Bulk-Boundary Correspondence and Singularity-Filling in Long-Range Free-Fermion Chains

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The bulk-boundary correspondence relates topologically protected edge modes to bulk topological invariants and is well understood for short-range free-fermion chains. Although case studies have considered long-range Hamiltonians whose couplings decay with a power-law exponent α, there has been no systematic study for a free-fermion symmetry class. We introduce a technique for solving gapped, translationally invariant models in the 1D BDI and AIII symmetry classes with $\alpha > 1$, linking together the quantized winding invariant, bulk topological string-order parameters, and a complete solution of the edge modes. The physics of these chains is elucidated by studying a complex function determined by the couplings of the Hamiltonian: in contrast to the short-range case where edge modes are associated to roots of this function, we find that they are now associated to singularities. A remarkable consequence is that the finite-size splitting of the edge modes depends on the topological winding number, which can be used as a probe of the latter. We furthermore generalize these results by (i) identifying a family of BDI chains with $\alpha < 1$ where our results still hold and (ii) showing that gapless symmetry-protected topological chains can have topological invariants and edge modes when $\alpha - 1$ exceeds the dynamical critical exponent.

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Introduction.—The bulk-boundary correspondence is a central concept in the study of topological phases of matter [1–17]. This relates topologically stable edge effects with topological features of the bulk Hamiltonian. A simple manifestation of this is in certain translation-invariant quantum chains with time-reversal symmetry, where the Hamiltonian on a periodic chain can be used to define a winding number which counts the number of topologically protected Majorana zero modes localized at the edge [1,4,18–21]. Research on this topic has predominantly focused on the short-range case where lattice Hamiltonians couple sites up to some finite range. In the past decade there has been significant interest in quantum systems with long-range interactions [22,23]. This has been motivated by proposals for, and progress in, experimental systems, such as Ref. [24] for effective free-fermion chains. Here long range typically means that couplings decay as a power of the distance [i.e., Hamiltonian terms acting between sites at distance r are $O(r^{-\alpha})$]. Interesting physical effects have been observed including algebraically localized edge modes and the breakdown of the entanglement area law [25] and conformal symmetry at criticality [26].

Regarding topological edge modes in such long-range chains, most results in the literature concern the canonical Kitaev chain [27] with additional long-range hopping or pairing terms [22,28–36]. (For interacting studies, see Refs. [37,38].) The long-range Kitaev chain sits in the BDI symmetry class of free-fermion Hamiltonians [4,8,18,39], and it is straightforward to see that for $\alpha > 1$ the bulk winding number remains well defined [30]. Very recently, Ref. [40] treated the free-fermionic phase diagram in great generality and gave a proof that the short-range phase classification is preserved in the long-range case with $\alpha > d$ (in general dimension and symmetry class). Work on the long-range Kitaev chain showed that topological edge modes exist, but only in particular models. This leaves open important questions for topological Majorana zero modes in long-range chains: when do they exist, what is their connection to the bulk invariant, and what are their localization properties at the edge?

Here, we present the first systematic study of a whole symmetry class, giving rise to a detailed bulk-boundary correspondence in long-range chains. We focus on the exemplary BDI class as mentioned above, although the

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results carry over for the AIII class [41] which famously includes the Su-Schrieffer-Heeger chain [58].

We show that the bulk invariant corresponds exactly to the number of topological edge modes and give a rigorous method to find the edge-mode wave functions. Additionally, we find that the bulk string-order parameters for the short-range case continue to reveal the bulk topology. We complement these results by outlining a principle for calculating the finite-size energy splittings for the zero modes in long-range chains, that we call *singularity-filling*. Together with our analysis of the localization properties of the edge modes, this brings a number of disparate results in the literature into a coherent picture.

The methods we use are from the mathematical theory of Toeplitz determinants (see, e.g., Ref. [59]), a key technique in the analysis of the two-dimensional Ising model [60]. We expect this approach to long-range chains to be fruitful more generally.

We use the standard notation $g(n) = O(h(n))$ when $g(n) \leq \text{const} \times h(n)$ for n sufficiently large, and $g(n) = \Theta(h(n))$ when $g(n) = O(h(n))$ and $h(n) = O(g(n))$.

Model.—Consider the BDI class of translation-invariant spinless free fermions with time-reversal symmetry:

$$H_{\mathrm{BDI}} = \frac{i}{2} \sum_{m,n \in \text{sites}} t_{m-n} \tilde{\gamma}_n \gamma_m. \tag{1}$$

Here $\gamma_n = c_n + c_n^\dagger$ $[\tilde{\gamma}_n = i(c_n^\dagger - c_n)]$ are the real [imaginary] Majorana fermions constructed from spinless complex fermionic modes c_n on each site. The real coupling coefficients t_n are called α-decaying [40] if $t_n \leq \text{const}(1 + |n|)^{-\alpha}$. Assuming absolute summability of the t_n (implied by $\alpha > 1$) we can solve the closed chain by a Fourier transformation and Bogoliubov rotation [42]. This information is summarized by the continuous complex function:

$$f(z) = \sum_{n=-\infty}^{\infty} t_n z^n, \quad z = e^{ik}, \quad 0 \leq k < 2\pi. \tag{2}$$

The eigenmode with momentum k is defined by the phase of $f(e^{ik})$ and has energy $\varepsilon_k = |f(e^{ik})|$. Thus, the Hamiltonian (1) is gapped when $f(z) \neq 0$ on the unit circle. In that case, the argument of $f(z)$ is well defined, and we have the winding number

$$\omega = \lim_{\varepsilon \to 0} \left\{ \arg[f(e^{i(2\pi - \varepsilon)})] - \arg[f(e^{i\varepsilon})] \right\} \in \mathbb{Z}. \tag{3}$$

This is the bulk topological invariant, which cannot change without a gap closing if we enforce the absolute-summability condition.

Bulk-boundary correspondence and edge-mode wave function.—We now consider the Hamiltonian (1) with open boundary conditions (we keep only the couplings that do not cross the boundary). We first consider the limit of a half-infinite chain, where edge modes have zero energy (later we study finite-size splitting).

In this limit, the edge-mode wave functions are zero eigenvectors of a Toeplitz operator, which can be solved using the Wiener-Hopf method. More directly, define a real Majorana zero mode as $\gamma_L = \sum_{n=0}^{\infty} g_n \gamma_n$ that satisfies $[\gamma_L, H_{\mathrm{BDI}}] = 0$. Evaluating the commutator gives us a Wiener-Hopf sum equation, which is straightforwardly solved [61] using results of McCoy and Wu [60], leading to the following result.

Theorem 1 (Bulk-boundary correspondence).—Take a half-infinite open chain H_{BDI}, where the related bulk Hamiltonian has winding number ω and absolutely summable couplings, then there exist exactly $|\omega|$ zero-energy edge modes.

More constructively, writing $f(z) = z^\omega b_+(z) b_-(z)$ [here $b_\pm(z)$ are the Wiener-Hopf factors defined below], then for $\omega > 0$ we have ω linearly independent normalizable real edge modes given by $\gamma_L^{(m)} = \sum_{n=0}^{\infty} g_n^{(m)} \gamma_n$ with $g_n^{(m)} = (b_-(1/z)^{-1})_{n-m}$ for $0 \leq m \leq \omega - 1$.

For $\omega < 0$ the same results hold upon substituting $\gamma_n \to \tilde{\gamma}_n$ and $b_-(1/z)^{-1} \to b_+(z)^{-1}$.

Here and throughout we use the notation that $(h(z))_n = (2\pi i)^{-1} \int_{S^1} h(z) z^{-(n+1)} dz$ is the nth Fourier coefficient of a function $h(z)$. Key to our result is a canonical form called the Wiener-Hopf decomposition. First define $f_0(z) = z^{-\omega} f(z)$, which is nonvanishing on the unit circle and has a continuous logarithm $\log(f_0)(z)$. We fix the normalization of H_{BDI} such that the zeroth Fourier coefficient $(\log(f_0))_0 = 0$. Then we can always write

$$f(z) = z^\omega b_+(z) b_-(z), \tag{4}$$

where the Wiener-Hopf factor given by $b_\pm(z) = e^{\sum_{n=1}^{\infty} (\log(f_0))_{\pm n} z^{\pm n}}$ is analytic strictly inside (outside) the unit disk. We note that z^ω encodes the winding around the unit circle and hence the topological invariant of the system. Multiplying $f(z)$ by z^m shifts [62] the hopping $t_n \to t_{n-m}$, such that $f_0(z)$ defines a topologically trivial "version" of the system. This is analogous to the trivial insulator and the Kitaev chain being related by a shift.

Theorem 1 extends the bulk-boundary correspondence from the short-range to the long-range case: the bulk winding number counts edge modes everywhere in the space of Hamiltonians with absolutely summable couplings [($\alpha > 1$)-decay implies absolute summability, but examples like the Weierstrass function [65,66] can be used to construct families with $0 < \alpha \leq 1$]. Our result is also constructive: we have the edge-mode wave function in terms of Fourier coefficients of a particular function. To construct the exact edge mode, one needs to first calculate the Wiener-Hopf decomposition. However, we will see below that this can often be bypassed if one is interested only in the asymptotic edge-mode profile.

In short-range models we expect exponentially localized edge modes, corresponding to roots of $f(z)$ [21,42,67]. Based on Theorem 1 we see that the localization follows from analytic properties of the Wiener-Hopf factors. If $(b_\pm(z^{\pm 1}))^{-1}$ is analytic to some distance outside the unit circle, we will see exponential decay (this appears consistent with previous such observations in the long-range Kitaev chain at fine-tuned points [33]). Exponential localization was also observed in Ref. [38], but for a different reason—there the short-range (parity-odd) edge modes cannot couple to the long-range density-density interactions in perturbation theory due to fermion parity symmetry. In our long-range case, the edge modes are generically algebraically decaying and guaranteed to be normalizable due to the Wiener-Lévy theorem [60,65].

Example.—Consider $f(z) = z^\omega \mathrm{Li}_\alpha(z)\mathrm{Li}_\alpha(1/z)$, where $\mathrm{Li}_\alpha(z) = \sum_{k=1}^\infty z^k/k^\alpha$ is the polylogarithm of order $\alpha > 1$. The couplings t_n are α-decaying and, moreover, $t_n = \Theta(n^{-\alpha})$ for $n \to \pm\infty$.

One can read off $b_+(z) = \mathrm{Li}_\alpha(z)/z$ and $b_-(z) = z\mathrm{Li}_\alpha(1/z)$. Suppose $\omega = 1$, then we have one edge mode with

$$g_n = \frac{1}{2\pi i}\int_{S^1} \frac{z^{-n}}{\mathrm{Li}_\alpha(z)} dz = -\frac{1}{\zeta(\alpha)^2 n^\alpha}(1 + o(1)); \quad (5)$$

the second equality is derived using contour integration and known asymptotics for $\mathrm{Li}_\alpha(z)$ on the real line (assuming $\alpha \notin \mathbb{N}$) [42,68].

For $\omega = 2$, we see we have two edge modes, with the same leading order behavior. This means we can take the difference $n^{-\alpha} - (n-1)^{-\alpha} = \Theta(n^{-\alpha-1})$, and have a faster decaying strictly localized mode (see Theorem 2).

Singularity-filling for wave functions.—While the bulk-boundary correspondence of Theorem 1 is our most general result, we can give additional results in a broad class of $(\alpha > 1)$-decaying models. We say that $1/f(1/z)$ has singularities at $\{k_s\}_{1\le s\le r}$ if it has asymptotic Fourier coefficients $(1/f(1/z))_n = \sum_{s=1}^r e^{ink_s} n^{-\Omega_{k_s}}(a_s + o(1))$ as $n \to +\infty$. We call $\Omega_{k_s} > 1$ the order of the singularity at k_s, and assume the $o(1)$ term is "nice"; i.e., it can be expressed as a sum of inverse powers of n [as is the case in Eq. (5)]. We also assume that $\Omega_{\min} = \min_s\{\Omega_{k_s}\} \notin \mathbb{Z}$. This implies that $1/f(1/z)$ has $\delta_0 = \lfloor \Omega_{\min} - 1 \rfloor$ continuous derivatives [42,66,69].

Theorem 2 (Edge mode from singularity-filling).—Consider the setup as in Theorem 1 with $\omega > 0$, and suppose in addition that $1/f(1/z)$ has singularities as defined above. Define $\nu_1, \ldots, \nu_\omega$ by the ω lowest levels $\mathcal{E}_s(n) = \Omega_{k_s} + n$ over all singularities s and $n \in \mathbb{Z}_{\ge 0}$ ("singularity-filling") and define $\nu_\star = \delta_0 + \Omega_{\min} - 1$.

We can find a basis of mutually anticommuting edge modes $\hat{\gamma}_L^{(p)} = \sum_{n=0}^\infty \hat{g}_n^{(p)} \gamma_n$, where $\hat{g}_n^{(p)} = O(n^{-\tilde{\nu}_p})$, for $\tilde{\nu}_p = \min\{\nu_p, \nu_\star\}$.

For $\omega < 0$ analogous results hold where we now take $\gamma_n \to \tilde{\gamma}_n$ and $f(1/z) \to f(z)$.

The idea of the proof is as in the $\omega = 2$ example following Eq. (5): we take linear combinations of edge modes that cancel the dominant asymptotic term(s), and then use the Gram-Schmidt process (with respect to the anticommutator) to construct anticommuting modes [21,42]. We note that if the Fourier coefficients of the Wiener-Hopf factors themselves have a nice expansion, then singularity-filling will hold with no limiting $\nu_\star$ [42].

Example.—The long-range Kitaev chain corresponds to

$$f_{\mathrm{LRK}}(z) = \mu + J[\mathrm{Li}_\alpha(z) + \mathrm{Li}_\alpha(1/z)] + \Delta[\mathrm{Li}_\beta(z) - \mathrm{Li}_\beta(1/z)]. \quad (6)$$

This model was studied for various choices of couplings in Refs. [22,29,31–33,36]. Computing $(1/f(1/z))_n$ gives the asymptotic behavior of the edge-mode wave function in the $\omega = 1$ case: $g_n = O(n^{-\Omega_0})$ for $\Omega_0 = \min(\alpha, \beta)$, agreeing with results in the literature [42]. There are no other singularities, so Theorem 2 implies that, for $0 < \delta\omega < \lfloor \Omega_0 - 2 \rfloor$, $f(z) = z^{\delta\omega} f_{\mathrm{LRK}}(z)$ will have $\omega = 1 + \delta\omega$ edge modes with a basis decaying as $n^{-\Omega_0}, n^{-(\Omega_0+1)}, \ldots, n^{-(\Omega_0+\delta\omega)}$.

Singularity-filling for finite-size splitting.—We now consider finite-size energy splittings for the edge modes. This quantity was considered in previous case studies of long-range Kitaev chains [28,33,36], but has not, to our knowledge, been explored in long-range systems with multiple edge modes (i.e., $|\omega| > 1$).

In analogy with the singularity-filling for edge-mode wave functions above, we have a conjecture for the finite-size splittings for the edge modes. In this case, the levels associated to singularities go up in steps of two.

Conjecture 1 (Splitting from singularity-filling).—Take an open chain H_{BDI} of size L, where the related bulk Hamiltonian has winding number $\omega > 0$ and $1/f(1/z)$ has singularities as defined above.

We conjecture that the ω finite-size edge modes have splittings $\varepsilon_1 = \Theta(L^{-\nu_1}), \ldots, \varepsilon_\omega = \Theta(L^{-\nu_\omega})$ where the ν_k are the ω lowest levels $\mathcal{E}'_s(n) = \Omega_{k_s} + 2n$ for $n \in \mathbb{Z}_{\ge 0}$.

For $\omega < 0$ analogous results hold where we replace $f(1/z) \to f(z)$.

This conjecture is based on numerical experiments (see Fig. 1) and theoretical results (see below). The underlying theory indicates that for a family $f(z) = z^\omega f_0(z)$, there may exist an $\omega_{\max}$ such that this holds only for $\omega < \omega_{\max}$. In fact, given $\Omega_{\min} > 5$, and an assumption on the spectrum, we can prove the conjecture up to $\omega_{\max} = 3$. However,

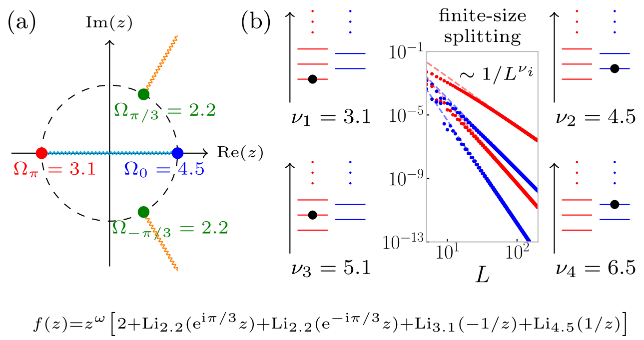

FIG. 1. Finite-size splitting from singularities. (a) As an example of our general results, we consider a long-range chain whose hopping coefficients define the complex function $f(z)$ [Eq. (2)] with singularities of $f(1/z)^{-1}$ depicted. According to Conjecture 1, the power-law exponents associated to these singularities dictate the finite-size energy splitting of the $|\omega|$ Majorana edge modes. (b) We illustrate this for $\omega = 4$, where we show the numerically obtained splittings for system size L. Their power-law decays $\sim 1/L^{\nu_i}$ are accurately predicted by the "singularity-filling" of Conjecture 1. For $\omega > 0$ the singularities associated to branch cuts inside the unit disk matter [i.e., $\Omega_0 = 4.5$ (blue) and $\Omega_\pi = 3.1$ (red)]; for $\omega < 0$ this is reversed [42].

empirically we expect the conjecture to hold more generally, as observed in Fig. 1.

The conjecture allows us to understand how finite-size effects hybridize the edge modes. For $\omega = 1$ we see that the predicted splitting comes from the dominant singularity $\varepsilon_1 = \Theta(L^{-\Omega_{\min}})$. Since this has the same asymptotics as the edge-mode wave function, this agrees with an intuitive connection between the spatial profile of the wave function and the induced splitting from the boundaries [42] that does not generically hold for the higher-winding case. For $\omega = 2$ we expect to have two edge modes, one with $\varepsilon_1 = \Theta(L^{-\Omega_{\min}})$ and one with either $\varepsilon_2 = \Theta(L^{-(\Omega_{\min}+2)})$ or $\varepsilon_2 = \Theta(L^{-\Omega_{\text{next}}})$, depending on which has the slower decay. In the case of higher winding numbers, our conjecture predicts the hybridization of the boundary modes, which is not in direct correspondence to the maximally localized basis identified in Theorem 2.

We can also make quantitative predictions without detailed calculation. Suppose we know for $\omega = 1$ that we have an edge mode with splitting $\Theta(L^{-\nu})$, then for $\omega = 2$ we infer that the second edge mode will have splitting $\Theta(L^{-\nu'})$ where $\nu \leq \nu' \leq \nu + 2$. For $f(z) = z^n f_{\text{LRK}}(z)$ we have a singularity at $z = 1$ only, and hence conjecture that splittings form a sequence $L^{-\Omega_0}, L^{-(\Omega_0+2)}, \ldots, L^{-(\Omega_0+2n)}$.

To justify the conjecture, consider models $f(z) = z^\omega f_0(z)$ with open boundary conditions; each such model has a corresponding single-particle (block Toeplitz) matrix, with determinant equal to $\prod_{j=1}^{L}(-\varepsilon_j^2)$, where ε_j are single-particle energies. Assuming $(\alpha > 1)$-decay, it can be shown, using Toeplitz determinants, that for the trivial model $f_0(z)$ this product is finite in the limit $L \to \infty$, while for $\omega \neq 0$, the corresponding determinant decays to zero with L [with power depending on ω and Fourier coefficients of $1/f(z)$ [42]]. Our method is to use the scaling of this determinant to predict the edge-mode splitting. For example, for $\omega = 1$ we interpret

$$\prod_{j=1}^{L}(-\varepsilon_j^2) = \text{const} \times L^{-\nu}(1 + o(1)), \tag{7}$$

as predicting a single edge mode with finite-size splitting $\varepsilon_1 = \Theta(L^{-\nu})$. For multiple edge modes (and $\omega > 0$), we further assume inductively that the $\omega - 1$ edge modes shared between the models $z^\omega f_0(z)$ and $z^{\omega-1} f_0(z)$ have the same energy splitting power law in each model, and hence the additional decay in the determinant for $z^\omega f_0(z)$ comes from the ωth edge mode [70].

This is plausible since for periodic boundaries the models defined by $f(z)$ have spectrum independent of ω, and we expect the system with open boundaries to differ from the bulk only "near the edge." With finite-range interactions we believe this could be proved using results about eigenvalues of banded block Toeplitz matrices [71]; for long-range chains we take it as an assumption that the scaling to zero with L comes only from edge modes rather than the bulk band. In an earlier work the idea appeared in reverse: utilizing the existence of exponentially localized edge modes in short-range chains to predict asymptotics of block Toeplitz determinants [72].

We thus convert the question of finite-size edge-mode splitting to a question about asymptotics of Toeplitz determinants. While there are several assumptions required to connect this theory to the edge mode splittings, the underlying singularity-filling picture for Toeplitz determinant asymptotics is in many cases fully rigorous. We outline some of these results in the Supplemental Material [42]; see Refs. [59,69,73,74] for important information.

Novel topological probe.—A remarkable consequence is that the finite-size splitting of the lowest energy mode depends on the total number of edge modes. In fact, we can turn this into a probe of ω: by perturbing a short-range chain $f_s(z)$ (with winding ω) by a long-range test function, its finite-size splitting exponent will allow us to find ω (note that this is the scaling of the lowest one-particle energy; no further information about the spectrum is required). An example test function would be $f_{\text{LRK}}(z)$, with $\Delta = 0$. Then for the function $f(z) = f_s(z) + \epsilon f_{\text{LRK}}(z)$, for ϵ small, our picture gives a finite-size splitting $L^{-[\alpha+2(|\omega|-1)]}$.

String-order parameters.—We now consider the periodic chain. Define the finite fermion parity string by $\mathcal{O}_0(n) = \prod_{m=1}^{n-1} i\tilde{\gamma}_m\gamma_m$. Then consider further string operators, $\mathcal{O}_\kappa(n)$, of the form $\mathcal{O}_0(n)\gamma_n\gamma_{n+1}\cdots\gamma_{n+\kappa}$ for $\kappa > 0$ and $\mathcal{O}_0(n)\tilde{\gamma}_n\cdots\tilde{\gamma}_{n+|\kappa|-1}$ for $\kappa < 0$ (up to phase factors).

It is know that the set of $\mathcal{O}_\kappa(n)$ form order parameters for the gapped phases in the short-range case [75]. In the long-range case we have the following.

Theorem 3 (String order).—Consider a gapped ($\alpha > 1$)-decaying $H_{\rm BDI}$, in the thermodynamic limit with periodic boundaries, and write $f(z)/|f(z)| = z^{\omega} e^{W(z)}$. Then,

$$\lim_{N\to\infty} |\langle \mathcal{O}_\kappa(1)\mathcal{O}_\kappa(N)\rangle| = \delta_{\kappa\omega} e^{\sum_{k\geq 0} k W_k W_{-k}}. \qquad (8)$$

Thus the $\mathcal{O}_\kappa$ act as order parameters in the long-range case. The idea of the proof is as follows: the string-correlation functions $\langle \mathcal{O}_\kappa(1)\mathcal{O}_\kappa(N)\rangle$ are Toeplitz determinants generated by $z^{-\kappa} f(z)/|f(z)|$. The function $f(z)/|f(z)|$ generates the *correlation matrix* of the chain, and it was proved in Ref. [40] that for an α-decaying chain with $\alpha > 1$, the correlation matrix is $(\alpha - \varepsilon)$-decaying for any $\varepsilon > 0$. This is sufficient regularity for us to use the results of Ref. [73] to prove Theorem 3 [42].

Gap closing and edge modes at critical points.—For $H_{\rm BDI}$ with finite-range couplings, topological edge modes can persist at critical points [21,76]. We give some results in this direction for the long-range case.

Suppose we have a gapless bulk mode with dynamical critical exponent $z_{\rm dyn}$. In the continuum limit, the dimension of the long-range term in the action $\delta S \sim \int \tilde{\psi}(x)\psi(y) \times (x-y)^{-\alpha} dt dx dy$ is $(z_{\rm dyn} + 1 - \alpha)$, which is irrelevant for $\alpha > z_{\rm dyn} + 1$. On the lattice, we hence expect that for gapless models of the form $f_{\rm crit}(z) = (z-1)^{z_{\rm dyn}} f_{\rm gap}(z)$ [which has the aforementioned low-energy description if $f_{\rm gap}(z)$ is nonvanishing on the unit circle], the edge modes will be stable as long as $f(z)$ is $(\alpha > z_{\rm dyn} + 1)$-decaying. Indeed, our Theorem 1 can be adapted to show that this $f_{\rm crit}(z)$ has ω localized edge modes where ω is the winding number of $f_{\rm gap}(z)$. This follows from expanding $(z-1)^{z_{\rm dyn}}$ in $f_{\rm crit}(z)$, and interpreting this as a sum of $(z_{\rm dyn} + 1)$ gapped Hamiltonians, all sharing the same ω edge modes as per Theorem 1.

The above functional form can arise by interpolating between topologically distinct gapped Hamiltonians. For instance, between two phases with winding numbers $\omega = 1$ and $\omega = 2$, there will generically be a single gap closing with a linearly dispersing mode if $\alpha > 2$. More precisely, if this occurs at momentum $k = 0$, then $f_{\rm gap}(z) \coloneqq [f(z)/(z-1)]$ should define a gapped model with $\omega = 1$. We can then apply the above discussion to infer the existence of the localized edge mode at criticality. We have confirmed this for an explicit example [42].

Outlook.—We have shown how general analytic methods can be used to establish the bulk-boundary correspondence in a class of long-range chains and give insights into edge-mode localization and finite-size splitting. This included examples with $\alpha < 1$ and certain gapless models.

Key questions remain within this class. What happens in the general case when $\alpha < 1$ and the integer winding classification breaks down? Can we establish general stability results in critical lattice models, and do these coincide with our field-theoretic analysis? We expect extensions of analytic techniques used above to provide further insights. Moreover, it is worth exploring how broadly our results can be generalized, including to other free-fermion classes (beyond BDI and AIII) [4,8,18] and higher-dimensional models.

The extension to long-range multiband cases would be interesting, likely requiring block Toeplitz operators. In the short-range BDI and AIII classes, edge modes were constructed in Ref. [67], where the bulk topological index is the winding of the determinant of a chiral block of the Hamiltonian. Symmetric matrix-Wiener-Hopf factorizations (for the short-range case) have been used to study fermionic zero modes [77,78]; these ideas should be useful also in long-range systems. In Ref. [79] Wiener-Hopf techniques are used to exclude edge modes in long-range bosonic systems.

From the mathematical side, it would be most interesting to find a proof of the singularity-filling conjecture. It would be interesting to see if this picture generalizes beyond the studied cases, perhaps even to interacting models with algebraically decaying edge modes, and whether their finite-size splitting also depends on the value of the topological invariant.

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