Hierarchy of Multipartite Nonlocality and Device-Independent Effect Witnesses

Peter Bierhorst[*] and Jitendra Prakash[†]
Mathematics Department, University of New Orleans, New Orleans, Louisiana 70148, USA

(Received 31 January 2023; accepted 19 May 2023; published 22 June 2023)

According to recent new definitions, a multiparty behavior is genuinely multipartite nonlocal (GMNL) if it cannot be modeled by measurements on an underlying network of bipartite-only nonlocal resources, possibly supplemented with local (classical) resources shared by all parties. The new definitions differ on whether to allow entangled measurements upon, and/or superquantum behaviors among, the underlying bipartite resources. Here, we categorize the full hierarchy of these new candidate definitions of GMNL in three-party quantum networks, highlighting the intimate link to device-independent witnesses of network effects. A key finding is the existence of a behavior in the simplest nontrivial multipartite measurement scenario (three parties, two measurement settings, and two outcomes) that cannot be simulated in a bipartite network prohibiting entangled measurements and superquantum resources—thus witnessing the most general form of GMNL—but can be simulated with bipartite-only quantum states with an entangled measurement, indicating an approach to device-independent certification of entangled measurements with fewer settings than in previous protocols. Surprisingly, we also find that this (3,2,2) behavior, as well as the others previously studied as device-independent witnesses of entangled measurements, can all be simulated at a higher echelon of the GMNL hierarchy that allows superquantum bipartite resources while still prohibiting entangled measurements. This poses a challenge to a theory-independent understanding of entangled measurements as an observable phenomenon distinct from bipartite nonlocality.

DOI: 10.1103/PhysRevLett.130.250201

Quantum nonlocality [1] is a fascinating phenomenon that can be convincingly demonstrated in experiments of two spatially separated parties [2–5]. Quantum mechanics also predicts nonlocal effects in experiments of three or more spatially separated parties. Naturally, a three-party experiment should only be considered *genuinely multiparty nonlocal* (GMNL) if it exhibits some nonlocal behavior beyond the two-party type, ruling out scenarios where, for instance, two parties observe nonlocality with each other while the third party's statistics are not correlated with the first two in any way.

A first approach to defining genuine multipartite nonlocality, introduced by Svetlichny [6] and later refined [7,8], proposes that a probability distribution of experimental outcomes be considered GMNL if it cannot be expressed as a convex mixture of distributions where each one factors into a product of at-most-bipartite nonlocal distributions. However, this definition admits anomalies [9–11]: for instance, if one measuring party simultaneously participates in two parallel but unrelated two-party Clauser-Horne-Shimony-Holt (CHSH [12]) experiments, one with the second party and the other with the third party, the combined statistics of all three parties will be classified as GMNL according to Svetlichny-type definitions.

Recently, some authors [10,11,13] have proposed new definitions of GMNL based on whether a behavior can be simulated by an underlying network of bipartite nonlocal resources, possibly with access to local or classical resources shared by all parties (shared randomness). Figure 1 gives a schematic representation of such an underlying network for the three-party scenario, where a bipartite resource such as ω_{AB} shared by Alice and Bob could be an entangled Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$, but three-way nonclassical states such as the GHZ state [14] are disallowed. According to the new paradigm, a three-party behavior is considered GMNL if it *cannot* be induced by an

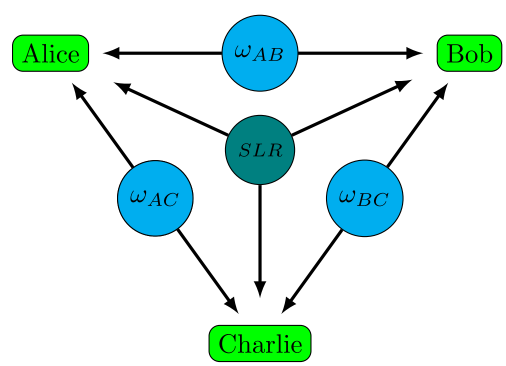


FIG. 1. A bipartite network model for a tripartite scenario. Tripartite behaviors that *cannot* be induced by an underlying bipartite network model of the above form—bipartite nonclassical sources (ω) possibly supplemented with classical randomness shared by all three parties (SLR)—are considered genuinely multipartite nonlocal (GMNL) according to recent new definitions [10,11,13].

underlying network like that of Fig. 1. The parallel-CHSH-experiment example of the previous paragraph would here be ruled (only) bipartite nonlocal.

The new definitions [10,11,13] differ on the impositions made on the underlying network. The strictest of these definitions—that is, the one which would categorize the *largest* class of behaviors as (only) bipartite nonlocal—is that of Coiteux-Roy *et al.* [11]. This definition allows the parties to perform entangled measurements, and also allows superquantum nonsignaling bipartite resources [such as Popescu-Rohrlich (PR) boxes [15]] in the underlying bipartite network. Tripartite behaviors ruling out this class, which can be achieved with appropriate measurements on the three-way entangled GHZ state [11,16], naturally also rule out other definitions with more restrictions on the networks such as a definition disallowing entangled measurements [13], a definition disallowing superquantum bipartite resources [10], or a fourth candidate definition disallowing both. Recent experimental results [16–18] provide initial evidence, subject to fair-sampling-type assumptions, for the existence of three-party behaviors that cannot be modeled by even the most general underlying bipartite networks of Ref. [11]. Different perspectives on what phenomena transcend that of (only) bipartite nonlocality motivate a closer study of the new definitions of GMNL that are less restrictive than that of Ref. [11]. To illustrate, observe that device-independent and self-testing witnesses of entangled measurements [19–21] are fundamentally multipartite phenomena, requiring at a minimum two distant parties and a third "entangling" party in between: any strictly two-party setup involving entangled measurements on different subsystems can always be easily simulated by a higher dimensional setup that does not employ entangled measurements [see Supplemental Material (SM), Sec. 1 [22]]. Constraints derived under notions of GMNL that disallow entangled measurements will indeed be intimately linked to device-independent certificates of entangled measurements, a crucial tool for teleportation and entanglement swapping protocols in quantum networks [23]. A device-independent perspective suggests disallowing superquantum nonsignaling bipartite resources (i.e., PR boxes [15]) among the ω sources in Fig. 1 as nonphysical, but a more foundational perspective seeking a better theory-independent understanding of the nature of entangled measurements, which have recently been argued to remain poorly understood [24], recommends consideration of the GMNL paradigm where superquantum resources are allowed. We will consider both viewpoints.

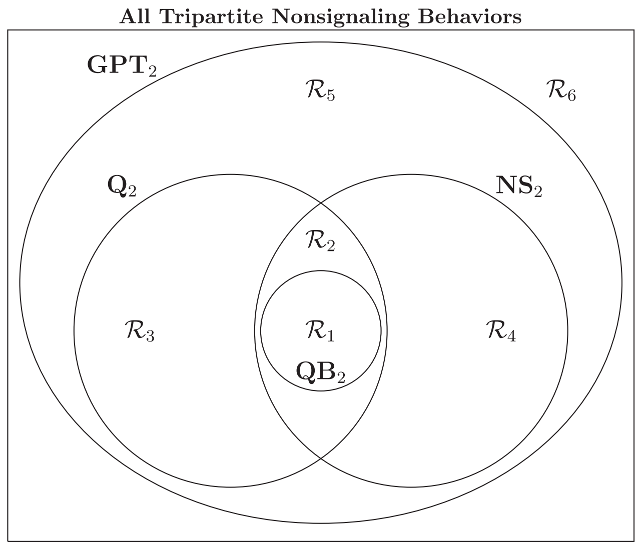
In this Letter, we study the full hierarchy of new definitions of GMNL and classify their interrelationships for the tripartite scenario. A main result of this work is the demonstration of a quantum behavior, using entangled measurements on bipartite-only quantum states, that witnesses the most general form of multiparty nonlocality—that disallowing entangled measurements and superquantum resources in the Fig. 1 network—in the simplest possible (3,2,2) scenario of three measuring parties, two measurement settings per party, and two possible outcomes for each measurement. This behavior demonstrates an important separation between different definitions of GMNL, while also providing a promising approach to the task of device-independent certification of entangled measurements with the fewest possible number of settings and outcomes—reducing the number of settings from previous scenarios achieving this task [19–21]. Note that as this behavior is not considered GMNL according to the stricter definition of Ref. [11], the non-fan-out inflation technique [25] used in Refs. [11,16] is inapplicable for demonstrating the weaker notion of GMNL studied here, and our proof uses a different approach invoking self-testing [26].

This (3,2,2) behavior demonstrates GMNL according to the definition where the ω in Fig. 1 are limited to quantum-achievable resources. We continue the study by asking whether this behavior is still GMNL according to a paradigm in which superquantum resources (i.e., nonsignaling Popescu-Rohrlich boxes [15]) are allowed for the underlying bipartite network, while still prohibiting entangled measurements and superquantum generalizations thereof. We find—perhaps surprisingly—that bipartite PR box networks can simulate the (3,2,2) behavior discussed above *without* appealing to entangled measurements (or superquantum generalizations of the notion). Hence, this behavior exhibits only bipartite nonlocality according to the GMNL definition allowing nonsignaling resources in Fig. 1.

Motivated by this finding, we asked whether such a model exists for the more complicated behavior introduced by Ref. [19], which has been studied in various forms [20,21] as the canonical behavior certifying the presence of an entangled measurement in a fully device-independent manner. Similarly, we find a model for the behavior of Ref. [19] using a network of bipartite PR boxes without entangled measurements. Hence, none of these behaviors bear a theory-independent signature of the phenomenon of entangled measurements (i.e., without reference to the axioms of quantum mechanics), raising questions about exactly what such a signature might be, or if it exists.

We now give a precise formulation of the bipartite network model in which we rigorously derive our results. The three parties Alice, Bob, and Charlie of Fig. 1 make choices of measurements represented by respective random variables X, Y, and Z, and record measurement outcomes A, B, and C. An experiment is then characterized by the behavior $P(A,B,C|X,Y,Z)$, the settings-conditional outcome distribution. Behaviors $P(ABC|XYZ)$ that can be induced by a network of the form in Fig. 1 are said to be not GMNL, where the precise class of behaviors singled out differs based on the nature of the bipartite sources ω_{PQ} and the form of the measurements allowed to Alice, Bob, and Charlie.

QB_2 is the smallest class of behaviors in the hierarchy of bipartite network models, which are summarized in Fig. 2.



Behavior Class	Superquantum Bipartite Sources	Entangled Measurements
QB_2	No	No
NS_2	Yes	No
Q_2	No	Yes
GPT_2	Yes	Yes

FIG. 2. Summary of features for the different models of an underlying network of bipartite-only systems in the tripartite scenario. The Venn diagram illustrates the containment relationships for the corresponding classes of behaviors.

Here, the bipartite sources ω_{PQ} are taken to be quantum states ρ_{PQ}, so that the joint quantum state of the system is of the form $\rho_{AB} \otimes \rho_{BC} \otimes \rho_{AC}$. The parties Alice, Bob, and Charlie apply quantum measurements [positive operator-valued measures (POVMs)] to their respective systems, but must separately measure subsystems shared with different players. This is a scenario of "quantum boxes" (motivating the choice of name QB$_2$), where quantum states are effectively input-output machines as entangling dynamics on the states are prohibited. Because of this, the POVM elements of Bob (for example), which act on the state space of the reduced system $\mathrm{tr}_{AC}[\rho_{AB} \otimes \rho_{BC}]$, are expressible in the separable form (Ref. [27], Proposition 6.5),

$$\sum_i c_i R_i^A \otimes R_i^C, \tag{1}$$

where each R_i^P is a rank-1 projector acting on the portion of Bob's state shared with player P and the c_i are positive real constants not greater than one (see SM, Sec. 2 [22]). The class of separable measurements of form Eq. (1) is in fact slightly larger than those measurements strictly admitting a quantum box description [28].

The framework QB$_2$ disallows superquantum resources for the ω_{PQ} in Fig. 1, which can be justified on practical grounds: superquantum correlations such as those of the PR boxes are generally expected to be nonphysical, and in the device-independent certification perspective the validity and completeness of quantum mechanics is generally assumed. Quantum behaviors outside of QB$_2$ require either entangled measurements or three-way entangled states, and so device independently witness the presence of at least one of these resources.

If one accepts the position that only quantum resources should be considered for the bipartite resources in Fig. 1, but that entangled measurements should be permitted, one arrives at the larger class of bipartite network behaviors Q_2. This corresponds to the notion of GMNL given in Definition 2 of Ref. [10]. Q_2 is precisely the boundary for a behavior exhibiting tripartite entangled states device independently; any tripartite quantum behavior lying outside this set certifies the presence of a three-way-entangled quantum state (in particular, a genuinely network 3-entangled state as defined in Ref. [29]).

Another option for extending the class QB$_2$ is to allow for superquantum resources such as PR boxes [15] while instead maintaining the prohibition on entangled measurements. For the observable phenomenon of (bipartite) nonlocality, the most abstract definition of this phenomenon—that without any appeal to the axioms of quantum mechanics—involves black boxes that can violate Bell inequalities while respecting the no-signaling conditions. The framework NS$_2$ allows the classical manipulation whereby outputs of some of the bipartite boxes are used as inputs to other bipartite boxes, expanding the scope of simulable tripartite behaviors [30]. Finally, the largest class GPT$_2$ (standing for generalized probabilistic theories) allows for both superquantum bipartite sources and entangled measurements (and possibly superquantum generalizations thereof); this corresponds to the GMNL definition of Ref. [11].

The containment relationships of the four sets are summarized in Fig. 2. It is known that some of the containments are strict: region $\mathcal{R}_4$ can be seen to be nonempty due to the presence of PR box correlations in NS$_2$ while Tsirelson's bound [31] rules these out of Q_2, and the results of Refs. [11,16] demonstrate quantum behaviors in region $\mathcal{R}_6$. It is conjectured in Sec. V C of Ref. [32] that there are correlations outside NS$_2$ but inside GPT$_2$, but to date we are unaware of an argument definitively proving the existence of behaviors in either region $\mathcal{R}_3$ or $\mathcal{R}_5$.

The three-party behavior introduced by Ref. [19] and further studied by Refs. [20,21] as a device-independent certificate of entangled measurements can, due to this certifying property, be situated in the current context as lying outside QB$_2$ but inside Q_2; see Proposition 7 in Ref. [32] for an extended discussion. (Whether these behaviors are in region $\mathcal{R}_2$ or $\mathcal{R}_3$ requires further analysis; we answer this question later.) The behavior of Ref. [19] and all of its later-studied variants are characterized by having more than two setting choices for at least one of the parties. In contrast, the following result shows that a behavior in $Q_2 \backslash \mathrm{QB}_2$ can be found for the simplest possible (3,2,2) scenario. All behaviors in any simpler measurement scenario can always be simulated with bipartite resources and shared local randomness (see SM, Sec. 3 [22]).

Theorem 1.—There is a behavior $P(ABC|XYZ)$ in Q_2 with binary input and output random variables satisfying the conditions $P(B=0|Y=1)>0$, $P_{Y=1,B=0}(AC|XZ)$ maximally violates the CHSH inequality, and $P(A=B|X=0, Y=0)=1$. Furthermore, no behavior in QB_2 can satisfy these conditions.

The behavior in Q_2 is obtained as follows: Alice and Bob, and Bob and Charlie, each share a Bell pair $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. No Alice-Charlie state is used. On setting $Y=1$, Bob performs an (entangled) Bell state measurement on his two portions of the Bell pairs; conditioned on observing the outcome corresponding to $|\Phi^+\rangle$, which occurs with probability 1/4, Bob reports outcome $B=0$ and all other Bell measurement outcomes are binned into outcome $B=1$. Conditioned on $B=0$, Alice and Charlie possess $|\Phi^+\rangle$ on which they can perform measurements maximally violating the CHSH inequality with Alice measuring σ_z on setting $X=0$. On setting $Y=0$, Bob measures (only) his qubit shared with Alice in the same direction σ_z, which ensures $P(A=B|X=0, Y=0)=1$. The exact behavior $P(ABC|XYZ)$ is

$$\begin{cases} \frac{1+(-1)^{A\oplus B}\delta_{X,0}}{8} & \text{if } Y=0, \\ \frac{\delta_{B,1}}{4} + \frac{(-1)^B}{4}\mathrm{CHSH}(AC|XZ) & \text{if } Y=1, \end{cases}$$

where $\mathrm{CHSH}(AC|XZ) = 2 + (-1)^{A\oplus C\oplus XZ}\sqrt{2}/8$.

That such behaviors cannot exist in QB_2 follows by the following intuition, which we make precise and prove in Sec. 4 of SM [22]. Assume Bob can make only a separable measurement on setting $Y=1$ (the proof does not assume separability of any other measurement). This measurement cannot create new entanglement between Alice and Charlie, but Alice and Charlie must be measuring an entangled Bell state to maximally violate CHSH, and so this must be a Bell state they initially possess via ω_{AC}. Then since Bob is not entangled with ω_{AC}, from his perspective Alice is measuring a fully mixed state and it will be impossible for him to do any better than blind guessing when trying to align his outcome with Alice's for setting $Y=0$.

As in Ref. [19], our rigorous proof relies crucially on self-testing, but we encounter a notable complication in the need to link a conditional *post*-Bob-measurement CHSH violation to restrictions on Bob's ability to align with Alice's outcome when he chooses a different measurement setting, requiring a new argument that necessarily cedes improved (but not perfect) prospects for Bob to align outcomes with Alice. Our proof applies in full generality, i.e., assuming only POVMs (see SM Sec. 6 [22], where we borrow an argument from Ref. [33] instead of the standard one [34] for POVM-to-projective-measurement dilation) on potentially mixed states, and while we do assume a maximal violation of the CHSH inequality, this leads to a robust upper bound (strictly less than 1) on $P(A=B|X=0, Y=0)$ such that robustness results for self-testing [35] provide a clear approach for lifting the argument to experimentally testable constraints, and thereby a device-independent witness of entangled measurements in the simplest possible (3,2,2) scenario.

We extend our analysis by asking whether this behavior lies in region $\mathcal{R}_2$ or $\mathcal{R}_3$. One might be tempted to think that the (3,2,2) behavior described above cannot be simulated in NS_2, due to the well-known prohibition on "nonlocality swapping" [36,37]. However, see Theorem 2.

Theorem 2.—There exists a behavior in NS_2 meeting the conditions of Theorem 1.

Proof.—Figure 3(a) gives an example of a PR box network that results in the behavior $P(ABC|XYZ)$ given by

$$\begin{cases} \frac{\delta_{A,B}}{4} & \text{if } Y=0, \\ \frac{\delta_{B,1}}{4} + \frac{(-1)^B}{2}\mathrm{PR}(AC|XZ) & \text{if } Y=1, \end{cases}$$

where $\mathrm{PR}(AC|XZ) = (\delta_{A\oplus C,XZ}/2)$. This behavior satisfies the conditions of Theorem 1 with the modification that $P_{Y=1,B=0}(AC|XZ)$ violates the CHSH inequality beyond the Tsirelson's bound to the nonsignaling maximum of 4. A convex mixture of this behavior with classical behaviors can induce violations of the CHSH inequality to only the quantum maximum of $2\sqrt{2}$. ■

A possible idea for why the (3,2,2) behavior might fail to bear a theory-independent signature of an entangled measurement is that tripartite quantum behaviors outside QB_2 only signify *either* the presence of entangled measurements *or* three-way entangled sources, and it is only with the additional assumption of the absence of three-way entangled sources that the (3,2,2) behavior certifies entangled measurements specifically. Indeed, Jordan's lemma ensures that any (3,2,2) behavior can be simulated with (nonentangled) measurements on qubits [38] and we provide in SM Sec. 5 an explicit example satisfying the conditions of Theorem 1 with a GHZ state [22]. The assumption of the absence of three-way entangled sources is also required in Refs. [20,21] for noise-robust device-independent certification of entangled measurements, and while the assumption can be well motivated physically in appropriate setups, it can be argued to technically represent a weakening to a semi-device-dependent scenario. However, this assumption is *not* invoked in the original argument concerning the noise-free behavior of Ref. [19]. But we find that even the original behavior of Ref. [19] is simulable with networks of bipartite PR boxes.

In the scenario of Ref. [19], reformulated as a Bell game, Alice and Charlie still have binary settings and outcomes, but now Bob has three settings $Y \in \{0,1,2\}$, each with

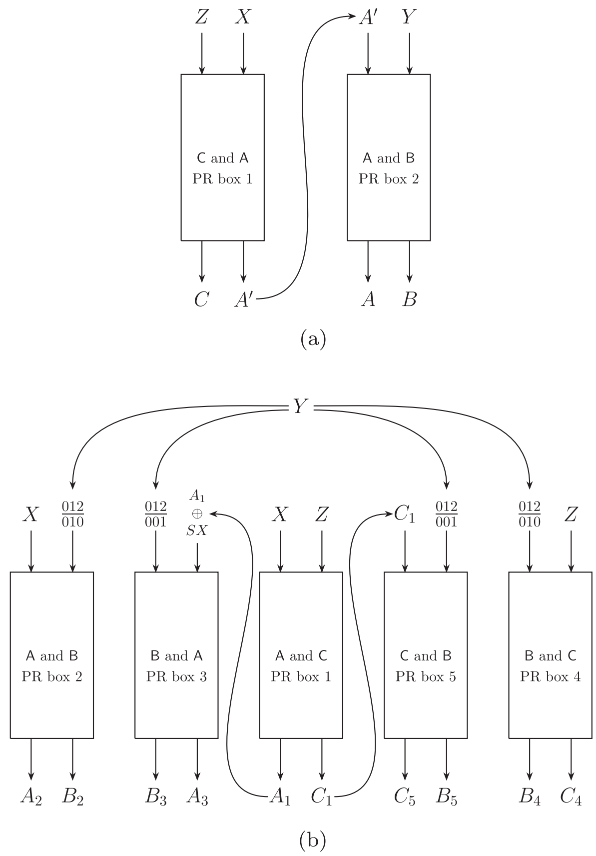


FIG. 3. Each rectangle denotes a PR box. The PR box behavior is the unique bipartite behavior satisfying $A \oplus B = XY$ and $P(A|X) = P(B|Y) = 1/2$ for binary inputs X, Y (denoted with inflowing arrows above) and outputs A, B (outflowing arrows). Text inside the box, for example, in the leftmost box of part (a), "C and A" means that the box is shared by Charlie and Alice, with input Z and output C of Charlie on the left-hand part of the box and input X and output A' of Alice on the right-hand part of the box. (a) A network of PR boxes spoofing entangled measurements by witnessing the claims of Theorem 2. (b) A network of PR boxes spoofing entangled measurements by witnessing the claims of Theorem 3. Alice's outcome is $A = A_2 \oplus A_3$, Charlie's outcome is $C = C_4 \oplus C_5$, and Bob's outcome is $(B_A, B_C) = (B_2 \oplus B_3, B_4 \oplus B_5)$ for $Y \in \{0, 1\}$ and $(B_A, B_C) = (S, B_2 \oplus B_3 \oplus B_4 \oplus B_5)$ for $Y = 2$. S is a fully random binary bit shared by Alice and Bob, which can be implemented with a sixth PR box with constant inputs 0. The fraction $\frac{012}{010}$ above PR box 2 means that if Bob receives $Y = 0$ (resp., 1 and 2), he inputs 0 (resp., 1 and 0) in that box. A similar convention holds for the other boxes.

four outcomes modeled as a binary pair $B = (B_A, B_C)$. When $Y \in \{0, 1\}$, two subgames are won if $A \oplus B_A = XY$ and $C \oplus B_C = ZY$; these are two parallel CHSH games played by Alice-Bob and Bob-Charlie. When $Y = 2$, the winning condition is $A \oplus C = XZ \oplus (XB_A \oplus B_C)$; this constitutes four variants of an Alice-Charlie CHSH game, corresponding to each potential value of B. As argued in Ref. [19], a strategy utilizing bipartite Bell states and a Bell basis measurement for $Y = 2$ can win all the CHSH games to the quantum maximum [$\cos^2(\pi/8) \approx 85\%$], whereas no strategy without an entangled measurement can do so even if tripartite entangled states are available. However, a network of PR boxes as in Fig. 3(b) fulfills the following theorem.

Theorem 3.—The Bell game of Rabello *et al.* [19] described above can be won with probability 1 by a behavior in NS_2.

The results of this Letter provide a minimally complex approach to witnessing entangled measurements, situated in the wider context of classifying different notions of genuine multipartite nonlocality. The techniques of Theorem 1 may also be useful in other paradigms: for example, in the triangle network *without* global shared randomness, the behavior of Ref. [39] was conjectured to require entangled measurements but was only recently proven to do so [40]. And the results of Theorems 2 and 3 indicate that claims of nonsimulability by PR boxes for behaviors invoking entangled measurements (such as is suggested for the behavior in Ref. [39], but this remains unproven) must be carefully evaluated. Whether any tripartite behaviors exist in region $\mathcal{R}_3$ of Fig. 2 remains an open question with important implications for a theory-independent understanding of entangled measurements.

The authors acknowledge Elie Wolfe for stimulating conversations leading to consideration of the (3,2,2) behavior studied here, and for helpful comments on the manuscript. This work was partially supported by NSF Award No. 2210399 and AFOSR Award No. FA9550-20-1-0067.

[*]plbierho@uno.edu
[†]jprakash@uno.edu

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics **1**, 195 (1964).

[2] B. Hensen *et al.*, Loophole-free Bell inequality violation using electron spins separated by 1.3 km, Nature (London) **526**, 682 (2015).

[3] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne *et al.*, Strong Loophole-Free Test of Local Realism, Phys. Rev. Lett. **115**, 250402 (2015).

[4] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer *et al.*, Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons, Phys. Rev. Lett. **115**, 250401 (2015).

[5] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, Event-Ready Bell Test Using Entangled Atoms Simultaneously Closing Detection and Locality Loopholes, Phys. Rev. Lett. **119**, 010402 (2017).

[6] G. Svetlichny, Distinguishing three-body from two-body nonseparability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[7] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Definitions of multipartite nonlocality, Phys. Rev. A **88**, 014102 (2013).

[8] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational Framework for Nonlocality, Phys. Rev. Lett. **109**, 070401 (2012).

[9] P. Contreras-Tejada, C. Palazuelos, and J. I. de Vicente, Genuine Multipartite Nonlocality is Intrinsic to Quantum Networks, Phys. Rev. Lett. **126**, 040501 (2021).

[10] D. Schmid, T. C. Fraser, R. Kunjwal, A. B. Sainz, E. Wolfe, and R. W. Spekkens, Understanding the interplay of entanglement and nonlocality: Motivating and developing a new branch of entanglement theory, arXiv:2004.09194v2.

[11] X. Coiteux-Roy, E. Wolfe, and M.-O. Renou, No Bipartite-Nonlocal Causal Theory Can Explain Nature's Correlations, Phys. Rev. Lett. **127**, 200401 (2021).

[12] J. Clauser, A. Horne, A. Shimony, and R. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[13] P. Bierhorst, Ruling out bipartite nonsignaling nonlocal models for tripartite correlations, Phys. Rev. A **104**, 012210 (2021).

[14] D. Greenberger, M. Horne, and A. Zeilinger, Going beyond Bell's theorem, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kaftos (Kluwer Academic, Dordrecht, 1989), pp. 69–72.

[15] S. Popescu and D. Rohrlich, Quantum nonlocality as an axiom, Found. Phys. **24**, 379 (1994).

[16] Y.-L. Mao, Z.-D. Li, S. Yu, and J. Fan, Test of Genuine Multipartite Nonlocality, Phys. Rev. Lett. **129**, 150401 (2022).

[17] L. Huang, X.-M. Gu, Y.-F. Jiang, D. Wu, B. Bai, M.-C. Chen, Q.-C. Sun, J. Zhang, S. Yu, Q. Zhang, C.-Y. Lu, and J.-W. Pan, Experimental Demonstration of Genuine Tripartite Nonlocality under Strict Locality Conditions, Phys. Rev. Lett. **129**, 060401 (2022).

[18] H. Cao, M.-O. Renou, C. Zhang, G. Massé, X. Coiteux-Roy, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, and E. Wolfe, Experimental Demonstration that No Tripartite-Nonlocal Causal Theory Explains Nature's Correlations, Phys. Rev. Lett. **129**, 150402 (2022).

[19] R. Rabelo, M. Ho, D. Cavalcanti, N. Brunner, and V. Scarani, Device-Independent Certification of Entangled Measurements, Phys. Rev. Lett. **107**, 050502 (2011).

[20] J.-D. Bancal, N. Sangouard, and P. Sekatski, Noise-Resistant Device-Independent Certification of Bell State Measurements, Phys. Rev. Lett. **121**, 250506 (2018).

[21] M.-O. Renou, J. Kaniewski, and N. Brunner, Self-Testing Entangled Measurements in Quantum Networks, Phys. Rev. Lett. **121**, 250507 (2018).

[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.250201 for more information.

[23] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[24] N. Gisin, Entanglement 25 years after quantum teleportation: Testing joint measurements in quantum networks, Entropy **21**, 325 (2019).

[25] E. Wolfe, R. Spekkens, and T. Fritz, The inflation technique for causal inference with latent variables, J. Causal Infer. **7**, 20170020 (2019).

[26] I. Šupić and J. Bowles, Self-testing of quantum systems: A review, Quantum **4**, 337 (2020).

[27] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).

[28] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A **59**, 1070 (1999).

[29] M. Navascués, E. Wolfe, D. Rosset, and A. Pozas-Kerstjens, Genuine Network Multipartite Entanglement, Phys. Rev. Lett. **125**, 240505 (2020).

[30] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Nonlocal correlations as an information-theoretic resource, Phys. Rev. A **71**, 022101 (2005).

[31] B. S. Tsirelson, Some results and problems on quantum Bell-type inequalities, Hadronic J. Suppl. **8**, 329 (1993).

[32] X. Coiteux-Roy, E. Wolfe, and M.-O. Renou, Any physical theory of nature must be boundlessly multipartite nonlocal, Phys. Rev. A **104**, 052207 (2021).

[33] A. Peres, ed., *Quantum Theory: Concepts and Methods* (Springer, Dordrecht, 2002).

[34] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[35] J. Kaniewski, Analytic and Nearly Optimal Self-Testing Bounds for the Clauser-Horne-Shimony-Holt and Mermin Inequalities, Phys. Rev. Lett. **117**, 070402 (2016).

[36] A. J. Short, S. Popescu, and N. Gisin, Entanglement swapping for generalized nonlocal correlations, Phys. Rev. A **73**, 012101 (2006).

[37] J. Barrett, Information processing in generalized probabilistic theories, Phys. Rev. A **75**, 032304 (2007).

[38] L. Masanes, Extremal quantum correlations for N parties with two dichotomic observables per site, arXiv:quant-ph/0512100.

[39] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine Quantum Nonlocality in the Triangle Network, Phys. Rev. Lett. **123**, 140401 (2019).

[40] P. Sekatski, S. Boreiri, and N. Brunner, Partial self-testing and randomness certification in the triangle network, arXiv:2209.09921.