Range-Controlled Random Walks

L. Régnier,[1] O. Bénichou,[1] and P. L. Krapivsky[2,3]
[1]*Laboratoire de Physique Théorique de la Matière Condensée, CNRS/Sorbonne Université, 75005 Paris, France*
[2]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[3]*Santa Fe Institute, Santa Fe, New Mexico 87501, USA*

(Received 31 January 2023; revised 3 April 2023; accepted 9 May 2023; published 30 May 2023)

We introduce range-controlled random walks with hopping rates depending on the range $\mathcal{N}$, that is, the total number of previously distinct visited sites. We analyze a one-parameter class of models with a hopping rate $\mathcal{N}^a$ and determine the large time behavior of the average range, as well as its complete distribution in two limit cases. We find that the behavior drastically changes depending on whether the exponent a is smaller, equal, or larger than the critical value, a_d, depending only on the spatial dimension d. When $a > a_d$, the forager covers the infinite lattice in a finite time. The critical exponent is $a_1 = 2$ and $a_d = 1$ when $d \geq 2$. We also consider the case of two foragers who compete for food, with hopping rates depending on the number of sites each visited before the other. Surprising behaviors occur in 1D where a single walker dominates and finds most of the sites when $a > 1$, while for $a < 1$, the walkers evenly explore the line. We compute the gain of efficiency in visiting sites by adding one walker.

DOI: 10.1103/PhysRevLett.130.227101

The range $\mathcal{N}(t)$, that is, the number of distinct sites visited at time t, is a central observable of random walk theory. This quantity has been the subject of a large number of works in various fields, ranging from physics and chemistry to ecology [1–4]. A key result is that the average range of the symmetric nearest-neighbor random walk exhibits the following asymptotic behaviors [5]:

$$N(t) \equiv \langle \mathcal{N}(t) \rangle \sim \begin{cases} \sqrt{\frac{8ht}{\pi}} & d = 1 \\ \frac{\pi ht}{\ln ht} & d = 2 \\ ht/W_d & d > 2 \end{cases} \qquad (1)$$

where W_d are Watson integrals [6–9] and h the constant hopping rate [10]. The sublinear behavior in $d \leq 2$ dimensions is a direct consequence of the recurrence of random walks in low dimensions. Beyond the average, the ratio $\sqrt{\mathrm{Var}[\mathcal{N}(t)]}/N(t)$ is known to go to 0 in the large time limit when $d \geq 2$; it remains finite for $d = 1$. Thus the range $\mathcal{N}(t)$ is an asymptotically self-averaging random quantity when $d \geq 2$, namely, its distribution is asymptotically a Dirac delta function peaked at the average value. In 1D, the range is a non-self-averaging random quantity.

In addition to its central place in random walk theory, the range has proven to be a fundamental tool to quantify the efficiency of random explorations, as it is the case in foraging theory [11–17]. The minimal models involve a forager, described as a random walker (RW), that gradually depletes the resource contained in a medium as it moves. The medium is a d-dimensional lattice with a food unit at each site at $t = 0$. When the walker encounters a site containing food, it consumes it so that the amount of food collected at time t is the range $\mathcal{N}(t)$. This class of models accounts for the depletion of food induced by the motion of the forager, yet the movement of the walker is not affected by the consumption of resources. Depending on the situation, the food collected along the path can provide additional energy to search for food or, because it represents extra weight, to slow down the walk. As a result, there is a clear coupling between the range $\mathcal{N}(t)$ and the dynamical properties of the RW. No modeling of this effect has been proposed so far, even at a schematic level.

Here, we fill this gap and introduce range-controlled random walks as a model accounting for this coupling, for which the hopping rate is a monotonic function $h(\mathcal{N})$ of the range, either increasing or decreasing (see Fig. 1). For concreteness, we consider the case where the hopping rate is a power of the amount of collected food: $h(\mathcal{N}) = \mathcal{N}^a$. However, our results still apply when this algebraic dependence holds only asymptotically when $\mathcal{N} \gg 1$. In the context of search problems [17], models with positive [resp. negative] exponent a mimic the walker rewarded [resp. penalized] and accelerated [resp. decelerated] upon acquiring new targets (see Fig. 1). Because the coupling between the range and the dynamical properties of foragers is natural and because our modeling of this coupling is minimal, the model of range-controlled random walks quantifies the efficiency of foraging and appears relevant at broader scale to random explorations.

At the theoretical level, range-controlled random walks belong to the class of non-Markovian random walks, in which the memory of trajectory or some of its features influences the choice of destination sites. Representative

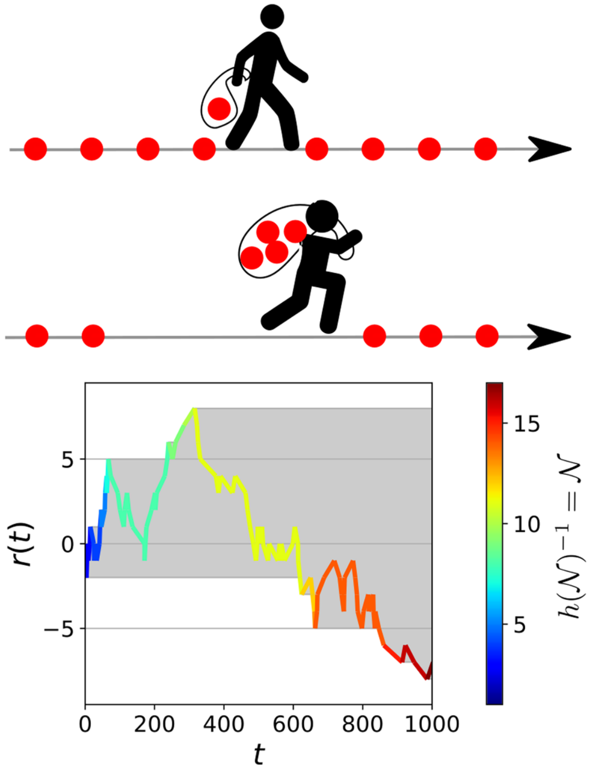

FIG. 1. Representation of a walker penalized upon acquiring new targets. Top: representation of the random walk slowed down by the load collected on distinct visited sites. Bottom: trajectory of a symmetric 1D nearest neighbor random walk in the particular case of hopping rates $h(\mathcal{N})$ varying as the inverse of the number of distinct sites visited, $\mathcal{N}^{-1}$. The grey area corresponds to the explored territory, and the coloring refers to the number of distinct sites visited.

examples comprise self-avoiding walks [18,19], true self-avoiding walkers [20–24], self-interacting random walks [25–36], and random walks with reinforcement such as the elephant walks [37–41]. In all of these models, the total hopping rate is kept constant [42]. Determining the range of non-Markovian random walks is notoriously difficult, and very few exact results are available.

Beyond this theoretical challenge, non-Markovian random walks with memory emerging from the interaction of the walker with the territory already visited are relevant in the case of living cells [45–49]. It has indeed been observed *in vitro* [50,51], both in 1D and 2D situations, that various cell types can chemically modify the extracellular matrix, which in turn deeply impacts their motility. In this context, range-controlled random walks appear as a minimal model where the modifications induced by the passage of cells are described in a mean-field way: all the complexity of the "perturbation," be it the concentration field of nutrients [52], the local orientation of matrix fibers [53], or hydrodynamics fields [54], is assumed to be encapsulated in the extension of the domain visited by the cell (i.e., the range of the associated random walk). It is then natural to mimic the "response" of the cell by a modification of a dynamical parameter, and we finally end up with a hopping rate $h(\mathcal{N})$ depending on the range as introduced above.

Main results.—In this Letter, we quantify the efficiency of d-dimensional range-controlled random walks by determining exact asymptotic expressions of their average range, as well as the full distribution in $d = 1$ and in the $d \to \infty$ limit. This allows us to unveil a surprising transition and show that the behavior of range-controlled random walks drastically changes depending on whether the exponent a is smaller, equal, or larger than the critical value, a_d, depending only on the spatial dimension: $a_1 = 2$ in $d = 1$ and $a_d = 1$ when $d \geq 2$. The explosive behavior occurs in the supercritical $a > a_d$ regime: The forager covers the entire infinite lattice in a finite time.

The behavior in the $a \leq a_d$ regime can be appreciated from the growth of the average number $N(t)$ of distinct visited sites. When $a < a_d$, the growth is algebraic with a logarithmic correction in 2D:

$$N(t) \sim C_d(a) \times \begin{cases} t^{\frac{1}{2-a}} & d = 1 \\ \left(\frac{t}{\ln t}\right)^{\frac{1}{1-a}} & d = 2\,. \\ t^{\frac{1}{1-a}} & d > 2 \end{cases} \qquad (2)$$

The amplitudes are

$$C_1(a) = \frac{2^{1+\frac{1}{2-a}}}{\Gamma(\frac{1}{2-a})} \int_0^\infty \frac{dv}{v} \left[\frac{\sqrt{v}}{\cosh v}\right]^{\frac{4}{2-a}}, \qquad a < 2 \qquad (3a)$$

$$C_2(a) = [\pi(1-a)^2]^{\frac{1}{1-a}}, \qquad a < 1 \qquad (3b)$$

$$C_d(a) = \left(\frac{1-a}{W_d}\right)^{\frac{1}{1-a}}, \qquad a < 1, \qquad d \geq 3. \qquad (3c)$$

In the critical regime $a = a_d$, the growth is exponential

$$\ln N(t) \sim \begin{cases} \lambda_1 t & d = 1, \quad a = 2 \\ \lambda_2 \sqrt{t} & d = 2, \quad a = 1 \\ \lambda_d t & d > 2, \quad a = 1 \end{cases} \qquad (4)$$

with growth rates

$$\lambda_1 = 1, \qquad \lambda_2 = \sqrt{2\pi}, \qquad \lambda_d = 1/W_d \qquad (d > 2). \qquad (5)$$

We discuss the competition between two foragers by determining the average number of distinct sites $N_2(t)$ visited by two foragers in 1D where their respective hopping rates depend on the number of distinct sites the walker visited before the other. In particular, by defining $N_1(t)$ as the average number of distinct sites visited by a single walker (without any other walker), we get an analytical value for the ratio at large times:

$$\lim_{t\to\infty}\frac{N_2(t)}{N_1(t)} = r_2(a) = \begin{cases} 2^{\frac{1-a}{2-a}} & a < 1 \\ 1 & a > 1. \end{cases} \tag{6}$$

This ratio quantifies the efficiency gain in finding new sites by adding one RW. In particular, we observe that for foragers accelerating fast enough with the number of distinct sites visited ($a > 1$), there is no gain in adding the second walker.

One forager.—Let $P_n(t) \equiv \mathbf{P}[\mathcal{N}(t) = n]$ be the range distribution and $\Pi_n(t) \equiv \mathbf{P}[\mathcal{N}(t) \geq n]$ the corresponding complementary cumulative distribution. In the 1D situation, an exact expression for the entire distribution of the range can be obtained. This exact solution relies on the observation that (see [55,56])

$$\{\mathcal{N}(t) \geqslant n\} = \left\{\sum_{k=1}^{n-1} \tau_k \leqslant t\right\} \tag{7}$$

where τ_k is the time elapsed between the visit of the k^{th} and $(k+1)^{\text{st}}$ site by the RW defined above. The key points are that (i) during this exploration, the walker has a constant hopping rate, k^a, and (ii) the τ_k's are independent random variables. Performing the Laplace transform $[\widehat{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt]$ of the probability of these events we get

$$\widehat{\Pi}_n(s) = \frac{1}{s}\prod_{k=1}^{n-1}\widehat{F}_k(s) \tag{8}$$

where $\widehat{F}_k$ is the Laplace transform of the distribution of the random variable τ_k. Here τ_k is the exit time from an interval of k sites starting on the boundary. At small s (corresponding to large time), $\widehat{F}_k$ is given by the exit time distribution of an interval of length $k+1$ starting at distance one of the border of a continuous Brownian motion with diffusion constant $D_k \equiv \frac{1}{2}k^a$ [4],

$$\widehat{F}_k(s) = \frac{\sinh\left(\sqrt{\frac{s}{D_k}}k\right) + \sinh\left(\sqrt{\frac{s}{D_k}}\right)}{\sinh\left(\sqrt{\frac{s}{D_k}}(k+1)\right)}. \tag{9}$$

This expression involves $k/\sqrt{D_k} \propto k^{1-a/2}$, and reveals the existence of three different regimes.

(i) In the subcritical regime $a < a_c = 2$, taking the limit $k \to \infty$ and $s \to 0$ while keeping $k^{2-a}s$ finite, gives

$$\widehat{F}_k(s) - 1 \sim -\sqrt{2s}k^{-a/2}\tanh\left(\sqrt{\frac{s}{2}}k^{1-a/2}\right) \tag{10}$$

and then (see Supplemental Material [57], SM, S1 and S2)

$$\begin{aligned} \widehat{P}_n(s) &= \widehat{\Pi}_n(s) - \widehat{\Pi}_{n+1}(s) \\ &\sim -\partial_n\widehat{\Pi}_n(s) \\ &\sim -\partial_n\left(\frac{1}{s\cosh\left(n^{1-a/2}s^{1/2}/\sqrt{2}\right)^{\frac{4}{2-a}}}\right). \end{aligned} \tag{11}$$

In particular, for $a = 0$, we recover the well-known range distribution [2] for a standard random walk. One can extract the average range, viz. Eqs. (2) and (3a), from the asymptotic behavior of the Laplace transform (11) in the $s \to 0$ limit. In addition, $P_n(t)$ acquires a scaling form $P_n(t) = t^{-[1/(2-a)]}\phi_a(x)$ $(x = n/t^{[1/(2-a)]})$, where ϕ_a is a function of the scaling variable x depending on the exponent a. Explicit analytical expressions are provided and displayed in the SM for accelerated and slowed down foragers.

(ii) In the critical regime, $a = a_c = 2$ in 1D, $\hat{F}_k(s) - 1 \sim -s/k$ and $P_n(t) \sim \delta(n - e^t)$ (see SM). Thus $\mathcal{N}(t)$ is asymptotically deterministic, chiefly characterized by exponentially growing average $e^{\lambda_1 t}$ with $\lambda_1 = 1$ as stated in Eq. (5).

(iii) In the supercritical regime, $a > a_c = 2$, the dynamics is explosive, and the entire infinite lattice is covered in a finite time (see SM).

In the $d \to \infty$ limit, the entire distribution of the range can also be obtained (see SM). In this case, the average N and the variance V of the number of distinct visited sites exhibit asymptotically identical growth:

$$N \sim [(1-a)t]^{\frac{1}{1-a}} \sim V. \tag{12}$$

This shows the self-averaging nature of $\mathcal{N}$ in the subcritical regime (for $d \to \infty$) as the standard deviation is negligible compared to the average.

For finite dimensions, when $a \leq a_d$, the asymptotic behavior of the average range can be obtained from heuristic arguments. In the case of varying hopping rates, we use (1) and a self-consistent estimate $h = N^a$ of the typical hopping rate. In 1D, for instance, this leads to $N \propto \sqrt{N^a t}$, from which $N \propto t^{[1/(2-a)]}$, in agreement with the exact solution provided in Eq. (2). Similarly we arrive at the announced growth laws (2) in higher dimensions. These results show that $a_d = 1$ when $d \geq 2$. We now turn to the determination of the amplitudes $C_d(a)$ (for $a < a_d$) and the growth rates λ_d (for $a = a_d$).

In $d > 2$ dimensions, the proper interpretation of (1) is that a RW hops to unvisited sites with probability that approaches $1/W_d$ [2]. Thus

$$N(t) \sim (W_d)^{-1}H(t) \tag{13}$$

where H is the average total number of hops. Using

$$H(t) \sim \int_0^t d\tau[N(\tau)]^a \sim (1-a)[C_d(a)]^a t^{\frac{1}{1-a}} \tag{14}$$

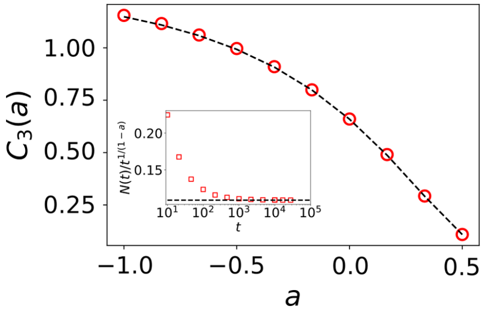

FIG. 2. Comparison of the coefficient $C_3(a)$ obtained in (3c) (black dashed line) and the results of numerical simulations (red circles) obtained by evaluating $\lim_{t\to\infty}[N(t)/t^{1/(1-a)}]$. The subfigure represents $N(t)/t^{1/(1-a)}$ (red squares) for increasing values of t in the particular case $a = 1/2$. It shows the convergence to $C_3(a)$ (black dashed line).

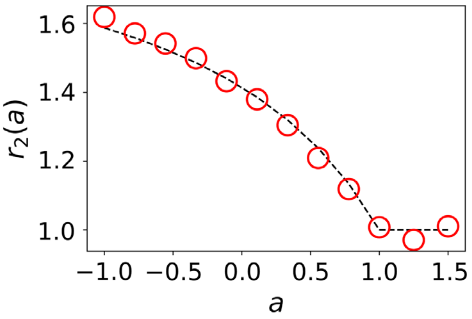

FIG. 3. Comparison of the theoretical prediction of the ratio $r_2(a) = \max(1, 2^{[(1-a)/(2-a)]})$ (black dashed lines) and numerical simulation $\lim_{t\to\infty}[N_2(t)/N_1(t)]$ (red circles, whose diameters correspond to the largest error bar estimation) of the average number of distinct sites visited by two foragers, $N_2(t)$, versus the average number of distinct sites visited by a single one, $N_1(t)$.

we arrive at $C_d(a) = (1-a)[C_d(a)]^a/W_d$ leading to the announced result (3c).

The relation (13) is asymptotically exact, but averaging the total number of hops

$$\mathcal{H} = \int_0^t d\tau [\mathcal{N}(\tau)]^a \tag{15}$$

gives (14) only if $\langle \mathcal{N}^a \rangle = \langle \mathcal{N} \rangle^a$. This is erroneous (when $a \neq 1$) if the random quantity $\mathcal{N}$ is non-self-averaging as it is in 1D. Since $\mathcal{N}$ is self-averaging if $d \geq 2$, the prediction (3c) is exact (see also the agreement with numerical simulations displayed in Fig. 2). In the critical regime, $a = 1$, we have $\dot{H} = N$ which we insert into (13) and obtain the differential equation $\dot{H} = H/W_d$, whose solution is $H \propto e^{t/W_d}$. The range is also exponential, confirming (4) and (5) for $d > 2$.

In 2D, the exact asymptotic is $N \sim \pi H/\ln H$, see (1). When $a < 1$, taking $h = N^a$ in Eq. (1) in the 2D case, we obtain $N \propto (N^a t/\ln N^a t)$ and $N \propto (t/\ln t)^{[1/(1-a)]}$. We note the constant prefactor $C_2(a)$, as defined in (2). Then similarly to Eq. (14),

$$\begin{aligned} H &\sim \int_0^t d\tau [N(\tau)]^a \\ &\sim (1-a)[C_2(a)]^a t^{\frac{1}{1-a}} (\ln t)^{-\frac{a}{1-a}}. \end{aligned} \tag{16}$$

Equating N to

$$\frac{\pi H}{\ln H} \sim \pi (1-a)^2 [C_2(a)]^a (t/\ln t)^{\frac{1}{1-a}}$$

fixes the amplitude and yields the announced result (3b). In the critical $a = 1$ regime, the growth is stretched exponential in 2D [see Eq. (4)]. Indeed, $H(t)$ asymptotically satisfies $\dot{H} = (\pi H/\ln H)$, whose solution is $H \propto e^{\sqrt{2\pi t}}$. This confirms (4) with $\lambda_2 = \sqrt{2\pi}$ in 2D. Thus we have established (4) and (5) in all dimensions (see SM Fig. 3 for the comparison with numerical simulations).

Two foragers.—We now discuss the competition of foragers. The forager with label j has the hopping rate $\mathcal{F}_j^a$, where $\mathcal{F}_j$ is the number of sites first visited by the forager. For one forager, $\mathcal{F}_1 = \mathcal{N}_1$ is just the range. For two foragers, $\mathcal{F}_1 + \mathcal{F}_2 = \mathcal{N}_2$ is the total range. Foragers do not directly interact, but their motion changes the environment that, in turn, affects the motion of the foragers.

To compare the two-forager and single-forager cases, we consider the ratio N_2/N_1 of the average numbers of distinct visited sites in both settings. The ratio $r_2(a)$ defined in Eq. (6) depends only on the exponent a and is nontrivial only in 1D, as $r_2(a) = 2$ for $d \geq 2$ as a consequence of [58], the number of common sites visited by the two RWs being asymptotically negligible compared to the number of distinct sites visited by each one of them in this case. Hereinafter we consider foraging in 1D in the nonexplosive regime, $a \leq 2$. For 1D RWs, the ratio $r_2(0) = \sqrt{2}$ is smaller than 2 reflecting the severe space limitation in 1D (see SM).

One has to differentiate between two regimes, $a < 1$ and $a > 1$. When $a < 1$, both foragers visit the 1D lattice equally (on average). Thus the rate of finding new sites is known by solving the problem at $a = 0$. Using a proper rescaling of the times by $(N_2/2)^a$ corresponding to the hopping rate of one of the forager [similarly to what we did with (1)], one establishes (6). When $a > 1$, the hypothesis that sites are equally visited breaks down. To understand the transition, suppose one walker (W_1) has found $k \gg 1$ sites, while he other (W_2) has found $\ell \ll k$ sites. If W_2 finds a new site at some time $t \gg 1$, which walker will be first to find a new site? The walker W_2 will find a new site in a typical time $\propto k/\ell^a$, as it is positioned at the border of the interval of at most $k + \ell \approx k$ distinct sites visited by W_1 and W_2. The position of W_1 is unknown, effectively uniform in the interval of k distinct sites visited, but it hops much faster than W_2 and so the average time of finding a new site is $k^2/k^a = k^{2-a}$. Thus, if $a > 1$, even though it is further away from the border than W_2, the walker W_1 will be the first to find a new site ($k^{2-a} \ll k/\ell^a$). This situation is stable if $a > 1$ (the dominant walker with the most distinct visited sites will become more and more dominating) and unstable if $a < 1$ (the subdominant walker catches up). The theoretical prediction is validated by numerical simulations (Fig. 3).

We introduced random walks with range-dependent hopping rates behaving as $\mathcal{N}^a$ when $\mathcal{N} \gg 1$. Our analysis provides the exact full distribution of the range in 1D, and also on a complete graph mimicking an infinite-dimensional setting. For a random walk on a hypercubic lattice $\mathbb{Z}^d$ with $d \geq 2$, we used a heuristic approach relying on results for the classical random walk ($a = 0$). We argued that this argument gives asymptotically exact results for the average range when $d \geq 2$. The above subcritical behaviors occur when $a < a_d$ with $a_1 = 2$ and $a_d = 1$ when $d \geq 2$. When $a > a_d$, the entire infinite lattice $\mathbb{Z}^d$ is covered in a finite time.

We thank the Erwin Schrödinger Institute for excellent working conditions.

[1] G. H. Weiss and R. J. Rubin, in *Advances in Chemical Physics*, edited by I. Prigogine and S. A. Rice (Wiley-Interscience, New York, 1983), Vol. 52, pp. 363–505.

[2] B. Hughes, *Random Walks and Random Environments, Vol. 1: Random Walks*, Oxford Science Publications (Clarendon Press, New York, 1996).

[3] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).

[4] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, UK, 2001).

[5] We consider hypercubic lattices, so, e.g., the prediction of (1) in 2D refers to the square grid.

[6] G. N. Watson, Q. J. Math. Oxford **10**, 266 (1939).

[7] M. L. Glasser and I. J. Zucker, Proc. Natl. Acad. Sci. U.S.A. **74**, 1800 (1977).

[8] A. J. Guttmann, J. Phys. A **43**, 305205 (2010).

[9] I. J. Zucker, J. Stat. Phys. **145**, 591 (2011).

[10] The hopping rate is $h = zD$ where z is the lattice coordination number and D the diffusion coefficient of the random walker. Hence $h = 2dD$ for the random walker on the hypercubic lattice $\mathbb{Z}^d$.

[11] H. Larralde, P. Trunfio, H. E. Stanley, and G. H. Weiss, Nature (London) **355**, 423 (1992).

[12] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley, and G. H. Weiss, Phys. Rev. A **45**, 7128 (1992).

[13] G. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, Nature (London) **401**, 911 (1999).

[14] G. M. Viswanathan, M. G. D. Luz, E. P. Raposo, and H. E. Stanley, *The Physics of Foraging: An Introduction to Random Searches and Biological Encounters* (Cambridge University Press, Cambridge, United Kingdom, 2011).

[15] D. Ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, UK, 2000).

[16] O. Bénichou, M. Coppey, M. Moreau, P. H. Suet, and R. Voituriez, J. Phys. Condens. Matter **17**, S4275 (2005).

[17] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, Rev. Mod. Phys. **83**, 81 (2011).

[18] J. Rudnick and G. Gaspari, *Elements of the Random Walk: An Introduction for Advanced Students and Researchers* (Cambridge University Press, Cambridge, New York, 2004).

[19] N. Madras and G. Slade, *The Self-Avoiding Walk* (Birkhäuser, New York, NY, 2013).

[20] D. J. Amit, G. Parisi, and L. Peliti, Phys. Rev. B **27**, 1635 (1983).

[21] L. Pietronero, Phys. Rev. B **27**, 5887 (1983).

[22] S. P. Obukhov and L. Peliti, J. Phys. A **16**, L147 (1983).

[23] L. Peliti and L. Pietronero, Riv. Nuovo Cimento **10**, 1 (1987).

[24] P. Grassberger, Phys. Rev. Lett. **119**, 140601 (2017).

[25] M. Perman and W. Werner, Probab. Theory Relat. Fields **108**, 357 (1997).

[26] B. Davis, Probab. Theory Relat. Fields **1**, 501 (1999).

[27] P. Pemantle and S. Volkov, Ann. Probab. **27**, 1368 (1999).

[28] R. Dickman and D. ben-Avraham, Phys. Rev. E **64**, 020102(R) (2001).

[29] I. Benjamini and D. Wilson, Electron. Commun. Probab. **8**, 86 (2003).

[30] M. P. W. Zerner, Probab. Theory Relat. Fields **133**, 98 (2005).

[31] T. Antal and S. Redner, J. Phys. A **38**, 2555 (2005).

[32] E. Kosygina and M. P. W. Zerner, Bull. Inst. Math. Acad. Sin. **8**, 105 (2013).

[33] D. Boyer and C. Solis-Salas, Phys. Rev. Lett. **112**, 240601 (2014).

[34] D. Campos and V. Méndez, Phys. Rev. E **99**, 062137 (2019).

[35] A. Barbier-Chebbah, O. Benichou, and R. Voituriez, Phys. Rev. E **102**, 062115 (2020).

[36] A. Barbier-Chebbah, O. Bénichou, and R. Voituriez, Phys. Rev. X **12**, 011052 (2022).

[37] G. M. Schütz and S. Trimper, Phys. Rev. E **70**, 045101(R) (2004).

[38] F. N. C. Paraan and J. P. Esguerra, Phys. Rev. E **74**, 032101 (2006).

[39] E. Baur and J. Bertoin, Phys. Rev. E **94**, 052134 (2016).

[40] B. Bercu and L. Laulin, Stoch. Processes Appl. **133**, 111 (2021).

[41] J. Bertoin, Trans. Am. Math. Soc. **375**, 1 (2022).

[42] Notable exceptions are the locally activated random walk model [43], where the hopping rate depends on the number of visits of a specific site, and the accelerated cover problem [44], where the hopping rate is a function of the typical time to visit a new site.

[43] O. Bénichou, N. Meunier, S. Redner, and R. Voituriez, Phys. Rev. E **85**, 021137 (2012).

[44] G. Maziya, L. Cocconi, G. Pruessner, and N. R. Moloney, Phys. Rev. Res. **2**, 023421 (2020).

[45] C. T. Mierke, Front. Phys. **9**, 749830 (2021).

[46] C. T. Mierke, P. Kollmannsberger, D. P. Zitterbart, G. Diez, T. M. Koch, S. Marg, W. H. Ziegler, W. H. Goldmann, and B. Fabry, J. Biol. Chem. **285**, 13121 (2010).

[47] C. T. Mierke, Rep. Prog. Phys. **77**, 076602 (2014).

[48] G. Charras and E. Sahai, Nat. Rev. Mol. Cell Biol. **15**, 813 (2014).

[49] C. T. Mierke, Rep. Prog. Phys. **82**, 064602 (2019).

[50] J. D'alessandro, A. Barbier-Chebbah, V. Cellerin, O. Benichou, R. M. Mège, R. Voituriez, and B. Ladoux, Nat. Commun. **12**, 4118 (2021).

[51] H. Flyvbjerg, Nat. Phys. **17**, 771 (2021).

[52] P. Macklin and J. Lowengrub, J. Theor. Biol. **245**, 677 (2007).

[53] D. K. Schlüter, I. Ramis-Conde, and M. A. Chaplain, Biophys. J. **103**, 1141 (2012).

[54] T. Heck, B. Smeets, S. Vanmaercke, P. Bhattacharya, T. Odenthal, H. Ramon, H. V. Oosterwyck, and P. V. Liedekerke, Comput. Methods Appl. Mech. Eng. **322**, 515 (2017).

[55] L. Régnier, M. Dolgushev, S. Redner, and O. Bénichou, Phys. Rev. E **105**, 064104 (2022).

[56] L. Régnier, M. Dolgushev, S. Redner, and O. Bénichou, Nat. Commun. **14**, 618 (2023).

[57] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.227101 for detailed calculations and numerical checks of the $1D$ model (single forager and two foragers), as well as supplemental calculations and figures in the case of infinite and finite ($d > 1$) dimensions.

[58] E. Ben-Naim and P. L. Krapivsky, J. Stat. Mech. (2022) 103208.