J. High Energy Phys. 08 (2022) 138.

[42] R. Fukuda and T. Kugo, Phys. Rev. D **13**, 3469 (1976).

[43] A. Ekstedt and J. Löfgren, J. High Energy Phys. 12 (2020) 136.

[44] A. Gynther and M. Vepsäläinen, J. High Energy Phys. 03 (2006) 011.

[45] J. Kripfganz, A. Laser, and M. G. Schmidt, Phys. Lett. B **351**, 266 (1995).

[46] S. R. Coleman and E. J. Weinberg, Phys. Rev. D **7**, 1888 (1973).

[47] J. Baacke and G. Lavrelashvili, Phys. Rev. D **69**, 025009 (2004).

[48] A. Ekstedt, Eur. Phys. J. C **82**, 173 (2022).

[49] N. K. Nielsen, Nucl. Phys. **B101**, 173 (1975).

[50] H. H. Patel and M. J. Ramsey-Musolf, J. High Energy Phys. 07 (2011) 029.

[51] J. R. Espinosa, M. Garny, T. Konstandin, and A. Riotto, Phys. Rev. D **95**, 056004 (2017).

[52] V. Guada, M. Nemevšek, and M. Pintar, Comput. Phys. Commun. **256**, 107480 (2020).

[53] C. Delaunay, C. Grojean, and J. D. Wells, J. High Energy Phys. 04 (2008) 029.

[54] D. Croon, O. Gould, P. Schicho, T. V. I. Tenkanen, and G. White, J. High Energy Phys. 04 (2021) 055.

[55] A. Ekstedt, J. High Energy Phys. 08 (2022) 115.

[56] O. Gould and T. V. I. Tenkanen, J. High Energy Phys. 06 (2021) 069.

[57] G. D. Moore, K. Rummukainen, and A. Tranberg, J. High Energy Phys. 04 (2001) 017.