Experimental Test of High-Dimensional Quantum Contextuality Based on Contextuality Concentration

Zheng-Hao Liu,[1,2,*] Hui-Xian Meng,[3,4] Zhen-Peng Xu,[5,6] Jie Zhou,[4,7] Jing-Ling Chen,[4,†] Jin-Shi Xu,[1,2,8,‡] Chuan-Feng Li,[1,2,8,§] Guang-Can Guo,[1,2,8] and Adán Cabello[9,10,∥]

[1]*CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, People's Republic of China*

[2]*CAS Centre For Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China*

[3]*School of Mathematics and Physics, North China Electric Power University, Beijing 102206, People's Republic of China*

[4]*Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China*

[5]*School of Physics and Optoelectronics Engineering, Anhui University, 230601 Hefei, People's Republic of China*

[6]*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany*

[7]*Centre for Quantum Technologies, National University of Singapore, 117543, Singapore*

[8]*Hefei National Laboratory, University of Science and Technology of China, Hefei 230088, People's Republic of China*

[9]*Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain*

[10]*Instituto Carlos I de Física Teórica y Computacional, Universidad de Sevilla, E-41012 Sevilla, Spain*

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Contextuality is a distinctive feature of quantum theory and a fundamental resource for quantum computation. However, existing examples of contextuality in high-dimensional systems lack the necessary robustness required in experiments. Here, we address this problem by identifying a family of noncontextuality inequalities whose maximum quantum violation grows with the dimension of the system. At first glance, this contextuality is the single-system version of multipartite Bell nonlocality taken to an extreme form. What is interesting is that the single-system version achieves the same degree of contextuality but uses a Hilbert space of *lower* dimension. That is, contextuality "concentrates" as the degree of contextuality per dimension increases. We show the practicality of this result by presenting an experimental test of contextuality in a seven-dimensional system. By simulating sequences of quantum ideal measurements with destructive measurements and repreparation in an all-optical setup, we report a violation of 68.7 standard deviations of the simplest case of the noncontextuality inequalities identified. Our results advance the investigation of high-dimensional contextuality, its connection to the Clifford algebra, and its role in quantum computation.

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Introduction.—In quantum theory, measurements cannot be considered as revealing preexisting properties that are independent of other compatible observables measured on the same system. This phenomenon is called contextuality or Kochen-Specker contextuality [1,2]. It constitutes a fundamental resource for some quantum information processing tasks [3,4] and some forms of universal quantum computation such as magic state distillation [5,6] and measurement-based quantum computation [7–9].

However, a fundamental problem is that arguably the most interesting forms of contextuality are experimentally inaccessible as they require high-dimensional quantum systems unavailable within current experimental platforms (for an extended discussion, see [10]). This problem affects extreme forms of contextuality [11,12], interesting temporal correlations [13,14], practical applications of contextuality such as dimension witnessing [15,16], self-testing [17,18], sequential measurements-based machine learning [19], and topologically protected quantum computation [20]. To attack this problem, one way is by looking for new high-dimensional systems [10]. Another complementary approach is to identify forms of contextuality that are much more robust to noise.

The objective of this work is to produce robust contextuality in high-dimensional quantum systems. The strategy we follow is looking for extreme forms of Bell nonlocality, which are multipartite versions of contextuality, and using the graph-theoretical approach to quantum correlations [21] to study single-particle versions of them. We observe that we can preserve the degree of contextuality but use a smaller dimensional quantum system. That is, there is a kind of "concentration" of contextuality in the transition between the multipartite and the single-particle cases. As a consequence, sequential measurements on a high-dimensional indivisible system can lead to quantum correlations whose violations of the corresponding noncontextuality inequalities grow with the system dimension. Moreover, the violations require Hilbert spaces smaller than that of the composite system manifesting the same degree of contextuality. This enhances the contextual correlation's robustness to noise and allows us to experimentally observe contextuality in high-dimensional systems. To demonstrate our findings, we report the experimental results of a path-encoded photonic qudit of $d = 7$ that yields, when quantified by the quantum-classical ratio [12], the highest degree of contextuality ever observed on a single system.

Method.—Bell nonlocality can be seen as a form of contextuality in which the requirement for compatibility is achieved using observables acting on spatially separated subsystems. Therefore, one can trivially convert every violation of a Bell inequality into a violation of a noncontextuality inequality that preserves both the degree of contextuality and the dimension of the Hilbert space [22,23]. A more intriguing approach is to associate every Bell operator with a graph indicating the exclusivity of a set of Bell experiment events [21], that is, to specify which pairs of events are impossible to happen simultaneously. Then, by identifying a contextuality witness that shares the same graph of exclusivity, we can achieve a greater quantum violation and/or employ smaller dimensional systems [24–27].

Our starting point is the observation that the n-qubit Mermin-Ardehali-Belinskii-Klyshko (MABK) Bell inequalities [28–30] have maximum quantum violations that saturate the no-signaling bound and grow exponentially with the number of qubits. Using the graphs of exclusivity of each Bell MABK operator, we identify a family of noncontextuality inequalities that admit a single-particle realization. We then show that the minimal Hilbert space dimension required to achieve its maximal quantum violation is smaller than that needed to achieve the maximal quantum violation of the corresponding MABK inequalities. This phenomenon, hereafter called "contextuality concentration," is not limited to the MABK inequalities but also occurs for the bipartite three-settings Bell inequality [26,31] and, as shown here, for the Bell inequalities for graph states [32–34]. Our emphasis on the MABK inequalities is motivated by their high degree of nonlocality, resistance to noise, and low requirement of critical detection efficiency [35].

Extreme contextuality in high dimensions.—The MABK inequalities for the n-party, two-setting, two-outcome, or $(n,2,2)$ Bell scenarios with $n \geq 3$ odd can be written as follows [28]:

$$\mathcal{M}_n = \langle M_n \rangle \overset{\text{NCHV}}{\leq} 2^{(n-1)/2}, \qquad (1)$$

where $M_n = (1/2i)\sum_{\nu\in\{\pm 1\}} \nu \otimes_{j=1}^{n} [A_1^{(j)} + i\nu A_2^{(j)}]$ and the operators $A_k^{(j)}, k \in \{0,1,2\}$ have eigenvalues ± 1. The superscripts differentiate the index of qubits, $\langle\cdot\rangle$ indicates expectation value, and NCHV means the inequality holds for any noncontextual hidden-variable theory. Its maximal quantum violation, $\mathcal{M}_n = 2^{n-1}$, is achieved by choosing Pauli-like operators $[A_1^{(j)}, A_2^{(j)}] = 2iA_0^{(j)}$ that pairwise anticommute, and using the Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}_n\rangle = (\otimes_{j=1}^{n} |A_+^{(j)}\rangle + i \otimes_{j=1}^{n} |A_-^{(j)}\rangle)/\sqrt{2}$, with $|A_\pm^{(j)}\rangle$ being the ± 1 eigenstate of $A_0^{(j)}$.

To apply the graph-theoretical approach, we rewrite M_n as a linear combination of rank-1 projectors: $M_n = \sum_{k=1}^{2^{2n-2}} \Pi_k - \sum_{k=1}^{2^{2n-2}} \Pi'_k$. The exponent $2n-2$ is due to M_n having 2^{n-1} terms and each term having 2^{n-1} positive (negative) projectors. By keeping only the projectors with positive signs, a witness of contextuality can be expressed in terms of event probabilities. Explicitly, $\mu_n = \sum_{k=1}^{2^{2n-2}} \langle \Pi_k \rangle = \mathcal{M}_n/2 + 2^{n-2}$. Let us call G_n the graph of exclusivity of the events in μ_n; we illustrate the case of $n = 3$ in Fig. 1 and elaborate the procedure in Supplemental Material [36]. According to the graph-theoretical approach, the noncontextual bound and quantum maximum of μ_n are

$$\mu_n \overset{\text{NCHV}}{\leq} \alpha(G_n) = 2^{(n-3)/2} + 2^{n-2} \overset{\text{Q}}{\leq} \vartheta(G_n) = 2^{n-1}, \qquad (2)$$

where $\alpha(G_n)$ and $\vartheta(G_n)$ are the independence and Lovász numbers of G_n, respectively [21]. The first observation is that the gap between noncontextuality and quantum theory is $\vartheta/\alpha = 2 - 2/(1 + 2^{(n-1)/2})$, and thus increases with n.

We now proceed to show that the new graph-theoretical inequality, Eq. (2), is stronger than the MABK inequality, Eq. (1), in the sense that the quantum maximum of μ_n exploits only $2^n - 1$-dimensional Hilbert space—one less than in the n-qubit Bell scenario. The proof is by explicit construction. Let us denote the juxtaposition of the projectors in Eq. (2) as $\mathcal{A} = (\Pi_1 \Pi_2 \cdots \Pi_{2^{2n-2}})$; then,

$$\mathrm{rank}(\mathcal{A}) = 2^n - \dim(\text{solution space of } \mathcal{A}\mathbf{x} = \underbrace{\mathbf{0}\mathbf{0}\cdots\mathbf{0}}_{n}), \qquad (3)$$

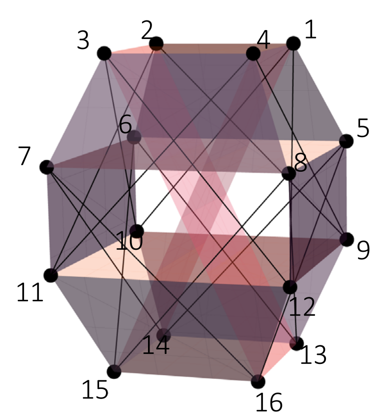

FIG. 1. The graph of exclusivity associated with the events in μ_n for $n = 3$. The points connected by a line represent pairs of mutually exclusive events. The four points on a colored quadrilateral represent four mutually exclusive events.

where $\mathbf{x}$ is a 2^n-dimensional ray. However, the only solution to $\mathcal{A}\mathbf{x} = \mathbf{00}\ldots\mathbf{0}$ is a phase-flipped GHZ state: $A_0^{(1)}|\mathrm{GHZ}_n\rangle = (\otimes_{j=1}^n |A_+^{(j)}\rangle - i \otimes_{j=1}^n |A_-^{(j)}\rangle)/\sqrt{2}$. To check the validity of the solution we observe that, as $A_0^{(1)}$ and $A_1^{(1)}(A_2^{(1)})$ anticommute, an additional $A_0^{(1)}$ in the state will cause every term in Eq. (1) to inverse sign. Therefore, for the phase-flipped GHZ state, $\mathcal{M}_n = -2^{n-1}$. Translating it into the event probabilities, we immediately find that μ_n evaluates to 0. The solution is unique because only the GHZ state maximally violates Eq. (1). Consequently, the projectors in $\mathcal{A}$ span only a $2^n - 1$-dimensional space and can be realized in a quantum system with the same dimension.

The above results show that some forms of multipartite nonlocality can be considered originating from contextuality in lower dimensions and, reciprocally, that some forms of multipartite nonlocality can be "concentrated" into single-particle contextuality with a dimension advantage. In addition, the maximal quantum violation of Eq. (2) can be efficiently obtained by the semidefinite program of Lovász optimization [16]; one possible realization for the $n = 3$ case is given in Supplemental Material [36]. This is in stark contrast with the situation in Bell nonlocality, where the maximal violation is not decidable even with a hierarchy of semidefinite programs [61].

High-dimensional contextuality without inequalities.—Just as nonlocality can be revealed by Hardy- and GHZ-type proofs [62,63] without using inequalities, the same can be done for contextuality [64–66]. However, no constructions of such proof are known for high-dimensional systems.

Here, we report a large class of logical contextuality from the exclusivity structures of the so-called graph states [67,68]. Using Clifford algebra, we prove in Supplemental Material [36] that, if the representation of an n-qubit graph state has an odd number of vertices and at least one universal vertex, the events corresponding to the GHZ-type nonlocality produced by the graph state will induce a graph of exclusivity that can be implemented in a $2^n - 1$-dimensional Hilbert space. Therefore, graph states are ideal candidates for showcasing examples of contextuality concentration. Moreover, our construction here can secure a 100% success probability of observing the Hardy-like events violating noncontextuality, thus paving the way for a robust experimental observation of logical contextuality in high-dimensional systems.

We develop the case $n = 3$, where the only graph state, up to local operations, is the GHZ state. In this case, the graph of exclusivity coincides with the one of the $n = 3$ MABK inequality. Subject to the exclusivity depicted by the graph in Fig. 1, the logical contextuality can be formulated as follows:

$$\frac{\sum_{i=1}^{4} P(1|i) = 1,\ \sum_{i=5}^{8} P(1|i) = 1,\ \sum_{i=9}^{12} P(1|i) = 1,}{1 \overset{\mathrm{Q}}{=} P_{\mathrm{suc}} := \sum_{i=13}^{16} P(1|i) \overset{\mathrm{NCHV}}{=} 0.} \tag{4}$$

Here, $P(1|i)$ denotes the probability that the measurement outcome of the observable Π_i is 1 and P_{suc} is the success probability for observing events forbidden in noncontextuality theories. The proof of Eq. (4) and the settings of projectors achieving the quantum maximum are deferred to Supplemental Material [36].

Experiment.—We present an experimental test of the simplest case of the noncontextuality inequalities in Eq. (2) with measurements on a seven-dimensional quantum system. The system is encoded in the photonic path degree of freedom and our experiment uses the techniques of spatial light modulation [69–71].

The main technical challenge of the experiment is to acquire the statistics of two-point sequential measurements with a photonic seven-dimensional system, which is an open technical problem for dimensions high enough for observing contextuality concentration. To this objective, we have devised a quantum-inspired procedure to realize a nondemolition measurement with destructive measurements followed by a repreparation of the postmeasurement state [72]. The procedure allowed us to emulate a sequence of two ideal measurements (i.e., yielding the same outcome when repeated and not disturbing compatible observables) of two rank-1 projectors Π_i and Π_j with simple prepare-and-measure experiments [73,74]. Concretely, it works as follows: first, perform a destructive measurement of Π_i. If the measurement yields the outcome 1, then prepare the state $|i\rangle$; if it yields the outcome 0, then prepare $|\psi\rangle - \langle i|\psi\rangle|i\rangle$, that is, the initial state with the $+1$ eigenstate of Π_i subtracted. Finally, measure Π_j on the reprepared state. Crucially, the context-independent, repeatable, and minimally disturbing nature of the procedure would allow us to justify the assumption of noncontextuality by resorting to classical physics. Therefore, violation of a noncontextuality inequality via the procedure certifies the experiment itself is indeed manifesting contextuality.

To witness the high-dimensional contextuality, we extracted three kinds of quantities from the results of the prepare-and-measure experiment: (i) the probabilities in Eq. (4) demonstrating the Hardy-type contextuality; (ii) the quality of exclusivity, i.e., $\{P(i,j|1,1)|[\Pi_i,\Pi_j]=0\}$ for establishing a lower bound of μ_3 with realistic measurements and check the measurement repeatability; and (iii) the absence of signaling between compatible measurements for confirming the ideality of the measurement and showing the observed effect was indeed due to contextuality, instead of disturbance. Explicitly, for estimating μ_3 under imperfect exclusivity, we used the fact that [72]

$$\mu_3 \geq \sum_k P(1|k) - \sum_{(i,j)} P(1,1|i,j), \tag{5}$$

where $k \in V(G_3)$ is an index associated with a vertex in G_3 and $(i,j) \in E(G_3)$ is an edge in G_3. The no-signaling condition between pairs of compatible observables Π_i and Π_j can be verified by checking if all the following signaling factors vanish:

$$\begin{aligned} \epsilon_{ij} &= P(1|i) - P(1,\_|i,j), \quad \epsilon'_{ij} = P(1|i) - P(\_,1|j,i), \\ \epsilon_{ji} &= P(1|j) - P(1,\_|j,i), \quad \epsilon'_{ji} = P(1|j) - P(\_,1|i,j). \end{aligned} \tag{6}$$

That is, the marginal probability of one measurement is statistically independent of the other, regardless of the sequence of two measurements performed. Here, $P(1,\_|i,j) = P(1|j)P(1|i=1,j) + P(0|j)P(1|i=0,j)$ denotes the marginal probability of Π_i yielding outcome 1 when Π_j is subsequently measured; similarly definitions hold for the other marginal probabilities.

Our experimental setup is illustrated in Fig. 2(a). Photons from an attenuated 800 nm laser were expanded and the wavefront resembled a Gaussian beam with a waist radius of 1.6 mm. Throughout the experiment, we used the spatial mode degree of freedom of the photons to register the seven-dimensional qudit, where the computational bases $|1\rangle$ through $|7\rangle$ are the angular states localized within a circular sector of the Gaussian beam. To generate these angular states and their superpositions, we casted photons on a spatial light modulator (SLM) that displayed a phase-only hologram of seven circular sectors [cf. Fig. 2(b)]. The hologram in each sector displayed a blazed grating with a different phase range; consequently, a fraction of photons underwent diffraction, resulting in their propagation direction being altered toward the second SLM. By adjusting the maximum phase variation of the grating to control the amplitude of photon wave functions in the seven sectors, we can realize the encoding of arbitrary qudit states [75], with an explicit example shown in Fig. 2(c).

The modulated photons then propagated through a $4f$ correlator, where the unwanted diffraction orders were filtered by a pinhole at the focal plane of the first lens. At the output plane of the correlator, a second SLM implemented the qudit measurement by employing the reverse transformation of the encoding process [76,77]. In this way, choices of the initial states and the measurement settings were realized by displaying different holograms on the first and second SLM, respectively. To check the precision of the setup, we prepared all holograms used in the experiment, measured the spatial wave function profiles directly before the second SLM with a charge-coupled device camera, and compared them with theoretical predictions (cf. Supplemental Material [36]). The results revealed an average Pearson correlation [78] of 95.5% between theoretical and experimental wave function profiles. Finally, a telescope shrank the beam waist, and the photons were collected by a single-mode fiber (SMF) to determine the detection probability of an initial state on a specific measurement basis with photon counting.

Our experimental results are presented in Fig. 3. For the observation of Hardy-like contextuality [Eq. (4)], we

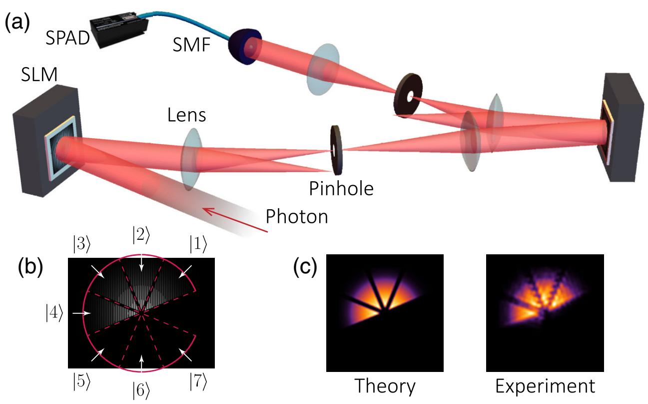

FIG. 2. Experimental setup. (a) Optical setup of the prepare-and-measure experiment. A spatial light modulator (SLM) encoded the initial state by modulating the photonic wavefront. A $4f$ correlator mapped the wavefront to the second SLM, which implemented the measurement. The photons are collected by a single-mode fiber (SMF) and sent to the single-photon avalanche detector (SPAD) for photon counting. (b) A sample hologram showing the encoding scheme. The corresponding state is $|\psi\rangle = (|1\rangle + |2\rangle + |3\rangle + |4\rangle)/2$. (c) An example of calculated versus measured wave function profile. The holograms and their characterization results for all states (projectors) needed in the experiment are shown in Supplemental Material [36].

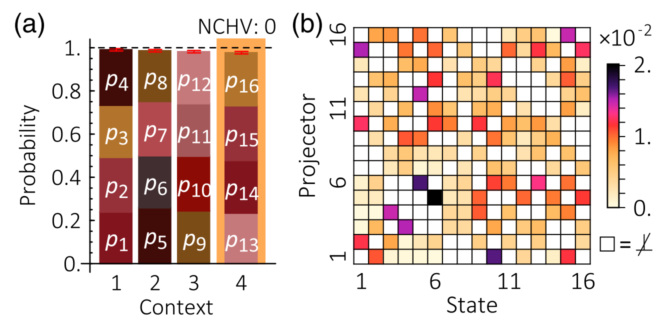

FIG. 3. Experimental results. (a) Stacked bar chart of event probabilities, $p_i = P(1|i)$, with the seven-dimensional photonic system. Given all prerequisites in former columns. The last column with orange background has noncontextual (quantum) predictions of ideally 0(1), therefore manifesting logical contextuality. The error bars denote the 1σ standard deviations calculated by assuming a Poissonian counting statistics. (b) Orthogonality between the projectors measured determined by preparing the nondegenerate eigenstate of every projector and measuring the detection probabilities on their corresponding compatible projectors. Only the grids corresponding to different compatible measurements are colored (the others are white).

displayed the hologram of the initial state on the first SLM and iterated the holograms on the second SLM over all the measurement basis. The total probabilities on the left-hand sides of the three constraints and the final nonclassical event were measured to exceed 97.6%, exhibiting a sharp contradiction with the prediction from noncontextuality. To ensure reasonable exclusivity of the compatible measurements in the experiment, pairs of holograms corresponding to orthogonal projectors were displayed on the two SLMs. The average detection probabilities for these settings were determined to be $\overline{P(1,1|i,j)} = 0.64\%$—almost vanishing as expected for ideal measurements. By substituting the recorded probabilities into Eq. (5) to compensate for the deviations from ideal exclusivity and test the quantitative noncontextuality inequality, we found that $\mu_3 \geq 3.821 \pm 0.012$, violating the prediction of noncontextuality by 68.7 standard deviations. Here, the standard deviations were estimated by assuming a Poisson distribution for the statistics and resampling the recorded data (cf. Supplemental Material [36]).

To verify the no-signaling condition, we prepared the conditional state after the first projector measurement for each edge in the exclusivity graph G_3, and then measured it with the second projector. Owing to the normalization of counting, the effect of a later measurement upon an earlier one, represented by ε_{ij} and ε_{ji}, would always be zero. The result of ε'_{ij} and ε'_{ji} in the no-signaling test for the 72 edges in G_3 is shown in Fig. 4. Most of the signaling factors are within 1 standard deviation from zero, with a quantitative characterization gave $\overline{|\varepsilon'|} = (1.17 \pm 1.39)\%$. The small nonzero values can be considered to originate from experimental imprecision and small drifting over time, and the overall results complied well with the no-signaling requirement.

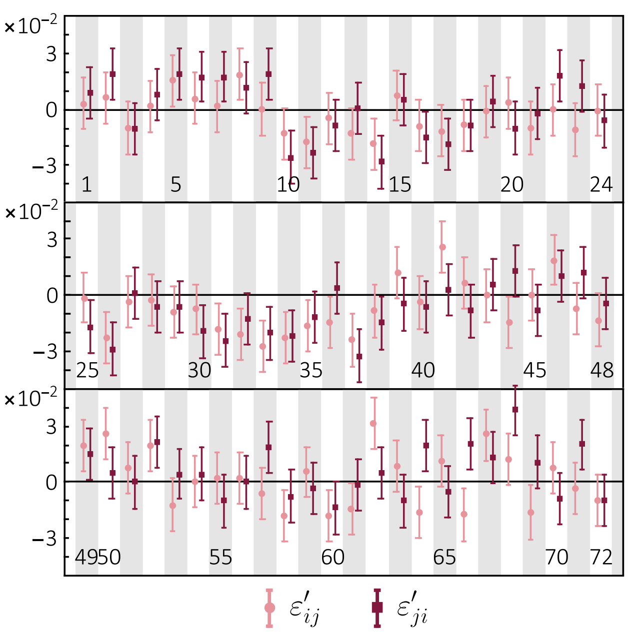

FIG. 4. Verification of no-signaling condition. The numbers stand for indices of edges $(i,j) \in G_3$, sorted by i then j with $i < j$. The data points denote signaling factors and the error bars denote the 1σ standard deviations calculated by assuming a Poissonian counting statistics.

Conclusions.—We have identified quantum correlations resulting from sequential measurements on a single particle that manifest extremely strong forms of contextuality in lower dimensions. These correlations exhibit large violations of noncontextuality inequalities and perfect success probabilities for single-shot detection of contextuality. An accompanying photonic experiment fleshed out the theoretical findings by simulating sequences of ideal quantum measurements with destructive measurements and observed the highest degree of contextuality on a single system (cf. Supplemental Material [36] for a comparison with previous studies). Although the present result only shows a dimension reduction of 1, we envisage that contextuality concentration can be scalable by using graph products, thereby offering additional advantages. Because contextuality in stabilizer subtheory-based exclusivity structures investigated here forms the backbone of quantum computation architectures [5–9,79–81] in many different physical systems [20,82–93], our work may stimulate the development of high-dimensional quantum information processing and novel quantum algorithms.

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Z.-H. L. and H.-X. M. contributed equally to this work.

*Present address: Center for Macroscopic Quantum States (bigQ), Department of Physics, Technical University of Denmark, Fysikvej, 2800 Kgs. Lyngby, Denmark.
†chenjl@nankai.edu.cn
‡jsxu@ustc.edu.cn
§cfli@ustc.edu.cn
‖adan@us.es

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