Unitarity of Symplectic Fermions in α Vacua with Negative Central Charge

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(Received 6 February 2023; accepted 23 May 2023; published 15 June 2023)

We study the two-dimensional free symplectic fermion theory with antiperiodic boundary condition. This model has negative norm states with a naive inner product. This negative norm problem can be cured by introducing a new inner product. We demonstrate that this new inner product follows from the connection between the path integral formalism and the operator formalism. This model has a negative central charge, $c = -2$, and we clarify how two-dimensional conformal field theory with negative central charge can have a non-negative norm. Furthermore, we introduce α vacua in which the Hamiltonian is seemingly non-Hermitian. In spite of non-Hermiticity, we find that the energy spectrum is real. We also compare a correlation function with respect to the α vacua with that of the de Sitter space.

DOI: 10.1103/PhysRevLett.130.241602

Introduction.—Unitarity, as a primary postulate of quantum theory, has played a central role in producing momentous achievements in physics such as the optical theorem for the S matrix and the Page curve in the black hole [1]. On the other hand, an open quantum system, which features nonunitarity, has been recently spotlighted. There, the physical properties of nonunitary models have been extensively investigated, such as topological phases [2–6] and phase transitions [7–11] of non-Hermitian systems.

In two-dimensional conformal field theory (CFT_2), the central charge is known to be a powerful criterion for unitarity; a CFT_2 with negative central charge has negative norm states and therefore is nonunitary. CFTs with negative central charge have provided fruitful laboratories to understand nonunitary physics.

The symplectic fermion theory, i.e., the anticommuting scalar field theory, has been thoroughly studied as an example of CFT_2 with central charge $c = -2$ [12,13] and as an example of a logarithmic CFT for the case of periodic boundary condition [14]. It was proposed in [15,16] that a new inner product can cure the negative norm states of the symplectic fermion; therefore this model with the new inner product was claimed to be unitary [15,16]. However, the tension between the negative central charge and the absence of the negative norm states has not been explicitly resolved. Recently the three-dimensional free symplectic fermion has attracted great interest as a holographic dual of higher spin gravity in four-dimensional de Sitter space [17–25]. In this de Sitter/CFT context, the new inner product has not been fully used.

In this Letter, we investigate the free symplectic fermion with antiperiodic boundary condition to show that CFT_2 with negative central charge can be unitary. Here, the *unitarity* means that the time evolution preserves the inner product. We review the new inner product that resolves the issue of the negative norm state in the model [15,16]. We demonstrate that this new inner product in the operator formalism follows from the path integral formalism. Therefore, if we define the model in the path integral, this inner product is not an *ad hoc* choice but the unique one for the operator formalism. The symplectic fermion, which is unitary with respect to the new inner product in spite of the negative central charge, is a counterexample of the well-known proposition that a CFT_2 with a negative central charge should have negative norm states. We clarify how the proposition can be avoided. Furthermore, we introduce $sl(2,\mathbb{R})$ invariant α vacua parameterized by an infinite set of real parameters α_n where n is a positive half-integer. In this α vacuum the Hamiltonian is seemingly non-Hermitian with respect to the new inner product so that the energy spectrum is not necessarily real. We find that the energy spectrum is real in spite of the non-Hermiticity. We observe that the two-point function with respect to the naive norm in the α vacua of the symplectic fermion has divergence similar to that in the two-point function of the antipodal points in the α vacua of de Sitter space [26–28].

Model.—We study the two-dimensional free symplectic fermion defined by the action

$$S = \int d^2x \partial_\mu \bar{\psi} \partial^\mu \psi, \tag{1}$$

where $\bar{\psi}(t,\sigma)$ and $\psi(t,\sigma)$ are anticommuting Grassmannian scalar fields [12–16]. In the spatial coordinate $\sigma \in S^1$, we consider the antiboundary condition $\psi(t,\sigma+\ell) = -\psi(t,\sigma)$ and $\bar{\psi}(t,\sigma+\ell) = -\bar{\psi}(t,\sigma)$. The conjugate momenta $\Pi = (\overleftarrow{\delta}L/\overleftarrow{\delta}\dot{\psi}) = \dot{\bar{\psi}}$ and $\bar{\Pi} = (\overrightarrow{\delta}L/\overrightarrow{\delta}\dot{\bar{\psi}}) = \dot{\psi}$ are obtained by the right and left derivative of the Lagrangian with respect to $\dot{\psi}$ and $\dot{\bar{\psi}}$, respectively, for consistency with the Hermitian adjoint. This leads to the symplectic one-form $\pi\dot{\psi} + \dot{\bar{\psi}}\,\bar{\pi}$ in the Legendre transformation to the Hamiltonian, and the canonical anticommutation relation reads

$$\{\psi(\sigma), \Pi(\sigma')\} = i\delta(\sigma-\sigma'), \quad \{\bar{\psi}(\sigma), \bar{\Pi}(\sigma')\} = -i\delta(\sigma-\sigma'). \tag{2}$$

Going over to Euclidean plane $z = e^{[2\pi(\tau - i\sigma)/\ell]}$ with Wick rotation $\tau = it$, we take the mode expansion of ψ and $\bar{\psi}$ as

$$\psi = \frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{1}{n} (b_n z^{-n} - c_{-n} z^n + \bar{b}_n \bar{z}^{-n} - \bar{c}_{-n} \bar{z}^n),$$
$$\bar{\psi} = \frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{1}{n} (-b_{-n} z^n + c_n z^{-n} - \bar{b}_{-n} \bar{z}^n + \bar{c}_n \bar{z}^{-n}), \tag{3}$$

where n runs over the positive half-integers due to the antiperiodic boundary condition. From the canonical anticommutation relation [Eq. (2)], we can obtain the anticommutation relation of the oscillators

$$\{b_n, b_m\} = |n|\delta_{n+m,0}, \qquad \{\bar{b}_n, \bar{b}_m\} = |n|\delta_{n+m,0},$$
$$\{c_n, c_m\} = -|n|\delta_{n+m,0}, \qquad \{\bar{c}_n, \bar{c}_m\} = -|n|\delta_{n+m,0}, \tag{4}$$

where the negative and positive modes serve as the creation and annihilation operators, respectively. For more details on the quantization, refer to the Supplemental Material [29]. Note that the anticommutation relations of c_n and $\bar{c}_n$ have a minus sign compared to those of b_n and $\bar{b}_n$. This minus sign in the anticommutation relation seemingly results in negative norm states. For example, using the anticommutation relation the usual norm of the excited state $c_{-n}|0\rangle$ can be shown to have the sign opposite to that of the vacuum,

$$\langle 0|c_n c_{-n}|0\rangle = -n\langle 0|0\rangle \qquad (n > 0). \tag{5}$$

Such nonunitarity from the negative norm state has been observed in the higher derivative systems [30–36]. In such higher derivative theories, by exchanging the role of the creation and annihilation oscillators, one can retrieve the non-negative norm at the cost of the energy spectrum unbounded from below, which leads to Ostrogradsky instability [33,35]. However for the fermionic oscillators the negative norm cannot be cured by exchanging the role of the creation and the annihilation oscillators.

$\mathcal{J}$ norm and unitarity.—The problem of the negative norm states can be resolved by introducing an operator $\mathcal{J}$ that is the exponentiation of the fermion number operator c_n and $\bar{c}_n$ [15,16]

$$\mathcal{J} \equiv e^{\pi i \sum_n \frac{1}{n}(c_{-n}c_n + \bar{c}_{-n}\bar{c}_n)}, \tag{6}$$

which is Hermitian and unitary, $\mathcal{J}^\dagger = \mathcal{J}$ and $\mathcal{J}^2 = 1$. The operator $\mathcal{J}$ commutes with the oscillators b_n and $\bar{b}_n$ while it anticommutes with the oscillators c_n and $\overline{c_n}$,

$$\mathcal{J} b_n \mathcal{J} = b_n, \qquad \mathcal{J} c_n \mathcal{J} = -c_n. \tag{7}$$

As in supersymmetry, one can define the $\mathcal{J}$ norm by inserting the operator $\mathcal{J}$

$$\langle \cdot \rangle_{\mathcal{J}} \equiv \langle \mathcal{J} \cdot \rangle. \tag{8}$$

Then the $\mathcal{J}$ norm of the excited states has positive $\mathcal{J}$ norm

$$\| c_{-n}|0\rangle \|_{\mathcal{J}} = \langle 0|c_n \mathcal{J} c_{-n}|0\rangle = n \| |0\rangle \|_{\mathcal{J}} \qquad (n > 0). \tag{9}$$

Since the Hermitian adjoint follows the inner product, one has to define a new Hermitian adjoint $\dagger_{\mathcal{J}}$ that is consistent with the new $\mathcal{J}$ norm,

$$\mathcal{O}^{\dagger_{\mathcal{J}}} \equiv \mathcal{J} \mathcal{O}^\dagger \mathcal{J}. \tag{10}$$

For the new $\mathcal{J}$-Hermitian adjoint $\dagger_{\mathcal{J}}$ we find it convenient to introduce a double-bracket notation. Namely, while a ket state with the double bracket is the same as the usual ket state, a bra state with a double bracket is defined via $\mathcal{J}$-Hermitian adjoint

$$|\Phi\rangle\rangle \equiv \Phi|0\rangle \overset{\dagger_{\mathcal{J}}}{\Rightarrow} \langle\langle \Phi| \equiv \langle 0|\Phi^{\dagger_{\mathcal{J}}}. \tag{11}$$

The inner product of double-bracket states is identical to the $\mathcal{J}$-inner product

$$\langle\langle \Phi|\mathcal{O}|\Psi\rangle\rangle = \langle \Phi|\mathcal{O}|\Psi\rangle_{\mathcal{J}}. \tag{12}$$

Hence double-bracket states also have the non-negative norm. Although the Hamiltonian of the symplectic fermion is Hermitian with the ordinary Hermitian adjoint, the Hermiticity of the Hamiltonian with $\mathcal{J}$-Hermitian adjoint is not straightforward in general, where more details can be found in the Supplemental Material [29]. And those double-bracket bra and ket states correspond to the biorthogonal basis for the non-$\mathcal{J}$-Hermitian Hamiltonian [37,38].

Connection to path integral.—We have introduced the $\mathcal{J}$-inner product to resolve the negative norm state problem. This might seem *ad hoc* to recover non-negative norm by modifying the theory. However, it turns out [36] that the path integral formalism of the symplectic fermion is consistent with the operator formalism with the $\mathcal{J}$ norm rather than the ordinary norm.

To understand the correspondence between the path integral and operator formalisms for the symplectic fermion, we define Fock states $|\{\nu,\mu\}\rangle\!\rangle \equiv (1/\mathcal{N}_{\{\nu,\mu\}}) \prod_{n>0} b_{-n}^{\nu_n} c_{-n}^{\mu_n} \bar{b}_{-n}^{\bar{\nu}_n} \bar{c}_{-n}^{\bar{\mu}_n} |0\rangle$ where n is a positive half-integer and $\nu_n, \mu_n, \bar{\nu}_n, \bar{\mu}_n \in \{0,1\}$. We normalize the state $|\{\nu,\mu\}\rangle\!\rangle$ in the double-bracket notation by choosing suitable normalization constant $\mathcal{N}_{\{\nu,\mu\}}$. Note that $\langle\!\langle\{\nu,\mu\}|$ is different from $\langle\{\nu,\mu\}|$ in general. Hence the identity operator can be expressed as

$$\boldsymbol{I} = \sum_{\{\nu,\mu\}} |\{\nu,\mu\}\rangle\!\rangle\langle\!\langle\{\nu,\mu\}| = \sum_{\{\nu,\mu\}} |\{\nu,\mu\}\rangle\langle\{\nu,\mu\}|\mathcal{J}. \tag{13}$$

In terms of ordinary bra and ket states, the operator $\mathcal{J}$ is inserted in the completeness relation, which makes this expression play the role of the identity operator. We also define the coherent state

$$|\eta,\zeta\rangle\!\rangle \equiv \prod_{n>0} e^{-\frac{1}{n}(\eta_n b_{-n} + \zeta_n c_{-n})} |0\rangle, \tag{14}$$

where η and ζ are complex numbers. Here, we omit the antiholomorphic part for simplicity, but one has to take it into account to connect to the path integral. Similarly one can define $\langle\!\langle\bar{\eta},\bar{\zeta}|$ by using $\mathcal{J}$-Hermitian adjoint. In terms of the coherent state, the identity operator can be expressed as

$$\begin{aligned}\boldsymbol{I} &= \int \prod_{n>0} e^{-\frac{1}{n}(\bar{\eta}_{-n}\eta_n + \bar{\zeta}_{-n}\zeta_n)} d\eta_n d\bar{\eta}_{-n} d\zeta_n d\bar{\zeta}_{-n} |\eta,\zeta\rangle\!\rangle\langle\!\langle\eta,\zeta| \\ &= \int \prod_{n>0} e^{-\frac{1}{n}(\bar{\eta}_{-n}\eta_n + \bar{\zeta}_{-n}\zeta_n)} d\eta_n d\bar{\eta}_{-n} d\zeta_n d\bar{\zeta}_{-n} |\eta,\zeta\rangle\langle\eta,\zeta|\mathcal{J}. \end{aligned} \tag{15}$$

In the coherent state representation of the identity operator, the operator $\mathcal{J}$ is also inserted when we express it in terms of the ordinary coherent state.

To make contact with the path integral, one can insert the completeness relation [Eq. (15)] into transition amplitude $\langle\!\langle\bar{\eta}_f,\bar{\zeta}_f|\eta_i,\zeta_i\rangle\!\rangle$ at each discretized time. The rest procedure is identical to the standard derivation of path integral except for the double-bracket notation, or equivalently we insert the operator $\mathcal{J}$ in the transition amplitude. For example, at finite temperature, one can have

$$\mathrm{Tr}(e^{-\beta H}) = \mathrm{tr}(\mathcal{J} e^{-\beta H}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_E[\psi,\bar{\psi}]}, \tag{16}$$

where the trace Tr runs over the double-bracket states while the trace tr runs over the states with single brackets. Here, $S_E[\psi,\bar{\psi}]$ denotes the Euclidean action for the symplectic fermion [Eq. (1)]. Note that the trace Tr corresponds to that of the biorthogonal basis. More detailed derivation can be found in the Supplemental Material [29]. One may express the identity operator [Eq. (15)] in terms of the ordinary coherent states $|\eta,\zeta\rangle$ and $\langle\eta,\zeta|$ without the $\mathcal{J}$ operator. However in this representation the measure becomes $e^{-(1/n)(\bar{\eta}_{-n}\eta_n - \bar{\zeta}_{-n}\zeta_n)}$. The asymmetry between η and ζ in the measure makes it difficult to repeat the standard derivation.

We have seen that the $\mathcal{J}$ norm follows from the path integral of the symplectic fermion. Therefore, the Fock states have a positive norm and positive energy, which implies the unitarity, at least, of the free theory.

Negative central charge and positive norm.—Let us now discuss the Virasoro symmetry of the symplectic fermion. Using the anticommutation relations [Eq. (4)] of the oscillators, the two-point function of the primary operator $\partial\bar{\psi}$ and $\partial\psi$ of conformal dimension 1 in the double-bracket notation is evaluated to yield

$$\langle\!\langle\partial\bar{\psi}(z)\partial\psi(w)\rangle\!\rangle = \frac{1}{8\pi}\frac{\sqrt{\frac{w}{z}} + \sqrt{\frac{z}{w}}}{(z-w)^2}. \tag{17}$$

Note that the correlation function with respect to the vacuum state with double bracket is identical to that of single bracket because the vacuum state $|0\rangle$ is invariant under the action of $\mathcal{J}$, i.e., $\mathcal{J}|0\rangle = |0\rangle$.

Using the two-point function [Eq. (17)], the operator product expansion of the energy-momentum tensor $T(z) = -4\pi{:}\partial\bar{\psi}\partial\psi$ is

$$T(z)T(w) \sim \frac{(-1)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{18}$$

Then we can read off the central charge $c = -2$ of the symplectic fermion. Note that the $\mathcal{J}$ norm is not essential in obtaining the central charge because the two-point function [Eq. (17)] is blind to the $\mathcal{J}$ operator insertion.

Now we consider the Virasoro generator. The nonzero mode reads

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m>0}(b_{n-m} + c_{n-m})(b_m - c_m) \\ &\quad + \frac{1}{2}\sum_{m>n}(b_{n-m} - c_{n-m})(b_m + c_m) \qquad (n \neq 0). \end{aligned} \tag{19}$$

While the usual Hermitian adjoint of L_n is L_{-n} (i.e., $L_n^\dagger = L_{-n}$), the $\mathcal{J}$-Hermitian adjoint of L_n is different from L_{-n}. On the other hand, the zero mode L_0 is $\mathcal{J}$-Hermitian

$$L_0 = \sum_{m>0}(b_{-m}b_m - c_{-m}c_m) - \frac{1}{8}, \tag{20}$$

where the vacuum energy density is chosen from the point-splitting regularization of the one-point function of the energy-momentum tensor $\langle\!\langle T(z)\rangle\!\rangle = -(1/8z^2)$. Using the anticommutation relations [Eq. (4)] one can explicitly double-check the central charge $c = -2$ from the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{21}$$

This result seemingly contradicts the well-known proposition that CFT_2 with negative central charge has negative norm states because the symplectic fermion does not have any negative norm state in spite of the negative central charge. We find that the standard proof for the proposition has a "loophole" for the case of the symplectic fermion.

The standard proof considers the norm of the state $L_{-n}|h\rangle$ $(n>0)$ where $|h\rangle$ is a primary state with conformal dimension h. Using the Virasoro algebra [Eq. (21)] one can show that the norm of the state $L_{-n}|h\rangle$ has opposite sign to that of $|h\rangle$ for sufficiently large n. However for the symplectic fermion one has to use $\mathcal{J}$-Hermitian adjoint as well as $\mathcal{J}$ norm, or equivalently, double-bracket states. Therefore the correct norm of the state $L_{-n}|h\rangle$ is

$$\langle\langle L_{-n}h|L_{-n}h\rangle\rangle = \langle\langle h|L_{-n}^{\dagger_{\mathcal{J}}}L_{-n}|h\rangle\rangle = \langle\langle h|\mathcal{J}L_n\mathcal{J}L_{-n}|h\rangle\rangle. \quad (22)$$

Since $L_n^{\dagger_{\mathcal{J}}} \neq L_{-n}$ for $n\neq 0$ as we observed above, one cannot use the Virasoro algebra to get the proposition. Therefore we conclude that the symplectic fermion is a "counterexample" for the nonunitarity of CFT_2 with negative central charge.

The unitarity of the symplectic fermion implies that physical quantities will be well-defined. For example, the entanglement entropy should be positive in unitary theory. For the case of negative central charge the entanglement entropy of a subsystem of length a is not proportional to the central charge; but it is given by [39–41]

$$S_{\mathrm{EE}}(a) = \frac{c_{\mathrm{eff}}}{3}\log\left(\frac{a}{\epsilon}\right), \quad (23)$$

where ϵ is the ultraviolet cutoff. The effective central charge denoted by c_{eff} is defined by $c_{\mathrm{eff}} = c - 24\Delta_{\min}$ where $\Delta_{\min}$ is the lowest holomorphic conformal dimension. For the case of the symplectic fermion the vacuum energy density in Eq. (20) implies that the identity operator has the lowest conformal dimension $\Delta_{\min} = -1/8$, and we have $c_{\mathrm{eff}} = 1$. Thus the entanglement entropy is positive as expected for a unitary theory. However the positive effective central charge does not always imply the unitarity. For instance, the Lee-Yang model with $c=-22/5$ has positive effective $c_{\mathrm{eff}} = 2/5$ in spite of the nonunitary.

α vacua.—The $\mathcal{J}$ Hermiticity of L_0 and the unitarity is nontrivial in the alternative mode expansion [Eq. (3)] of the ψ and $\bar{\psi}$. To see this issue, we define Bogoliubov generator $\mathcal{G}_\alpha$ by

$$\mathcal{G}_\alpha \equiv i\sum_{n>0}\frac{\alpha_n}{n}(b_{-n}c_{-n} + b_n c_n), \quad (24)$$

where $\alpha_n \in \mathbb{R}$ $(n = 1/2, 3/2, \ldots)$. Note that $\mathcal{G}_\alpha$ is Hermitian but not $\mathcal{J}$-Hermitian. The adjoint action of $\mathcal{J}$ on $\mathcal{G}_\alpha$ flips the sign of all αs, i.e., $\mathcal{J}\mathcal{G}_\alpha\mathcal{J} = \mathcal{G}_{-\alpha}$. Similarly we can define $\bar{\mathcal{G}}_\alpha$ for the antiholomorphic oscillators, but we omit the antiholomorphic contributions for simplicity, which is parallel to the holomorphic calculation. Since $\mathcal{G}_\alpha$ generates the canonical (Bogoliubov) transformation, one may take the mode expansion of ψ and $\bar{\psi}$ in terms of $\tilde{b}_n \equiv e^{-i\mathcal{G}_\alpha}b_n e^{i\mathcal{G}_\alpha}$ and $\tilde{c}_n \equiv e^{-i\mathcal{G}_\alpha}c_n e^{i\mathcal{G}_\alpha}$ instead of b_n and c_n. In this new mode expansion, L_0 is expressed as

$$L_0 = \sum_{n>0}[\cosh 2\alpha_n(\tilde{b}_{-n}\tilde{b}_n - \tilde{c}_{-n}\tilde{c}_n) + \sinh 2\alpha_n(\tilde{b}_{-n}\tilde{c}_{-n} + \tilde{c}_n\tilde{b}_n)], \quad (25)$$

up to vacuum energy density constant. The oscillators $\tilde{b}_n$ and $\tilde{c}_n$ also depend on αs, and the adjoint action of $\mathcal{J}$ on $\tilde{b}_n$ and $\tilde{c}_n$ flips the sign of α_ns, i.e., $\mathcal{J}\tilde{b}_n^{(\alpha)}\mathcal{J} = \tilde{b}_n^{(-\alpha)}$ and $\mathcal{J}\tilde{c}_n^{(\alpha)}\mathcal{J} = -\tilde{c}_n^{(-\alpha)}$. Hence it is more convenient to define a new operator $\tilde{\mathcal{J}} = \exp[\pi i\sum_n(1/n)\tilde{c}_{-n}^{(\alpha)}\tilde{c}_n^{(\alpha)}]$ instead of the original $\mathcal{J}$ operator.

Repeating the same procedure with $\tilde{\mathcal{J}}$-inner product and $\tilde{\mathcal{J}}$-Hermitian adjoint, we note that L_0 is not $\tilde{\mathcal{J}}$-Hermitian anymore though L_0 is still Hermitian. Therefore the eigenvalue of L_0 is not necessarily real. The non-$\tilde{\mathcal{J}}$-Hermiticity of L_0 can arise because the Bogoliubov transformation generated by $\mathcal{G}_\alpha$ is not $\mathcal{J}$-unitary transformation but a similarity transformation.

Using the bra and ket states with double brackets, the matrix elements of L_0 can be evaluated, and its eigenvalues are identical to the original real eigenvalues from the b_n and c_n oscillators, where more details can be found in the Supplemental Material [29]. A similar phenomenon has been observed in [36] with a quantum mechanical model where a non-$\mathcal{J}$-Hermitian Hamiltonian can have real eigenvalues when there exists a Bogoliubov transformation that makes the Hamiltonian $\mathcal{J}$-Hermitian. And if such a Bogoliubov transformation does not exist, the Hamiltonian develops complex energy spectrum. Since the Eq. (25) expression is obtained by the Bogoliubov transformation of the $\mathcal{J}$-Hermitian operator, it is not surprising to have real eigenvalues that are identical to the original ones.

Going back to the original $\mathcal{J}$ operator and the corresponding $\mathcal{J}$ norm, we consider the α vacuum $|\alpha\rangle\rangle$, which is annihilated by $\tilde{b}_n$ and $\tilde{c}_n$ for $n>0$. The α vacuum is related to the vacuum $|0\rangle$ by the Bogoliubov transformation

$$|\alpha\rangle\rangle = \frac{e^{-i\mathcal{G}_\alpha}}{\sqrt{\mathcal{N}}}|0\rangle = \prod_{n>0}\left(\frac{\cosh\alpha_n + \sinh\alpha_n\frac{1}{n}b_{-n}c_{-n}}{\sqrt{\cosh 2\alpha_n}}\right)|0\rangle, \quad (26)$$

where $\mathcal{N} = \prod_{n>0}\cosh 2\alpha_n$ is the normalization constant. Under the assumption that the vacuum $|0\rangle$ is invariant under the under the action of $\mathcal{J}$, the α vacuum is not, i.e., $\mathcal{J}|\alpha\rangle\rangle = |-\alpha\rangle\rangle$.

The operators $J^0 = \sum_{n>0}(1/2n)(b_{-n}b_n + c_{-n}c_n)$ and $J^\pm = \pm\sum_{n>0}(1/n)b_{\mp n}c_{\pm n}$ form $sl(2,\mathbb{R})$ algebra, and the vacuum $|0\rangle$ is $sl(2,\mathbb{R})$ invariant ground state. Since $\mathcal{G}_\alpha$ commutes with the $sl(2,\mathbb{R})$ generators, the α vacuum is also the $sl(2,\mathbb{R})$ invariant ground state.

Note that the α vacuum is the maximally entangled state of the Fock spaces $\mathcal{H}_b$ and $\mathcal{H}_c$ created by the oscillators b and c, respectively. The usual maximally entangled states obtained by Bogoliubov transformations do not need the nontrivial normalization constant $\mathcal{N}$, i.e., $\mathcal{N}=1$. However, the nontrivial normalization constant is necessary for the symplectic fermion because the Bogoliubov transformation in our case is not a unitary but similarity one. By tracing out the $\mathcal{H}_c$, the reduced density matrix ρ_b of the pure state $|\alpha\rangle\!\rangle$ reads

$$\rho_b = \bigotimes_{n>0}\left(\frac{\cosh^2\alpha_n}{\cosh 2\alpha_n}|0\rangle\langle 0| + \frac{\sinh^2\alpha_n}{\cosh 2\alpha_n} b_{-n}|0\rangle\langle 0|b_n\right). \quad (27)$$

If we choose a specific value of $\alpha_n = e^{-(\beta|n|/2)}$, the reduced density matrix is identical to the thermal density matrix of the fermi oscillator b with temperature β^{-1}, and the α vacuum corresponds to the thermofield dynamics state [42].

We now evaluate the two-point function of the primary operator with respect to the α vacuum. We find that

$$\langle\!\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle\!\rangle_\alpha = \frac{1}{4\pi zw}\sum_{n>0} n\left[\left(\frac{w}{z}\right)^n \frac{\cosh^2\alpha_n}{\cosh 2\alpha_n} - \left(\frac{z}{w}\right)^n \frac{\sinh^2\alpha_n}{\cosh 2\alpha_n}\right]. \quad (28)$$

If all α_n are set to the same value α, the two-point function is independent of α and it reproduces the two-point function [Eq. (17)] with respect to the vacuum state. On the other hand, one can also evaluate the two-point function with the naive norm without $\mathcal{J}$ insertion, which is not consistent with the path integral formulation and leads to negative norm states. With the naive norm, we have the ordinary Hermitian adjoint, and the ket state of α vacuum does not need the nontrivial normalization constant $\mathcal{N}$ as usual. Using the α vacuum [Eq. (26)] with $\mathcal{N}=1$, we obtain

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_\alpha = \frac{1}{4\pi zw}\sum_{n>0} n\Bigg[\left(\frac{w}{z}\right)^n \cosh^2\alpha_n + \left(\frac{z}{w}\right)^n \sinh^2\alpha_n + [(zw)^n + (zw)^{-n}]\sinh\alpha_n\cosh\alpha_n\Bigg]. \quad (29)$$

This result with naive norm contains the power series in zw in addition to that in z/w. This implies that the correlation function could diverge at $z=1/w$ as well as $z=w$. To see this divergence explicitly, we set all α_n parameters to be the same value α, and we obtain

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{\alpha_n=\alpha} = \frac{1}{8\pi}\frac{\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}}{(z-w)^2}\cosh(2\alpha) + \frac{1}{8\pi}\frac{1}{w^2}\frac{\sqrt{zw}+\sqrt{\frac{1}{zw}}}{(z-\frac{1}{w})^2}\sinh(2\alpha). \quad (30)$$

The divergence of the naive two-point function at $zw=1$ has been observed in the two-point function of the free scalar field in the α vacua of de Sitter space [26–28], which is demonstrated in Fig. 1. The two-point function with respect to the α vacuum of the de Sitter space diverges when one point is identical or antipodal to the other in the de Sitter space, where more detailed comparison can be found in the Supplemental Material [29]. The Bunch-Davies vacuum [43] is free of the divergence of the antipodal points, which is analogous to the vacuum $|\alpha=0\rangle = |0\rangle$ in the symplectic fermion with naive norm.

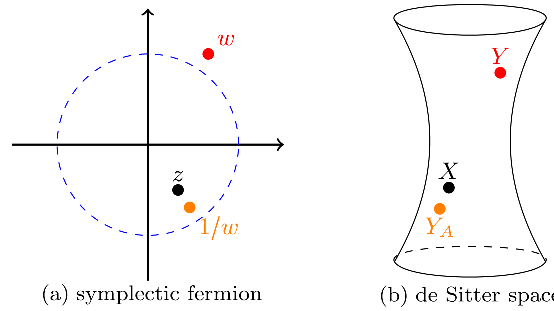

FIG. 1. Two-point function in the α vacuum of symplectic fermion and scalar field in the α vacuum of de Sitter space. (a) With the naive inner product, the two-point function $\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{\alpha_n=\alpha}$ with respect to α vacuum diverges as z approaches to $1/w$. (b) Two-point function of scalar field in the α vacuum of de Sitter space also diverges when X is close to the antipodal point Y_A of Y.

Discussion.—In this Letter, we have explained that the symplectic fermion is a unitary CFT_2 in spite of the negative central charge $c=-2$. The $\mathcal{J}$ norm following from the path integral formulation makes the theory unitary, and the corresponding $\mathcal{J}$-Hermitian adjoint plays a key role in avoiding the well-known proposition on the existence of negative norm states in the CFT_2 with negative central charge. We have also analyzed the $sl(2,\mathbb{R})$ invariant α vacua in which non-$\mathcal{J}$-Hermitian Hamiltonian can retrieve the real energy spectrum. We have compared the two-point function with the naive norm to that of de Sitter space.

The absence of the interaction played an important role in explicit demonstration of the unitary time evolution in the free symplectic model. However, for more general theories such as the Yang-Lee edge singularity [44–48] and the parity-time-symmetric Su-Schrieffer-Heegermodel [49], and in particular interacting ones, the non-negative norm itself would not be sufficient to deduce the unitary time evolution. Nevertheless, our work suggests a possibility of a new class of CFT_2 with negative central charge that could be salvaged from the illusionary nonunitarity by clarification of the inner product. It might be possible to explore a lattice model or an interacting continuum CFT_2 that is unitary but has negative central charge. The divergence of two-point function at the antipodal points in the α vacua of de Sitter space is still an open problem. In the symplectic fermion, the analogous problem is cured by the $\mathcal{J}$-inner product. Thus it is highly interesting to find an alternative inner product in the de Sitter space that might shed light on revisiting the α vacua issue in the de Sitter space.

We thank Changrim Ahn for discussion. S. R. is supported by the National Science Foundation under Grant No. DMR-2001181, and by a Simons Investigator Grant from the Simons Foundation (Grant No. 566116). J. Y. was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (Grants No. 2019R1F1A1045971 and No. 2022R1A2C1003182). This research was supported in part by the International Centre for Theoretical Sciences (ICTS) for the program "Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography" (code: ICTS/numstrings-2022/8). J. Y. is supported by an appointment to the JRG Program at the APCTP through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government. This is also supported by the Korean Local Governments—Gyeongsangbuk-do Province and Pohang City. This work is also supported by Korea Institute for Advanced Study (KIAS) grant funded by the Korean Government, and by the Gordon and Betty Moore Foundation through Grant No. GBMF8685 toward the Princeton theory program.

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