Quantum Uncertainty Principles for Measurements with Interventions

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Heisenberg's uncertainty principle implies fundamental constraints on what properties of a quantum system we can simultaneously learn. However, it typically assumes that we probe these properties via measurements at a single point in time. In contrast, inferring causal dependencies in complex processes often requires interactive experimentation—multiple rounds of interventions where we adaptively probe the process with different inputs to observe how they affect outputs. Here, we demonstrate universal uncertainty principles for general interactive measurements involving arbitrary rounds of interventions. As a case study, we show that they imply an uncertainty trade-off between measurements compatible with different causal dependencies.

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Introduction.—We learn about physical systems through measurement, and the uncertainty principle fundamentally limits what we can simultaneously learn [1]. Quantum mechanics states the existence of incompatible measurements (e.g., position and momentum of a free particle), such that predicting both outcomes to absolute precision is impossible [2–4]. Subsequent use of information theory then led to various entropic uncertainty relations that quantified uncertainty using entropic measures [5], culminating with universal uncertainty relations that provide general constraints of the joint probabilities of incompatible measurements [6–9].

Yet these relations pertain to only passive measurements, where a system is left to evolve freely before observation [see Fig. 1(a)]. In contrast, the most powerful means of learning involve intervention. When toddlers learn of their environment, they do not merely observe. Instead, they actively intervene—performing various actions, observing resulting reactions, and adapting future actions based on observations. Such "interactive measurements" are essential to fully infer causation, so we may know if one event caused another or if both emerged from some common causes [10]. Indeed, interactive measurements permeate diverse sciences, whether using reinforcement learning to identify optimal strategic behavior or sending data packets to probe the characteristics of a network [11–13]. Such interactive measurement processes also describe many quantum protocols, including quantum illumination, quantum-enhanced agents, and non-Markovian open systems [14–17].

Could uncertainty principles also fundamentally constrain such interactive measurements [see Figs. 1(b)–1(d)]? How would such principle interplay with interventions aimed to discern causal structure? Here, we explore these questions by deriving a universal uncertainty principle that constrains the joint measurement probabilities of interactive measurements. This principle then pinpoints when two interactive measurements are noncompatible—and quantifies the necessary trade-offs in the certainty of their measurement outcomes. Our results make no assumptions on the number of interventions or the causal structure of processes we probe and encompass previous uncertainty relations for states and channels as special cases [20–22]. We apply them to interactive measurements compatible with direct cause vs common cause, showing that they satisfy an uncertainty trade-off analogous to position and momentum.

Framework.—The premise of an interactive measurement consists of an agent that wishes to probe the dynamics of some unknown quantum process Φ. Here Φ can be modeled as an open quantum system, consisting of a system accessible to the agent with $\mathcal{H}$ coupled with some generally non-Markovian environment E [see blue shaded region in Fig. 1(d)]. Initially, the $\mathcal{H}$-E system is in some

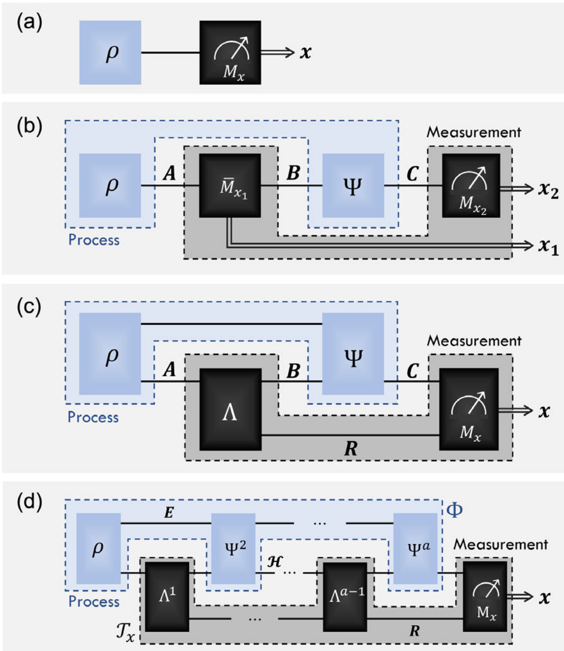

FIG. 1. Interactive measurements. Our uncertainty relations apply to all interactive measurements, including (a) passive measurements (framed by standard uncertainty relations) and (b) two-time measurements, where a quantum system first passes a quantum instrument that incorporates both a measurement outcome and the output state and later gets measured, as described by the framework of a pseudo-density matrix [18,19]. Our results also pertain to (c) non-Markovian interactive measurements that involve coherently interacting the system with a quantum register R and doing some joint measurement at a subsequent time step and, most generally, (d) any interactive measurement $\mathcal{T}_x$ with interventions at $a-1$ different time steps.

joint state ρ. At each time step k, the system and environment jointly evolve under Ψ^k. Φ is then completely defined by the set $\{\Psi^k\}_{k=2}^a$ and the initial state ρ, where a represents the number of time steps. In literature, Φ offers the most general representations of non-Markovian quantum stochastic processes [23] and is also closely related to concepts of higher-order quantum maps, adaptive agents, and causal networks [24–29].

Interactive measurements then represent the most general means for an agent to determine properties of Φ [see black shaded region in Fig. 1(d)]: the agent initializes some internal memory register R; between time steps (i.e., before Ψ^k with $2 \le k \le a$), the agent performs an *intervention*—some general quantum operation Λ^k that interacts the memory R with the accessible system $\mathcal{H}$; after $a-1$ such interventions, the agent finally makes a joint measurement with respect to some positive operator valued measure (POVM) $M \coloneqq \{M_x\}$ on the joint $\mathcal{H}$-R system to obtain some outcome x. Thus, each interactive measurement $\mathcal{T}$ is completely defined by a set of interventions $\{\Lambda^k\}_{k=1}^{a-1}$ and POVM M. Just as a conventional positive operator valued measurement on a quantum state induces some probability distribution over measurement outcomes, so does an interactive measurement on a quantum process. Analogous to eigenstates, we say Φ is an "eigencircuit" of $\mathcal{T}$ if Φ always yields a definite outcome when measured by $\mathcal{T}$.

We make two remarks. (1) The interactive measurements encompass *everything* an agent can possibly do causally. Notably, R can also store classical information; for example, making a projective measurement and conditioning future action on the system based on the result of these measurements. (2) Both Φ and $\mathcal{T}$ have succinct representations using Choi-Jamiołkowski operators, often referred to as quantum combs [25,26] or process tensors [17]. We provide a rigorous mathematical treatment in the Supplemental Material ([30] Secs. I B and I C).

Uncertainty principles.—In conventional quantum theory, certain observables are mutually incompatible. Given an observable $\mathcal{O}$ whose outcome o_k occurs with probability p_k, we can quantify the uncertainty by the Shannon entropy $H(\mathcal{O}) \coloneqq -\sum_k p_k \log p_k$. The entropic uncertainty principle then states that there exists mutually noncompatible observables $\mathcal{O}_1$ and $\mathcal{O}_2$, such that the joint uncertainty $H(\mathcal{O}_1) + H(\mathcal{O}_2)$ is always lower bounded by some state-independent constant $C > 0$ [50–52].

Can we identify similar uncertainty relations for general interactive measurements? We answer this question by employing majorization [53]. Consider two probability vectors $\mathbf{x}$ and $\mathbf{y}$, whose elements x_k and y_k are arranged in nonincreasing order. We say $\mathbf{x}$ is majorized by $\mathbf{y}$, written as $\mathbf{x} \prec \mathbf{y}$, if $\sum_{k=1}^i x_k \le \sum_{k=1}^i y_k$ holds for all index i. The rationale is that majorization maintains significant connections with entropy since $\mathbf{x} \prec \mathbf{y}$ implies that $H(\mathbf{x}) \ge H(\mathbf{y})$. In fact, $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \ge f(\mathbf{y})$ for a large class of functions known as "Schur-concave functions." Such functions align with those that remain nondecreasing under random relabeling of measurement outcomes and have been proposed as the most general class of uncertainty quantifiers [6]. Thus, majorization constraints on outcome probabilities for conventional quantum measurements are referred to as universal uncertainty relations [6–9]. Here, we establish such a universal uncertainty relation for general interactive measurements (see Supplemental Material [30] Sec. II D for the proof).

Lemma 1.—Consider two distinct interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ on some dynamical process Φ, with outcomes described by probability distributions $\mathbf{p}$ and $\mathbf{q}$. There then exists a probability vector $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ such that

$$\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q} \prec \mathbf{v}(\mathcal{T}_1, \mathcal{T}_2). \tag{1}$$

Here the vector-type bound $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ is independent of Φ and hence captures the essential incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$. Meanwhile, $\oplus$ represents the concatenation of vectors. For example, $(1,0) \oplus (1/2,1/2) = (1,0,1/2,1/2)$.

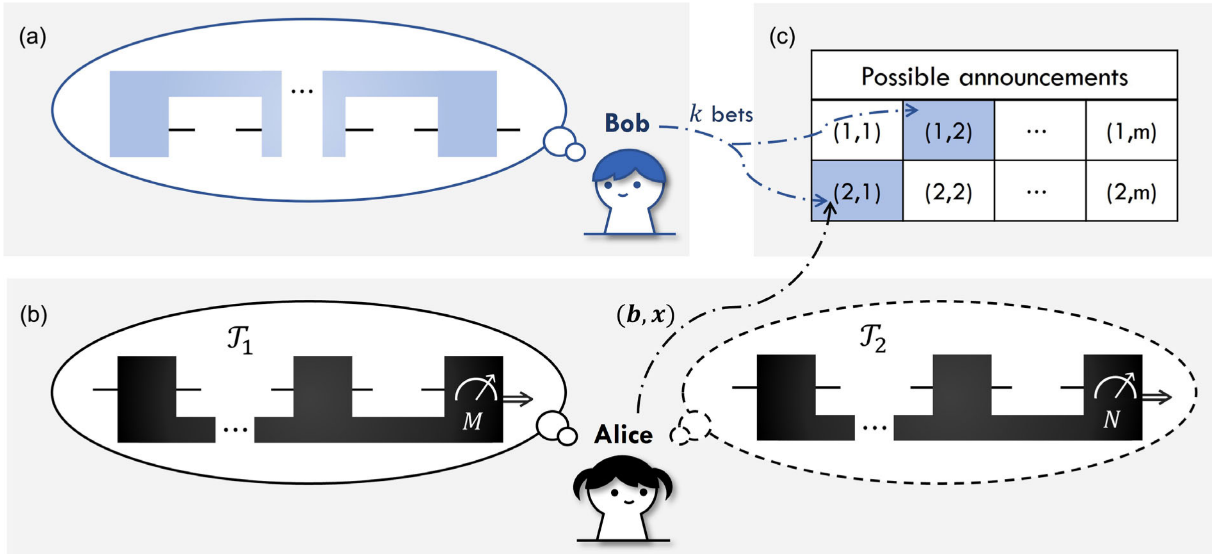

FIG. 2. Quantum roulette is a game that aids in interpreting lower bounds for the combined uncertainty of two general interactive measurements $\{\mathcal{T}_b\}_{b=1,2}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ are shown in (b). Now introduce a quantum "roulette table" with $2 \times m$ grid of cells (c), labeled (b, x) with $x = 1, \ldots, m$. In the kth-order game, Bob begins with k chips, of which he can allocate to k of these cells. Bob then supplies Alice with a dynamical process Φ (a). Alice selects a b at random and measures Φ with $\mathcal{T}_b$ to obtain outcome x. Bob wins if he has a chip on the cell (b, x). Lemma 1 and Theorem 1 then relate Bob's winning probabilities with the incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$.

Our result for interactive measurement is also universal in this sense. In particular, they imply an infinite family of uncertainty relations, namely, $f(\mathbf{p}/2 \oplus \mathbf{q}/2) \geq f(\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2))$ for any Schur-concave function f (including Rényi entropies). Choosing f as the Shannon entropy, Lemma 1 then results in entropic bounds for general interactive measurements (see Ref. [30] Sec. II D for details).

Theorem 1.—Given two interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ acting on some dynamical process Φ, the entropies of their measurement outcomes [54] satisfy

$$H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi \geq C(\mathcal{T}_1, \mathcal{T}_2), \tag{2}$$

where $C(\mathcal{T}_1, \mathcal{T}_2)$—measuring incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$—is non-negative and independent of Φ. $C(\mathcal{T}_1, \mathcal{T}_2)$ can be explicitly computed. It is strictly nonzero whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ have no common eigencircuit.

In the Supplemental Material ([30] Sec. II D), we illustrate a choice of $C(\mathcal{T}_1, \mathcal{T}_2)$ that reduces to $\log(1/c)$ when $\mathcal{T}_1$ and $\mathcal{T}_2$ are standard quantum measurements. Here c stands for the maximal overlap between measurements [5]. Meanwhile, just as there exist many alternative bounds beyond $\log(1/c)$ [55–66], there are many other valid bounds for $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ (see Ref. [30] Sec. II D). Here, we focus on a choice of $C(\mathcal{T}_1, \mathcal{T}_2)$ that can give tighter bounds in causal inference settings. More results are presented in the Supplemental Material ([30] Sec. II C).

Our formulations of $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ and $C(\mathcal{T}_1, \mathcal{T}_2)$ carry direct operational meaning in a guessing game, which we refer to as "quantum roulette." The two-party game consists of (1) Alice, an agent that probes any supplied dynamical process using one of two possible interactive measurements, $\mathcal{T}_1$ and $\mathcal{T}_2$, and (2) Bob, who can engineer various dynamical processes for Alice to probe (see Fig. 2). In each round, Alice and Bob begin with a roulette table, whose layout consists of all tuples (b, x), where $b \in \{1, 2\}$ and x are all possible measurement outcomes of $\mathcal{T}_1$ and $\mathcal{T}_2$. Bob begins with k chips, which he can use to place bets on k of the possible tuples and supplies Alice with any Φ of his choosing. Alice will then select some $b \in \{1, 2\}$ at random and probe Φ with $\mathcal{T}_b$. She finally announces both b and the resulting measurement outcome x. Bob wins if one of his chips is on (b, x).

Let p_k denote Bob's maximum winning probability. Naturally $p_0 = 0$ and p_k increases monotonically with k, tending to 1. We define a probability vector $\mathbf{w}$ with elements $w_k = p_k - p_{k-1}$, $k = 1, 2, \ldots$, representing the increase in Bob's probability of winning with k rather than $k - 1$ chips. In the Supplemental Material ([30] Sec. II D), we show that $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2) \coloneqq \mathbf{w}$ and $C(\mathcal{T}_1, \mathcal{T}_2) \coloneqq 2H(\mathbf{w}) - 2$ are bounds for $\mathbf{p}/2 \oplus \mathbf{q}/2$ and $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$, respectively.

This game gives an operational criterion of noncompatibility for interactive measurements. When two observables are compatible, $H(\mathbf{w}) = 1$. This aligns with the scenario that $\mathbf{w} = (0.5, 0.5, 0, \ldots, 0)$, which occurs when Bob's success rate is limited only by his uncertainty of which measurement Alice makes. That is, placing one counter ensures Bob can correctly predict the outcome of $\mathcal{T}_1$ and two counters gives him perfect prediction regardless of b. We see this is only possible if $\mathcal{T}_1$ and $\mathcal{T}_2$ share at least one

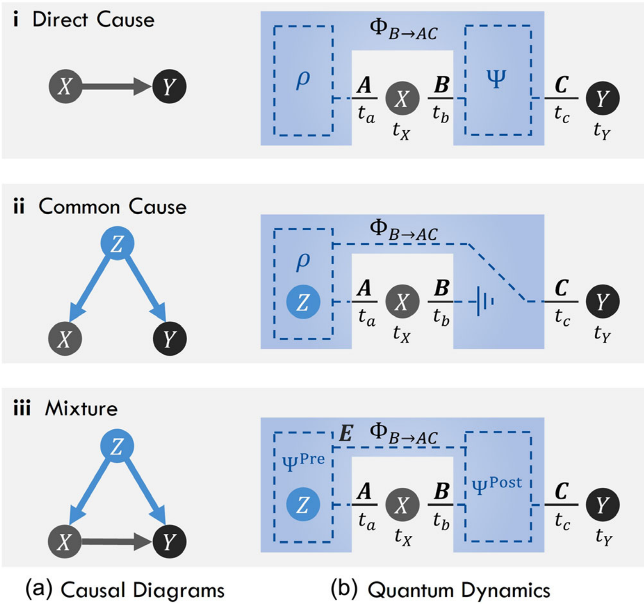

FIG. 3. Quantum description of causal structures. There are three possible causal structures for two events X and Y (a), all of which can be expressed by a quantum dynamic process $\Phi_{B\to AC}$ (b). (i) Direct cause, $\Phi_{B\to AC}$ involves preparing a state A to be observed at X, whose output is sent directly to Y via quantum channel from B to C. (ii) Common cause, correlations between X and Y can be attributed to measurements on some preprepared correlated state ρ_{AC} (event Z). Most generally (iii), $\Phi_{B\to AC}$ consists of a state-preparation process $\Psi^{\rm Pre}_{\mathbb{C}\to AE}$ and a postprocessing quantum channel $\Psi^{\rm Post}_{BE\to C}$ [(b-iii); E is an ancillary system]. This then corresponds to a (possibly coherent) mixture of direct and common cause.

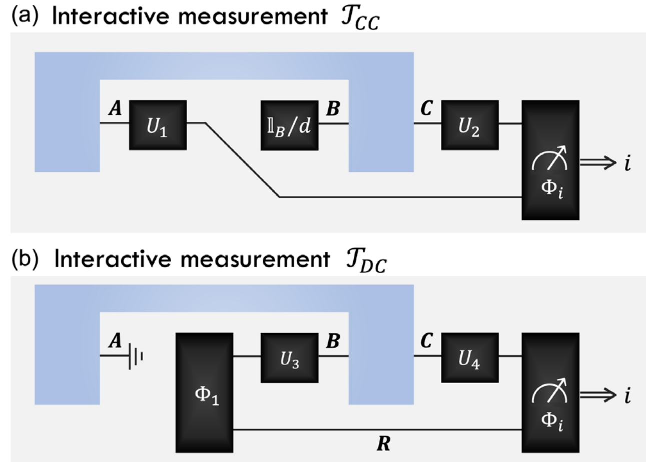

FIG. 4. Maximal common cause (CC) and direct cause (DC) indicators. We introduce (a) $\mathcal{M}_{\rm CC} = \{\mathcal{T}_{\rm CC}(U_1, U_2)\}$ and (b) $\mathcal{M}_{\rm DC} = \{\mathcal{T}_{\rm DC}(U_3, U_4)\}$ as two respective families of interactive measurements with a single intervention. Here, systems A, B, and C are d-level quantum systems (qudits), and each U_k, $k = 1, 2, 3, 4$ is some single-qudit unitary, and $|\Phi_1\rangle \coloneqq \sum_{k=0}^{d-1} |kk\rangle/\sqrt{d}$. Measurements are done with respect to a maximally entangling basis $\{\Phi_i\}_i$ with d^2 possible outcomes. The two measurement families are incompatible and satisfy the causal uncertainty relation in Eq. (3).

common eigencircuit. Thus, $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ is strictly greater than 0 whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ share no common eigencircuit.

Causal uncertainty relations.—The central relevance of interventions in causal inference makes it an appropriate illustrative example [67]. Consider the case where Φ represents a d-level system (the accessible qudit) that evolves while in possible contact with other systems (e.g., a non-Markovian environment E). Now suppose an agent, Alice, can access this qudit at two different points in time, say t_X and t_Y. In general, the quantum process Φ can fall under three scenarios [68]: (i) The system at t_X is a "direct cause" of the system at t_Y: the qudit at t_Y is the output of some quantum map acting on the qubit at t_X [Fig. 3(b-i)]. (ii) The system at t_X and t_Y share a "common cause": the qudit at t_X is correlated with an environmental qudit E. E is measured at time t_Y [Fig. 3(b-ii)]. (iii) A mixture of both, corresponding to a general non-Markovian quantum process [Fig. 3(b-iii)].

We now introduce two families of interactive measurements: $\mathcal{M}_{\rm CC}$ and $\mathcal{M}_{\rm DC}$, as depicted in Fig. 4. Each $\mathcal{T}_1 \in \mathcal{M}_{\rm CC}$ is a maximal common-cause indicator, such that its eigencircuits imply that X and Y are actually two arms of some maximally entangled state [Fig. 3(b-ii)]. Meanwhile, each $\mathcal{T}_2 \in \mathcal{M}_{\rm DC}$ is a maximal direct-cause indicator, whose eigencircuit involves a lossless channel from X to Y [i.e., Fig. 3(b-i), where Ψ is unitary]. In the Supplemental Material ([30] Sec. III A), we establish the following "causal uncertainty relation":

$$H(\mathcal{T}_1) + H(\mathcal{T}_2) \geq 2 \log d, \tag{3}$$

for any $\mathcal{T}_1 \in \mathcal{M}_{\rm CC}$ and $\mathcal{T}_2 \in \mathcal{M}_{\rm CC}$. Here $H(\mathcal{T}_i)$ $(i = 1, 2)$ is the Shannon entropy of the probability distribution associated with outcomes when $\mathcal{T}_i$ is measured. Furthermore, this bound can be saturated.

Consider the application of this uncertainty to a specific parametrized quantum circuit $\Phi_{\alpha,\beta}$ [Fig. 5(a)] describing a single qubit undergoing non-Markovian evolution. Figure 5(b) then demonstrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$ for various values of α and β, including cases where they saturate the lower bound of 2. We also note that, unlike classical processes—which must be either purely common cause, purely direct cause, or a probabilistic mixture of both—quantum processes can feature richer causal dependencies [69]. Figure 5(c) depicts this for the cross section of $\alpha = \pi/4$. Such circuits include the coherent superposition of direct and common cause as a special case. Our causal uncertainty relation also applies to these uniquely quantum causal structures.

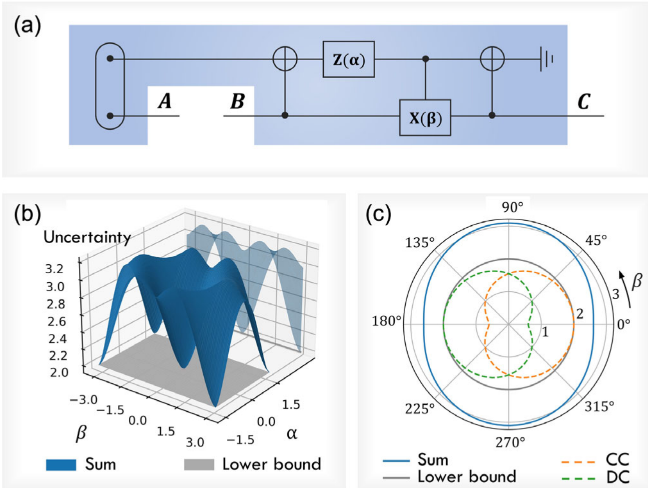

FIG. 5. Causal uncertainty relations on non-Markovian dynamics. Consider a single qubit—bottom rail of the circuit in (a)—undergoing non-Markovian evolution. Here the systems are initialized in a maximally entangled state, $Z(\theta)$ and $X(\theta)$ represent single-qubit rotation gates in X and Z axis. (b) Illustrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$, where $\mathcal{T}_1 \in M_{\mathrm{CC}}$ and $\mathcal{T}_2 \in M_{\mathrm{DC}}$ are, respectively, common- and direct-cause indicators in Fig. 4 with all U_k set to the identity. Observe that this never goes below the fundamental lower bound of 2 (gray plane). (c) Illustrates $H(\mathcal{T}_1)$ (green dashed), $H(\mathcal{T}_2)$ (red dashed), and their sum (blue solid) for $\alpha = -\pi/4$ and various values of β, corresponding to various coherent superpositions of common- and direct-cause circuits.

Discussion.—The most powerful means of learning involves interactive measurement—a procedure in which we can intervene by injecting (possible entangled) quantum states into the process over multiple time steps before observing the final output. Here, we derive entropic uncertainty relations that govern all interactive measurements, bounding their joint uncertainty whenever such measurement outcomes are noncompatible. In the context of causal inference, they predict a uniquely quantum entropic trade-off between measurements that probe for direct and common cause. More generally, our relations encompass all possible means for an agent to interact and learn about a target quantum system and thus include previously studied uncertainty relations on states and channels as special cases.

One potential application of such relations is the metrology of unknown quantum processes with memory [70–72]. In practice, full tomography of a general quantum process can be extremely costly. Even a single non-Markovian qubit measured at two different times requires 54 different interactive measurements [73]. Our result may help us ascertain specific properties of a process while avoiding this costly procedure. In the Supplemental Material ([30] Sec. IV B), we illustrate how our causal uncertainty relations imply that a single interactive measurement can rule out specific causal structures. Indeed, quantum illumination and adaptive sensing can both cast as measuring desired properties of a candidate quantum process and thus could benefit from such an approach.

Interactive measurements through repeated interventions also emerge in other settings [74–76]. In quantum open systems, sequential intervention provides a crucial toolkit for characterizing non-Markovian noise [77–80]. Meanwhile, in reinforcement learning, quantum agents that continuously probe an environment show enhancements in enacting or learning complex adaptive behavior [12,81,82]. Investigating uncertainty relations specific to such contexts has exciting potential, perhaps revealing new means of probing non-Markovian dynamics or fundamental constraints on how well an agent can simultaneously optimize two different rewards.

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[1] W. Heisenberg, Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik, Z. Phys. **43**, 172 (1927).
[2] E. H. Kennard, Zur quantenmechanik einfacher bewegungstypen, Z. Phys. **44**, 326 (1927).
[3] H. Weyl, *Gruppentheorie und Quantenmechanik* (S. Hirzel, Leipzig, 1928).

[4] H. P. Robertson, The uncertainty principle, Phys. Rev. **34**, 163 (1929).

[5] D. Deutsch, Uncertainty in Quantum Measurements, Phys. Rev. Lett. **50**, 631 (1983).

[6] S. Friedland, V. Gheorghiu, and G. Gour, Universal Uncertainty Relations, Phys. Rev. Lett. **111**, 230401 (2013).

[7] Z. Puchała, Ł. Rudnicki, and K. Życzkowski, Majorization entropic uncertainty relations, J. Phys. A **46**, 272002 (2013).

[8] L. Rudnicki, Z. Puchała, and K. Życzkowski, Strong majorization entropic uncertainty relations, Phys. Rev. A **89**, 052115 (2014).

[9] Z. Puchała, Ł. Rudnicki, A. Krawiec, and K. Życzkowski, Majorization uncertainty relations for mixed quantum states, J. Phys. A **51**, 175306 (2018).

[10] J. Pearl, *Causality*, 2nd ed. (Cambridge University Press, Cambridge, England, 2009).

[11] G. Lample and D. S. Chaplot, Playing fps games with deep reinforcement learning, in *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence, AAAI'17* (AAAI Press, San Francisco, 2017), pp. 2140–2146.

[12] G. D. Paparo, V. Dunjko, A. Makmal, M. A. Martin-Delgado, and H. J. Briegel, Quantum Speedup for Active Learning Agents, Phys. Rev. X **4**, 031002 (2014).

[13] H. X. Nguyenand P. Thiran, Active measurement for multiple link failures diagnosis in ip networks, in *Passive and Active Network Measurement*, edited by C. Barakat and I. Pratt (Springer, Berlin, Heidelberg, 2004), pp. 185–194.

[14] T. J. Elliott, M. Gu, A. J. P. Garner, and J. Thompson, Quantum Adaptive Agents with Efficient Long-Term Memories, Phys. Rev. X **12**, 011007 (2022).

[15] S. Lloyd, Enhanced sensitivity of photodetection via quantum illumination, Science **321**, 1463 (2008).

[16] S. Seah, S. Nimmrichter, and V. Scarani, Nonequilibrium dynamics with finite-time repeated interactions, Phys. Rev. E **99**, 042103 (2019).

[17] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Non-Markovian quantum processes: Complete framework and efficient characterization, Phys. Rev. A **97**, 012127 (2018).

[18] J. F. Fitzsimons, J. A. Jones, and V. Vedral, Quantum correlations which imply causation, Sci. Rep. **5**, 18281 (2015).

[19] C. Marletto, V. Vedral, S. Virzì, E. Rebufello, A. Avella, F. Piacentini, M. Gramegna, I. P. Degiovanni, and M. Genovese, Theoretical description and experimental simulation of quantum entanglement near open time-like curves via pseudo-density operators, Nat. Commun. **10**, 182 (2019).

[20] K. Kraus, A. Böhm, J. Dollard, and W. Wootters, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, Lecture Notes in Physics (Springer, Berlin, Heidelberg, 1983).

[21] M. Ziman, Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments, Phys. Rev. A **77**, 062112 (2008).

[22] Y. Xiao, K. Sengupta, S. Yang, and G. Gour, Uncertainty principle of quantum processes, Phys. Rev. Res. **3**, 023077 (2021).

[23] S. Milzand K. Modi, Quantum stochastic processes and quantum non-Markovian phenomena, PRX Quantum **2**, 030201 (2021).

[24] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Transforming quantum operations: Quantum supermaps, Europhys. Lett. **83**, 30004 (2008).

[25] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Quantum Circuit Architecture, Phys. Rev. Lett. **101**, 060401 (2008).

[26] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, Phys. Rev. A **80**, 022339 (2009).

[27] A. Bisioand P. Perinotti, Theoretical framework for higher-order quantum theory, Proc. R. Soc. A **475**, 20180706 (2019).

[28] G. Gour, Comparison of quantum channels by superchannels, IEEE Trans. Inf. Theory **65**, 5880 (2019).

[29] J. Wechs, H. Dourdent, A. A. Abbott, and C. Branciard, Quantum circuits with classical versus quantum control of causal order, PRX Quantum **2**, 030335 (2021).

[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.240201 for full proofs and mathematical details of our theorem, an improved entropic uncertainty relation, applications in causal inference, and corresponding numerical experiments, which includes Refs. [31–46].

[31] M. A. Nielsenand I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, England, 2010).

[32] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, England, 2013).

[33] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).

[34] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rep. Math. Phys. **3**, 275 (1972).

[35] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. **10**, 285 (1975).

[36] P. Taranto, F. A. Pollock, S. Milz, M. Tomamichel, and K. Modi, Quantum Markov Order, Phys. Rev. Lett. **122**, 140401 (2019).

[37] S. Seah, S. Nimmrichter, D. Grimmer, J. P. Santos, V. Scarani, and G. T. Landi, Collisional Quantum Thermometry, Phys. Rev. Lett. **123**, 180602 (2019).

[38] A. Abbas, D. Sutter, C. Zoufal, A. Lucchi, A. Figalli, and S. Woerner, The power of quantum neural networks, NATO ASI series Series F, Computer and system sciences **1**, 403 (2021).

[39] K. Bharti, A. Cervera-Lierta, T. H. Kyaw, T. Haug, S. Alperin-Lea, A. Anand, M. Degroote, H. Heimonen, J. S. Kottmann, T. Menke, W.-K. Mok, S. Sim, L.-C. Kwek, and A. Aspuru-Guzik, Noisy intermediate-scale quantum algorithms, Rev. Mod. Phys. **94**, 015004 (2022).

[40] J. Preskill, Quantum Computing in the NISQ era and beyond, Quantum **2**, 79 (2018).

[41] F. Cicaleseand U. Vaccaro, Supermodularity and subadditivity properties of the entropy on the majorization lattice, IEEE Trans. Inf. Theory **48**, 933 (2002).

[42] I. Białynicki-Birulaand J. Mycielski, Uncertainty relations for information entropy in wave mechanics, Commun. Math. Phys. **44**, 129 (1975).

[43] L. Vandenbergheand S. Boyd, Semidefinite programming, SIAM Rev. **38**, 49 (1996).

[44] S. Boydand L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, England, 2004).

[45] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, Phys. Rev. A **88**, 022318 (2013).

[46] D. Ebler, S. Salek, and G. Chiribella, Enhanced Communication with the Assistance of Indefinite Causal Order, Phys. Rev. Lett. **120**, 120502 (2018).

[47] G. Chiribella, M. Wilson, and H. F. Chau, Quantum and Classical Data Transmission through Completely Depolarizing Channels in a Superposition of Cyclic Orders, Phys. Rev. Lett. **127**, 190502 (2021).

[48] G. Rubino, L. A. Rozema, D. Ebler, H. Kristjánsson, S. Salek, P. Allard Guérin, A. A. Abbott, C. Branciard, i. c. v. Brukner, G. Chiribella, and P. Walther, Experimental quantum communication enhancement by superposing trajectories, Phys. Rev. Res. **3**, 013093 (2021).

[49] V. V. Shende, I. L. Markov, and S. S. Bullock, Minimal universal two-qubit controlled-not-based circuits, Phys. Rev. A **69**, 062321 (2004).

[50] K. Kraus, Complementary observables and uncertainty relations, Phys. Rev. D **35**, 3070 (1987).

[51] H. Maassenand J. B. M. Uffink, Generalized Entropic Uncertainty Relations, Phys. Rev. Lett. **60**, 1103 (1988).

[52] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, The uncertainty principle in the presence of quantum memory, Nat. Phys. **6**, 659 (2010).

[53] A. Marshall, I. Olkin, and B. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer Series in Statistics (Springer, New York, 2010).

[54] Denote the probability distribution of outcomes when $\mathcal{T}_1$ is measured as $\mathbf{p}$, then the uncertainty of $\mathcal{T}_1$ can be quantified by Shannon entropy, i.e., $H(\mathcal{T}_1)_\Phi \coloneqq H(\mathbf{p})$.

[55] J. Sánches-Ruiz, Optimal entropic uncertainty relation in two-dimensional Hilbert space, Phys. Lett. A **244**, 189 (1998).

[56] G. Ghirardi, L. Marinatto, and R. Romano, An optimal entropic uncertainty relation in a two-dimensional Hilbert space, Phys. Lett. A **317**, 32 (2003).

[57] J. I. de Vicenteand J. Sánchez-Ruiz, Improved bounds on entropic uncertainty relations, Phys. Rev. A **77**, 042110 (2008).

[58] M. Tomamicheland R. Renner, Uncertainty Relation for Smooth Entropies, Phys. Rev. Lett. **106**, 110506 (2011).

[59] P. J. Coles, R. Colbeck, L. Yu, and M. Zwolak, Uncertainty Relations from Simple Entropic Properties, Phys. Rev. Lett. **108**, 210405 (2012).

[60] P. J. Colesand M. Piani, Improved entropic uncertainty relations and information exclusion relations, Phys. Rev. A **89**, 022112 (2014).

[61] L. Rudnicki, Majorization approach to entropic uncertainty relations for coarse-grained observables, Phys. Rev. A **91**, 032123 (2015).

[62] Y. Xiao, N. Jing, S.-M. Fei, and X. Li-Jost, Improved uncertainty relation in the presence of quantum memory, J. Phys. A **49**, 49LT01 (2016).

[63] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, Rev. Mod. Phys. **89**, 015002 (2017).

[64] Y. Xiao, A framework for uncertainty relations, Ph.D. thesis, Leipzig University, Leipzig, Germany, 2017.

[65] P. J. Coles, V. Katariya, S. Lloyd, I. Marvian, and M. M. Wilde, Entropic Energy-Time Uncertainty Relation, Phys. Rev. Lett. **122**, 100401 (2019).

[66] Y. Xiao, Y. Xiang, Q. He, and B. C. Sanders, Quasi-fine-grained uncertainty relations, New J. Phys. **22**, 073063 (2020).

[67] H. Reichenbach, M. Reichenbach, and H. Putnam, *The Direction of Time*, California Library Reprint Series (University of California Press, Berkeley, 1956).

[68] K. Ried, M. Agnew, L. Vermeyden, D. Janzing, R. W. Spekkens, and K. J. Resch, A quantum advantage for inferring causal structure, Nat. Phys. **11**, 414 (2015).

[69] J.-P. W. MacLean, K. Ried, R. W. Spekkens, and K. J. Resch, Quantum-coherent mixtures of causal relations, Nat. Commun. **8**, 15149 (2017).

[70] Y. Yang, Memory Effects in Quantum Metrology, Phys. Rev. Lett. **123**, 110501 (2019).

[71] A. Altherrand Y. Yang, Quantum Metrology for Non-Markovian Processes, Phys. Rev. Lett. **127**, 060501 (2021).

[72] W. Górecki, A. Riccardi, and L. Maccone, Quantum Metrology of Noisy Spreading Channels, Phys. Rev. Lett. **129**, 240503 (2022).

[73] A. Feixand Č. Brukner, Quantum superpositions of common-cause and "direct-cause" causal structures, New J. Phys. **19**, 123028 (2017).

[74] O. Gühne, M. Kleinmann, A. Cabello, J.-A. Larsson, G. Kirchmair, F. Zähringer, R. Gerritsma, and C. F. Roos, Compatibility and noncontextuality for sequential measurements, Phys. Rev. A **81**, 022121 (2010).

[75] M. Gu, K. Wiesner, E. Rieper, and V. Vedral, Quantum mechanics can reduce the complexity of classical models, Nat. Commun. **3**, 762 (2012).

[76] D. Tan, S. J. Weber, I. Siddiqi, K. Mølmer, and K. W. Murch, Prediction and Retrodiction for a Continuously Monitored Superconducting Qubit, Phys. Rev. Lett. **114**, 090403 (2015).

[77] L. Li, M. J. Hall, and H. M. Wiseman, Concepts of quantum non-Markovianity: A hierarchy, Phys. Rep. **759**, 1 (2018).

[78] S. Cialdi, C. Benedetti, D. Tamascelli, S. Olivares, M. G. A. Paris, and B. Vacchini, Experimental investigation of the effect of classical noise on quantum non-Markovian dynamics, Phys. Rev. A **100**, 052104 (2019).

[79] G. A. L. White, C. D. Hill, F. A. Pollock, L. C. L. Hollenberg, and K. Modi, Demonstration of non-Markovian process characterisation and control on a quantum processor, Nat. Commun. **11**, 6301 (2020).

[80] S. Virzì, A. Avella, F. Piacentini, M. Gramegna, T. c. v. Opatrný, A. G. Kofman, G. Kurizki, S. Gherardini, F. Caruso, I. P. Degiovanni, and M. Genovese, Quantum Zeno and Anti-Zeno Probes of Noise Correlations in Photon Polarization, Phys. Rev. Lett. **129**, 030401 (2022).

[81] J. Thompson, A. J. P. Garner, V. Vedral, and M. Gu, Using quantum theory to simplify input–output processes, npj Quantum Inf. **3**, 6 (2017).

[82] T. J. Elliott, M. Gu, A. J. P. Garner, and J. Thompson, Quantum Adaptive Agents with Efficient Long-Term Memories, Phys. Rev. X **12**, 011007 (2022).