Construction of Non-Hermitian Parent Hamiltonian from Matrix Product States

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There are various research strategies used for non-Hermitian systems, which typically involve introducing non-Hermitian terms to preexisting Hermitian Hamiltonians. It can be challenging to directly design non-Hermitian many-body models that exhibit unique features not found in Hermitian systems. In this Letter, we propose a new method for constructing non-Hermitian many-body systems by generalizing the parent Hamiltonian method into non-Hermitian regimes. This allows us to build a local Hamiltonian using given matrix product states as its left and right ground states. We demonstrate this method by constructing a non-Hermitian spin-1 model from the asymmetric Affleck-Kennedy-Lieb-Tasaki state, which preserves both chiral order and symmetry-protected topological order. Our approach opens up a new paradigm for systematically constructing and studying non-Hermitian many-body systems, providing guiding principles for exploring new properties and phenomena in non-Hermitian physics.

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Introduction.—Non-Hermitian physics has attracted much attention both theoretically [1–10] and experimentally [11–17] for describing open systems [18], such as photonics [19] and acoustics [20,21] with gain and loss, as well as quasiparticles in interacting or disordered systems [22,23]. It has also revealed nontrivial properties that have no Hermitian counterpart [24–29].

However, many recent studies revealing nontrivial properties of non-Hermitian systems have focused on the single-particle picture [30–32]. One reason for this is that many powerful numerical methods for Hermitian quantum many-body models, such as the density matrix renormalization group [33,34] and quantum Monte Carlo methods [35], cannot be directly applied to non-Hermitian systems. Some modified algorithms also suffer from unstable convergence [36,37] and incapability near exceptional points [38–40].

Therefore, it is interesting to consider the opposite question: can we construct a non-Hermitian Hamiltonian from a pair of easily engineered states that preserve desired properties, rather than having to extract various properties from a given Hamiltonian? In Hermitian systems, this task can be achieved using the parent Hamiltonian method [41,42], which allows for the construction of a local, gapped Hamiltonian whose ground state is represented by a matrix product state (MPS) [43–47]. However, this method cannot be directly applied to non-Hermitian systems.

In this Letter, we present a method for constructing non-Hermitian parent Hamiltonians (NH-PHs) by generalizing the conventional Hermitian approach. We provide criteria for states that can be used to establish a NH-PH and derive the explicit form of the Hamiltonian. As an example, we construct a non-Hermitian model from asymmetric Affleck-Kennedy-Lieb-Tasaki (AKLT) states [48] and examine its physical properties in the thermodynamic limit using the generalized infinite time-evolving block decimation (iTEBD) method [49,50]. We find that the model has two nontrivial orders: chiral order detected by a local order parameter and symmetry-protected topological (SPT) order [51–53] detected by a string order parameter [54].

Non-Hermitian parent Hamiltonian.—The expectation value of any observable $\langle\hat{O}\rangle$ for a general non-Hermitian system can be evaluated in different ways [55–58]. Here, we choose the formalism discussed in [55] to calculate the expectation as

$$\langle\hat{O}\rangle_{LR} = \langle L|\hat{O}|R\rangle/\langle L|R\rangle, \tag{1}$$

which has a clear geometric interpretation [59]. Here, $|R\rangle$ and $|L\rangle$ are the ground states of H and $H^\dagger$, respectively, which are defined as the eigenstates with the lowest real parts of the eigenvalues. As a result, many more novel properties emerge in non-Hermitian systems since we have more degrees of freedom in choosing $\langle L|$ independent of $|R\rangle$ than in the Hermitian case, where expectation values are evaluated under $\langle\hat{O}\rangle_{RR} = \langle R|\hat{O}|R\rangle/\langle R|R\rangle$. A natural question arises: can such a system be constructed, i.e., can we find a non-Hermitian Hamiltonian that has the given $\langle L|$ and $|R\rangle$ as its corresponding ground states? In the following, we answer this question for MPSs, which satisfy the entanglement area law and can describe ground states of one-dimensional (1D) local and gapped Hamiltonians [60,61]. The same argument can be easily extended to higher dimensions.

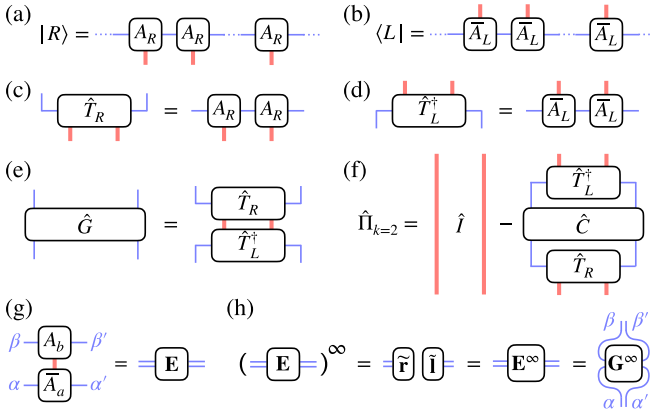

FIG. 1. Construction of non-Hermitian parent Hamiltonian for $k = 2$. (a),(b) Translation-invariant MPS. (c),(d) Local tensors $\hat{T}_R = |\mathbf{p}\rangle\mathbf{T}_R(\mathbf{r}|$ and $\hat{T}_L^\dagger = |\mathbf{l})\mathbf{T}_L^\dagger\langle\mathbf{p}|$. (e) The metric operator $\hat{G} = \hat{T}_L^\dagger\hat{T}_R$. (f) The local projector $\hat{\Pi} = \hat{I} - \hat{T}_R\hat{C}\hat{T}_L^\dagger$. (g) The transfer matrix $\mathbf{E}$ constructed from tensors A_b and A_a^*. (h) The RG fixed point $\mathbf{E}^\infty$ and the corresponding $\mathbf{G}^\infty$.

In this context, we consider 1D translation-invariant (TI) and injective MPS [42] $|R\rangle$ and $|L\rangle$ shown in Figs. 1(a) and 1(b), written as

$$|R(L)\rangle = \sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[A_{R(L)}^{[i_1]},\ldots,A_{R(L)}^{[i_N]}\right]|i_1,\ldots,i_N\rangle, \quad (2)$$

with the same virtual bond dimension D and physical bond dimension d. A detailed explanation of MPSs is shown in the Supplemental Material [50]. As shown in Figs. 1(c) and 1(d), tensors after contracting k neighboring sites can be regarded as maps from virtual to physical degrees of freedom, i.e., $\hat{T}_R = |\mathbf{p}\rangle\mathbf{T}_R(\mathbf{r}|$ and $\hat{T}_L^\dagger = |\mathbf{l})\mathbf{T}_L^\dagger\langle\mathbf{p}|$. Here, $|\mathbf{p}\rangle$ is the collective physical basis and $|\mathbf{r})$ [$(\mathbf{l}|$] is the collective virtual basis of $|R\rangle$ ($\langle L|$). $\mathbf{T}_R$ and $\mathbf{T}_L$ are coefficient matrices. The local support spaces $\mathcal{H}_R$ and $\mathcal{H}_L$ are the images of $\hat{T}_R$ and $\hat{T}_L$, respectively. With the injectivity condition, we can choose a large enough k such that $d^k \geq D^2$ and $\dim\mathcal{H}_R = \dim\mathcal{H}_L = D^2$ [42].

We aim to find a local Hamiltonian in the form $\hat{H} = \sum_i \hat{\Pi}_i$, where each $\hat{\Pi}_i = \hat{I} - \hat{P}_i$ acts on k local sites and ensures that $\langle L|$ and $|R\rangle$ are zero-energy modes. Here, $\hat{P}_i$ is a projector with $\hat{P}_i^2 = \hat{P}_i$. In other words, we require $\hat{P}_i$ and $\hat{P}_i^\dagger$ to be projectors onto $\mathcal{H}_R$ and $\mathcal{H}_L$, respectively,

$$\hat{P}_i\hat{T}_R = \hat{T}_R, \qquad \hat{T}_L^\dagger\hat{P}_i = \hat{T}_L^\dagger. \quad (3)$$

Meanwhile, we require $\mathrm{rank}\hat{P} = \mathrm{rank}\hat{P}^\dagger = D^2$. Therefore, the most general form of a projector can be written as (the site index i has been omitted for simplicity)

$$\hat{P} = |\mathbf{p}\rangle\mathbf{T}_R\mathbf{C}\mathbf{T}_L^\dagger\langle\mathbf{p}| = \hat{T}_R\hat{C}\hat{T}_L^\dagger, \quad (4)$$

where $\mathbf{C}$ is a $D^2 \times D^2$ matrix and $\hat{C} = |\mathbf{r})\mathbf{C}(\mathbf{l}|$ is its operator form. The matrix elements are determined by Eq. (3), $\hat{T}_R\hat{C}\hat{T}_L^\dagger\hat{T}_R = \hat{T}_R$. As $\mathrm{rank}\hat{T}_R = \mathrm{rank}\hat{T}_L^\dagger = D^2$, this gives

$$\hat{C} = (\hat{T}_L^\dagger\hat{T}_R)^{-1} \equiv \hat{G}^{-1}. \quad (5)$$

Therefore, the metric operator $\hat{G} = |\mathbf{l})\mathbf{G}(\mathbf{r}|$ shown in Fig. 1(e) must be invertible and can fully determine the k-local Hamiltonian shown in Fig. 1(f)

$$\hat{\Pi} = \hat{I} - \hat{T}_R\hat{C}\hat{T}_L^\dagger = |\mathbf{p}\rangle\mathbf{I}\langle\mathbf{p}| - |\mathbf{p}\rangle\mathbf{T}_R\mathbf{G}^{-1}\mathbf{T}_L^\dagger\langle\mathbf{p}|. \quad (6)$$

In Fig. S1 in the Supplemental Material [50], we verify Eq. (3) in a straightforward way.

We note that $\hat{G}^{-1}$ is simply the operator used to biorthogonalize $\mathcal{H}_R$ and $\mathcal{H}_L$. In other words, if we perform the transformations $\mathbf{T}_R \to \mathbf{T}_R\mathbf{G}^{-1}$ and $\mathbf{T}_L \to \mathbf{T}_L$, $\hat{T}_L$ and $\hat{T}_R$ become orthogonal operators

$$\hat{T}_L'^\dagger\hat{T}_R' = |\mathbf{l})\mathbf{T}_L^\dagger\mathbf{T}_R\mathbf{G}^{-1}(\mathbf{r}| = \hat{I}. \quad (7)$$

On the other hand, the ability to perform biorthogonalization guarantees the existence of NH-PHs, as proved in the Supplemental Material [50]. As a specific example, when we set $\hat{T}_L = \hat{T}_R$, our method reproduces the conventional Hermitian projector, but with a clearer physical interpretation.

By referencing the proof in Ref. [42], we see that the given right (left) state is guaranteed to be the unique zero-energy eigenstate of $\hat{H}$ ($\hat{H}^\dagger$) by construction. However, the zero mode is not necessarily the ground state due to non-Hermiticity. Specifically, we have $\langle\hat{H}\rangle = \langle\psi|\sum_i\hat{h}_i|\psi\rangle = \sum_i\langle\psi|\hat{h}_i|\psi\rangle \geq \sum_i E_{0,i}$ for a Hermitian parent Hamiltonian, where $E_{0,i}$ is the ground state energy of the local term $\hat{h}_i$. Thus, the total system energy is bound by the local ground-state energies. However, this inequality no longer holds in the non-Hermitian regime. $\Re(\langle\psi|\hat{h}_i|\psi\rangle)$ can be even smaller than $\Re(E_{0,i})$ (which equals to 0 in our NH-PH), implying the existence of a negative energy eigenstate. As a consequence, the bound on the total energy disappears, and the common ground state of local projectors is not necessarily the global ground state. This phenomenon often occurs when non-Hermitian effects are significant but can be reduced by increasing the interaction length k, as demonstrated in the following example.

$\mathcal{PT}$ symmetry.—In the following, we consider non-Hermitian systems with $\mathcal{PT}$ symmetry. These systems are particularly noteworthy because their spectra only contain real numbers or conjugate pairs [1,62,63], and they can be easily implemented and maintained in our NH-PH by designing $|R\rangle$ and $\langle L|$.

We construct a pesudo-Hermitian Hamiltonian [5] satisfying that $\hat{\mathcal{P}}\hat{H}^\dagger\hat{\mathcal{P}}^{-1} = \hat{H}$ and $\hat{\mathcal{T}}\hat{H}^\dagger\hat{\mathcal{T}}^{-1} = \hat{H}$, where

$\hat{\mathcal{P}}$ and $\hat{\mathcal{T}} = e^{i\pi\hat{S}_y}\hat{K}$ are the parity symmetry and time-reversal symmetry operators, respectively. This results in the $\mathcal{PT}$ joint symmetry $\hat{\mathcal{P}}\hat{\mathcal{T}}\hat{H}(\hat{\mathcal{P}}\hat{\mathcal{T}})^{-1} = \hat{H}$. Meanwhile, the above condition requires that the ground state of H and $H^\dagger$ be connected by similar transformations $\hat{\mathcal{P}}$ or $\hat{\mathcal{T}}$. To construct such a non-Hermitian Hamiltonian, we need a TI MPS that does not preserve $\hat{\mathcal{P}}$ or $\hat{\mathcal{T}}$ symmetry itself, but satisfies the joint symmetry condition $\sum_j (e^{-i\pi\hat{S}_y})_{i,j}(A^{[j]})^* \propto M^{-1}(A^{[i]})^{\mathrm{T}}M$. Here, $\hat{\mathcal{P}}$ is realized by exchanging two virtual indices for a TI MPS, and M is an arbitrary gauge on the virtual indices of the right ground state $|R\rangle$. The left ground state is chosen as $|L\rangle = \hat{\mathcal{P}}|R\rangle$, whose tensors are given by $A'^{[i]} = (A^{[i]})^{\mathrm{T}}$.

Asymmetric AKLT model.—We use the asymmetric AKLT state as the right ground state $|R\rangle = |\Phi_\mu\rangle$ [48], which satisfies the aforementioned conditions. This state can be represented by an MPS with the following nonzero elements:

$$A^{[1]}_{\mu,\downarrow\uparrow} = -\sqrt{\mu}, \qquad A^{[-1]}_{\mu,\uparrow\downarrow} = \sqrt{\mu},$$
$$A^{[0]}_{\mu,\uparrow\uparrow} = 1/\sqrt{2}, \qquad A^{[0]}_{\mu,\downarrow\downarrow} = -\mu/\sqrt{2}. \tag{8}$$

Its entanglement structure is similar to that of the AKLT state, with an asymmetric underlying valence bond $|\uparrow\downarrow\rangle - \mu|\downarrow\uparrow\rangle$ tending toward one side. It is worth noting that $|L\rangle \propto |\Phi_{1/\mu}\rangle$ since their local tensors are related by a gauge transformation on virtual indices $(A^{[i]}_\mu)^{\mathrm{T}} = -\mu\hat{\sigma}_y A^{[i]}_{1/\mu}\hat{\sigma}_y$ [50].

First, we focus on the region $\mu \in [0, 1]$ for simplicity, and will reveal the reason later. To calculate the expectation value of any observable $\langle O\rangle_{LR}$ in the thermodynamic limit, we need to evaluate the composed transfer matrix [44,45] defined in Fig. 1(g) using $A_b = A_\mu$ and $A_a = A^{\mathrm{T}}_\mu$. On the basis $\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$, we obtain

$$\mathbf{E}_{\mu,(\alpha\beta,\alpha'\beta')} = \left(\frac{1}{2}\right) \oplus \begin{pmatrix} -\frac{\mu}{2} & \mu \\ \mu & -\frac{\mu}{2} \end{pmatrix} \oplus \left(\frac{\mu^2}{2}\right), \tag{9}$$

whose eigenvalues are $\{\frac{1}{2}, -(3\mu/2), (\mu/2), (\mu^2/2)\}$ [50]. At $\mu = \frac{1}{3}$, there is a "level crossing" transition for the dominant eigenvector of $\mathbf{E}_\mu$. We will study this transition from the renormalization group (RG) perspective and conclude that it is a unique phenomenon that can only occur in non-Hermitian systems.

Implementation of a RG aims to remove short-range entanglement and study long-range patterns. This can be achieved from the fixed point of $\mathbf{E}$ via grouping infinite local tensors [52], i.e., $\mathbf{E}^\infty = \lim_{k\to\infty}(\mathbf{E}/\lambda)^k$ with λ being the dominant eigenvalue. When $\mu > \frac{1}{3}$, the fixed point transfer matrix [50]

$$\mathbf{E}^\infty_{\mu,(\alpha\beta,\alpha'\beta')}\left(\mu > \frac{1}{3}\right) = \frac{1}{2}(0) \oplus \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus (0), \tag{10}$$

is the same as that of the conventional AKLT state up to a gauge, indicating that the non-Hermitian system is in the same AKLT phase for $\mu > \frac{1}{3}$. On the contrary, $\mathbf{E}^\infty_{(\alpha\beta,\alpha'\beta')}(\mu < \frac{1}{3}) = \mathrm{Diag}\{1, 0, 0, 0\}$ is equivalent to a transfer matrix constructed from two product states, where any local observable would have a trivial expectation value. Therefore, there is a quantum phase transition from the AKLT phase to the trivial phase at $\mu_c = \frac{1}{3}$, which can be detected by a chiral order parameter that will be introduced later.

At the same time, the corresponding metric matrix $\mathbf{G}^\infty_{(\alpha\alpha',\beta\beta')}(\mu < \frac{1}{3}) = \mathrm{Diag}\{1, 0, 0, 0\}$ shown in Fig. 1(h) is not invertible, implying that the ability to biorthogonalize the local Hilbert spaces $\mathcal{H}^\infty_R$ and $\mathcal{H}^\infty_L$ will be destroyed during the RG process. Thus, it is impossible to create a projector-form NH-PH for $k \to \infty$, even if $\mathbf{G}$ is invertible for finite k. In summary, this new kind of phase transition without a Hermitian counterpart originates from the mismatch between the left and right ground states at the RG fixed point.

Chiral order and SPT order.—The asymmetric underlying valence bonds $|\uparrow\downarrow\rangle - \mu|\downarrow\uparrow\rangle$ and $-\mu|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ tend in opposite directions in $|R\rangle$ and $|L\rangle$, implying an interesting chiral property. To detect this chiral order, two non-Hermitian order parameters are introduced $\hat{O}_{\mathrm{left}} = \frac{1}{2}\hat{S}^+_i\hat{S}^-_{i+1}$ and $\hat{O}_{\mathrm{right}} = \frac{1}{2}\hat{S}^-_i\hat{S}^+_{i+1}$. The chiral order parameter is then defined as $\hat{O}_{\mathrm{chiral}} = \hat{O}_{\mathrm{right}} - \hat{O}_{\mathrm{left}}$. As a comparison, we also consider $\hat{O}_{\mathrm{AF}} = \hat{S}^z_i\hat{S}^z_{i+1}$, which is commonly adopted to detect the conventional antiferromagnetic (AF) order [50]. The results for the non-Hermitian case are shown in Fig. 2(a). For $\frac{1}{3} < \mu < 3$, we obtain

$$\langle\hat{O}_{\mathrm{AF}}\rangle = -\frac{4}{9}, \quad \langle\hat{O}_{\mathrm{left}}\rangle = -\frac{4\mu}{9}, \quad \langle\hat{O}_{\mathrm{right}}\rangle = -\frac{4}{9\mu}. \tag{11}$$

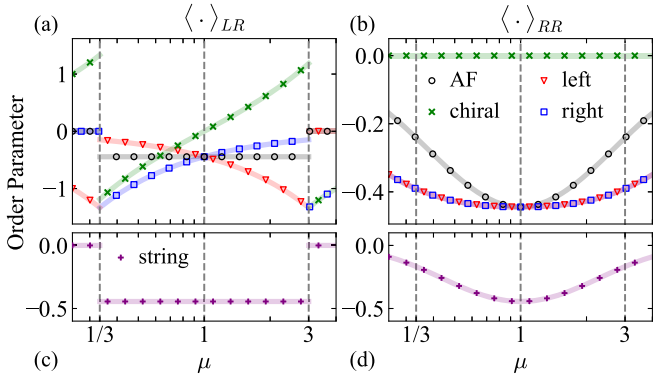

FIG. 2. Order parameters evaluated under different μ. The x axis is presented in a log scale. (a) and (c) The expectation values of chiral and string order parameters for non-Hermitian systems. (b) and (d) The same for Hermitian systems.

At the AKLT point $\mu = 1$, the state is isotropic with $\langle \hat{O}_{\mathrm{AF}} \rangle = \langle \hat{O}_{\mathrm{left}} \rangle = \langle \hat{O}_{\mathrm{right}} \rangle$. Meanwhile, $\mathrm{sgn}\langle \hat{O}_{\mathrm{chiral}} \rangle$ changes when μ passes by 1, demonstrating the chiral property of different directions. In contrast, if we choose $|L\rangle = |R\rangle$ in Fig. 2(b), $\langle \hat{O}_{\mathrm{chiral}} \rangle = 0$ for all values of μ. This is because the chiral order parameter $\hat{O}_{\mathrm{chiral}}$ is anti-Hermitian, meaning that $\Re(\langle \psi | \hat{O}_{\mathrm{chiral}} | \psi \rangle) = 0$ for any $|\psi\rangle$. As a result, such a nontrivial chiral order cannot be realized in Hermitian systems.

In addition, there is a duality $\mu \sim (1/\mu)$ for $H \sim H^\dagger$, which is induced by the parity operation, i.e., $|\uparrow\downarrow\rangle - \mu|\downarrow\uparrow\rangle \to -\mu|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle = -\mu(|\uparrow\downarrow\rangle - (1/\mu)|\downarrow\uparrow\rangle)$. Since H and $H^\dagger$ share the same energy spectrum, this relation directly gives the isotropic point $\mu = 1$ and explains why the transitions from nontrivial to trivial systems occur in pairs at $\mu = \frac{1}{3}$ and $\mu = 3$. For the same reason, the chiral order parameter in Fig. 2(a) is centrosymmetric.

Our system also exhibits nontrivial SPT order. We use $\hat{O}_{\mathrm{string}}(i,j) = \hat{S}_i^z (\prod_{k=i+1}^{j-1} e^{i\pi \hat{S}_k^z}) \hat{S}_j^z$, which was previously adopted for the conventional AKLT state [54,64], to detect the SPT order in our non-Hermitian system. Its expectation value can be calculated analytically in the thermodynamic limit [48,50], and the result is shown in Fig. 2(c). For $\frac{1}{3} < \mu < 3$, the system preserves perfect nondecaying string order $\langle \hat{O}_{\mathrm{string}} \rangle = -\frac{4}{9}$ for any string length, indicating that it is in the same SPT phase as the conventional AKLT model. Nevertheless, the string order vanishes for $\mu < \frac{1}{3}$ and $\mu > 3$, showing that it is similar to a trivial product state. This is consistent with previous discussions. In contrast, for the Hermitian system shown in Fig. 2(d), the string order parameter also saturates to a nonzero value for all μ [50], but the value becomes smaller as μ deviates from the AKLT point.

Parent Hamiltonian.—Here, we explicitly construct a TI non-Hermitian Hamiltonian to realize the aforementioned chiral and SPT orders with $k = 2$, i.e., with only nearest-neighbor interactions [50]

$$\begin{aligned}\hat{\Pi}(\mu)_i = {} & \frac{5}{12}\left(\frac{\mu}{2}\hat{S}_i^- \hat{S}_{i+1}^+ + \frac{1}{2\mu}\hat{S}_i^+ \hat{S}_{i+1}^- + \hat{S}_i^z \hat{S}_{i+1}^z\right) + \frac{2}{3} \\ & + \frac{1}{6}\left(\frac{\mu^2}{4}\hat{S}_i^{-2}\hat{S}_{i+1}^{+2} + \frac{1}{4\mu^2}\hat{S}_i^{+2}\hat{S}_{i+1}^{-2} - \hat{S}_i^{z2} - \hat{S}_{i+1}^{z2}\right) \\ & + \frac{1}{24}\left(\mu \hat{S}_i^{-z}\hat{S}_{i+1}^{+z} + \frac{1}{\mu}\hat{S}_i^{+z}\hat{S}_{i+1}^{-z}\right) + \frac{1}{4}\hat{S}_i^{z2}\hat{S}_{i+1}^{z2}, \end{aligned} \tag{12}$$

where $\hat{S}^{\pm z} = \hat{S}^{\pm}\hat{S}^z + \hat{S}^z\hat{S}^{\pm}$. It is obvious that $\hat{H}(\mu) = \sum_i \hat{\Pi}(\mu)_i$ does not preserve either $\hat{\mathcal{P}}$ (exchanging site i and $i+1$) or $\hat{\mathcal{T}}$ ($\hat{S}^z \to -\hat{S}^z$, $\hat{S}^+ \to -\hat{S}^-$, $\hat{S}^- \to -\hat{S}^+$) individually, but remains unchanged when $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$ are combined.

We use exact diagonalization to investigate the energy spectrum for small systems and find that the spectrum for open boundary condition is identical for all $\mu > 0$ [50], with fourfold degenerate ground states as a characteristic property of SPT [51]. We also calculate the spectrum under periodic boundary condition and show that the Hamiltonian remains gapped for a wide range of μ via finite-size scaling to $N \to \infty$, as shown in Fig. S5 in the Supplemental Material [50].

According to previous sections, the phase transitions occur at $\mu_c = \frac{1}{3}$ and $\mu_c = 3$ for $k \to \infty$. In this case, eigenvalues with negative real parts will not appear, and the invertibility of $\mathbf{G}^\infty$ is equivalent to the existence of a NH-PH with $|\Phi_\mu\rangle$ as its ground state. When using finite k, the NH-PH $\hat{H}_k(\mu)$ is still well defined, even when $\mu < \frac{1}{3}$ and $\mu > 3$, but it does not have $|\Phi_\mu\rangle$ as its unique ground state in these regions. Furthermore, the construction of NH-PHs may have unfavorable consequences, such as level crossing caused by the noncommutability of local projectors and non-Hermiticity, which shifts the critical points toward the intermediate phase for finite k.

To detect phase transitions, we generalize the modified iTEBD method [49] to analyze Hamiltonians with multisite interactions [50]. We find that $k = 2$ is sufficient to identify chiral and string orders for a wide range of μ in the intermediate phase. We evaluate the infidelity between the output state from iTEBD $|\Psi_\mu\rangle$ with $D = 12$ and the given asymmetric AKLT state $|\Phi_\mu\rangle$, which is defined as $\eta = 1 - \lim_{N\to\infty} |\langle \Phi_\mu | \Psi_\mu \rangle|^{1/N} = 1 - |\lambda_{\Phi\Psi}|$ with normalization conditions $\langle \Phi_\mu | \Phi_\mu \rangle = 1$ and $\langle \Psi_\mu | \Psi_\mu \rangle = 1$. The results shown in Figs. 3(a) and 3(b) indicate that the asymmetric AKLT state $|\Phi_\mu\rangle$ is, indeed, the ground state in the intermediate phase, but not for extreme values of μ near the regions $\mu < \frac{1}{3}$ and $\mu > 3$, although it is always a zero mode by construction. As we increase k, the critical point will converge to μ_c. Using $k = 3$ allows us to expand the region of NH-PHs and brings the critical points much closer to μ_c.

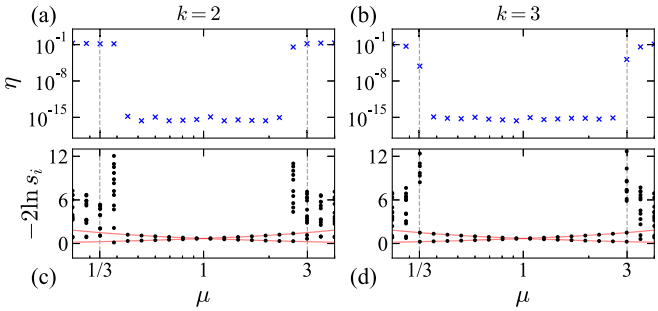

FIG. 3. Calculated ground state $|\Psi_\mu\rangle$ of $H_k(\mu)$ using the multisite iTEBD method with $D = 12$ for $k = 2$ and $k = 3$. (a),(b) Infidelity between $|\Psi_\mu\rangle$ and $|\Phi_\mu\rangle$. (c),(d) Entanglement spectrum of $|\Psi_\mu\rangle$ (black dots) and $|\Phi_\mu\rangle$ (red lines).

Moreover, we investigate the entanglement spectrum of $|\Psi_\mu\rangle$ in Figs. 3(c) and 3(d). In the intermediate phase, the ground state $|\Psi_\mu\rangle$ has only two nonzero elements in the entanglement spectrum, consistent with that of $|\Phi_\mu\rangle$ shown in red curves. On the contrary, for extreme μ, the algorithm cannot converge to a unique ground state and the entanglement spectrum is gapless. Numerical simulations for $H_k^\dagger(\mu)$, whose ground state is expected to be $|\Phi_{1/\mu}\rangle$, are shown in Fig. S7 in the Supplemental Material [50], where we obtain consistent results.

Conclusion.—In this Letter, we propose a general scheme to construct a non-Hermitian Hamiltonian from two different MPSs $\langle L|$ and $|R\rangle$ as left and right ground states. As an example, we demonstrate how to create a non-Hermitian model from asymmetric AKLT states that preserves both chiral and SPT orders, and identify a phase transition with a new origin without a Hermitian counterpart.

Our approach changes the paradigm of non-Hermitian physics, from top-down to bottom-up. We can now construct Hamiltonians with short-range interactions from states that preserve desired properties rather than extracting information from a given Hamiltonian. Compared to the conventional Hermitian parent Hamiltonian, our method offers more possibilities, as there are extra degrees of freedom in choosing two states, instead of one. It also establishes a duality between quantum states and Hamiltonians, liberates researchers from the constraints of specific systems, and provides a new perspective to study strongly correlated quantum many-body systems. We believe that there is a broader world in strongly correlated many-body systems in the non-Hermitian regime.

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