Nonequilibrium Phase Transition to Temporal Oscillations in Mean-Field Spin Models

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We propose a mean-field theory to describe the nonequilibrium phase transition to a spontaneously oscillating state in spin models. A nonequilibrium generalization of the Landau free energy is obtained from the joint distribution of the magnetization and its smoothed stochastic time derivative. The order parameter of the transition is a Hamiltonian, whose nonzero value signals the onset of oscillations. The Hamiltonian and the nonequilibrium Landau free energy are determined explicitly from the stochastic spin dynamics. The oscillating phase is also characterized by a nontrivial overlap distribution reminiscent of a continuous replica symmetry breaking, in spite of the absence of disorder. An illustration is given on an explicit kinetic mean-field spin model.

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The emergence of spontaneous oscillations at a collective scale in large assemblies of interacting units is one of the most striking features of nonequilibrium systems. Beyond the now well-understood synchronization of coupled oscillators [1,2], spontaneous oscillations also appear in diverse systems of interacting units where individual units do not oscillate in the absence of interactions, making the onset of oscillations a genuinely collective phenomenon. Such oscillations have been reported, for instance, in biochemical clocks [3–5], populations of biological cells [6,7], assemblies of active particles with nonreciprocal interactions [8,9], nonequilibrium spin systems [10–12], as well as population dynamics [13,14] and socioeconomic models [15,16].

In the thermodynamic limit, the onset of spontaneous oscillations is described by a deterministic Hopf bifurcation [17]. Yet, oscillations often occur in mesoscopic systems like biochemical clocks for which fluctuations play an important role [18], leading to a stochastic Hopf bifurcation [19,20] and to a finite coherence time of oscillations [21–25]. To provide a consistent theoretical ground, the emergence of spontaneous oscillations in large assemblies of interacting units has been characterized as a nonequilibrium thermodynamic phase transition by identifying the entropy production as a generalized thermodynamic potential whose derivative is discontinuous at the transition [4,26–33]. Similar results have also been obtained for the entropy production in population dynamics [13] and for a nonequilibrium free energy in the context of Turing pattern formation [34]. However, beyond singularities of thermodynamic potentials, the equilibrium theory of phase transitions and critical phenomena is based on the key concepts of spontaneous symmetry breaking and of associated order parameter [35]. Once the latter is identified, the generic Landau free energy can be determined unambiguously to characterize the phase transition at mean-field level. Recent nonequilibrium generalizations of Landau's theory include the description of relaxation effects [36,37] or multiple heat baths and oscillations driven by an oscillatory field [38].

In this Letter, we go beyond the thermodynamic approach to phase transitions with spontaneously emerging oscillations and show how to build a nonequilibrium generalization of the Landau free energy in a class of driven kinetic mean-field spin models based on the spontaneous breaking of spin-reversal symmetry and time-translation invariance. The generalized Landau free energy is obtained from the joint distribution of the magnetization and its smoothed stochastic time derivative, at odds with previous generalizations based on magnetization only [36–38]. Close to the phase transition to an oscillating phase, the nonequilibrium Landau free energy can be expressed in terms of a single order parameter, which is an effective Hamiltonian describing the oscillating dynamics of the magnetization. In addition, by evaluating the overlap distribution of spin configurations, we show that the oscillating phase is also characterized by an analog of the continuous replica symmetry breaking phenomenon observed in disordered systems [39].

We consider a generic class of nonequilibrium mean-field spin models with N spins $s_i \pm 1$ (and possibly auxiliary variables) and define the magnetization $m = N^{-1}\sum_{i=1}^{N} s_i$. We explore far-from-equilibrium regimes where, for large N, the magnetization $m(t)$ may exhibit oscillations, leading to a limit cycle [10–12,40–42]. In dynamical systems theory, a limit cycle may be generically described in the plane of a variable and its time derivative. We aim at building a generalized Landau theory describing finite size fluctuations around the average limit cycle. Thus, we need to characterize not only the fluctuations of magnetization, but also of its time derivative. Yet, directly considering the time derivative of $m(t)$ leads to diverging, white-noise type fluctuations that are not appropriate to build a Landau theory. Thus, rather, we aim at defining an observable attached to each microscopic configuration that would play the role of an appropriately smoothed out derivative of the magnetization. We denote as $\mathcal{C}$ the microscopic configuration of the system; $\mathcal{C}$ may correspond to the spin configuration $\mathcal{C} = (s_1, \ldots, s_N)$ [10,40], or may include additional binary variables, $\mathcal{C} = (s_1, \ldots, s_N, h_1, \ldots, h_M)$, see below. For a Markov jump dynamics with transition rate $W(\mathcal{C}'|\mathcal{C})$ from configuration $\mathcal{C}$ to configuration $\mathcal{C}'$, a stochastic derivative $\dot{m}(\mathcal{C})$ of the magnetization $m(\mathcal{C})$ can be defined as (see Supplemental Material [43])

$$\dot{m}(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} [m(\mathcal{C}') - m(\mathcal{C})] W(\mathcal{C}'|\mathcal{C}). \tag{1}$$

This definition is such that $d\langle m \rangle/dt = \langle \dot{m} \rangle$, where the average $\langle \ldots \rangle$ is defined as $\langle x \rangle = \sum_{\mathcal{C}} x(\mathcal{C}) P(\mathcal{C})$. The definition Eq. (1) of the derivative $\dot{m}$ is valid for any system size N and leads to fluctuations on a scale comparable to that of m.

To break detailed balance and possibly allow for oscillations, the configuration $\mathcal{C}$ is split into two groups of binary variables denoted as s_i^k ($k = a, b$) having different single-spin-flip dynamics (see Ref. [43] for details). These may correspond to two groups of spins in contact with different heat baths [10,40,49] or to the spin and field variables as in the explicit model described below. To detect temporal oscillations, we use, as global observables, the magnetization m and its stochastic time derivative $\dot{m}$ defined in Eq. (1). We consider the joint distribution $P_N(m, \dot{m}) = \sum_{\mathcal{C} \in \mathcal{S}(m,\dot{m})} P(\mathcal{C})$, where $\mathcal{S}(m, \dot{m})$ corresponds to the set of configurations $\mathcal{C}$ with $m(\mathcal{C}) = m$ and $\dot{m}(\mathcal{C}) = \dot{m}$. The coarse-grained transition rate corresponding to flipping any spin $s_i^k = \pm 1$ in group $k = a, b$, starting from a configuration $\mathcal{C} \in \mathcal{S}(m, \dot{m})$, is denoted as $N W_k^{\pm}(m, \dot{m})$. A global spin-reversal symmetry is assumed, yielding $W_k^{\pm}(-m, -\dot{m}) = W_k^{\mp}(m, \dot{m})$. Variations of m and $\dot{m}$ when flipping a spin $s_i^k = \pm 1$ ($k = a, b$) scale as $1/N$: $(\Delta m, \Delta \dot{m}) = \pm \mathbf{d}_k / N$. The coarse-grained master equation governing the evolution of $P_N(m, \dot{m})$ reads

$$\partial_t P_N(m, \dot{m}) = N \sum_{k,\sigma} \left[-W_k^{\sigma}(m, \dot{m}) P_N(m, \dot{m}) + W_k^{\sigma}\left((m, \dot{m}) - \frac{\sigma \mathbf{d}_k}{N} \right) P_N\left((m, \dot{m}) - \frac{\sigma \mathbf{d}_k}{N} \right) \right]. \tag{2}$$

From the theory of Markov jump processes with vanishing jump size [50], the stationary joint distribution $P_N(m, \dot{m})$ takes, for large N, a large deviation form [51]

$$P_N(m, \dot{m}) \sim \exp[-N \phi(m, \dot{m})], \tag{3}$$

which can be interpreted as a WKB approximation of the solution of the master equation (2) [50]. Using the large deviation form (3) in Eq. (2) and taking the limit $N \to \infty$, one ends up with the following equation for the steady-state rate function $\phi(m, \dot{m})$:

$$\sum_{k,\sigma} W_k^{\sigma}(m, \dot{m}) [e^{\sigma \mathbf{d}_k \cdot \nabla \phi(m,\dot{m})} - 1] = 0, \tag{4}$$

with $\nabla \phi = (\partial_m \phi, \partial_{\dot{m}} \phi)$. We are interested in an expansion of $\phi(m, \dot{m})$ close to its minimum (or minima) and, thus, assume $\nabla \phi$ to be small. At order $|\nabla \phi|^2$, Eq. (4) reads

$$\dot{m} \partial_m \phi + Y \partial_{\dot{m}} \phi + D_{11} (\partial_m \phi)^2 + D_{22} (\partial_{\dot{m}} \phi)^2 + D_{12} (\partial_m \phi)(\partial_{\dot{m}} \phi) = 0, \tag{5}$$

where Y and $\mathbf{D} = \{D_{ij}\}$ are defined as, using Eq. (1),

$$[\dot{m}, Y(m, \dot{m})] = \sum_{k,\sigma} \sigma \mathbf{d}_k W_k^{\sigma}(m, \dot{m}),$$
$$\mathbf{D}(m, \dot{m}) = \frac{1}{2} \sum_{k,\sigma} W_k^{\sigma}(m, \dot{m}) \mathbf{d}_k \cdot \mathbf{d}_k^T. \tag{6}$$

At the transition to spontaneous oscillations, $\phi(m, \dot{m})$ should change from a paraboloidlike shape to a "Mexican-hat" shape. To identify the parameter controlling the transition, we start with a quadratic approximation of $\phi(m, \dot{m})$ for small m and $\dot{m}$, and look for a change of curvature. At quadratic order in m and $\dot{m}$, Eq. (4) takes the same form as Eq. (5), but with constant coefficients $D_{ij} \geq 0$ and a linear function $Y(m, \dot{m}) = -v_0 m + u_0 \dot{m}$, assuming $v_0 > 0$ $[Y(0,0) = 0$ because $Y(-m, -\dot{m}) = -Y(m, \dot{m})]$. Assuming $\phi(m, \dot{m}) = (\gamma_1/2) m^2 + (\gamma_2/2) \dot{m}^2 + \gamma_3 m \dot{m}$ with small γ_i's close to the transition, one finds $\gamma_3 \sim \gamma_1^2 \ll \gamma_1$ and $u_0 \gamma_2 = -(D_{11} \gamma_1^2 / v_0 + D_{22} \gamma_2^2) < 0$. Thus, the sign of $\gamma_2 = \partial^2 \phi / \partial \dot{m}^2 (0,0)$ is the opposite of the sign of u_0. Hence, u_0 is the control parameter of the phase transition: $u_0 = 0$ corresponds to the critical point, and time-translation invariance is broken for $u_0 > 0$, when $\dot{m} = 0$ is no longer stable.

For $u_0 > 0$, the quadratic approximation is not enough to describe the minima of $\phi(m, \dot{m})$, and higher order terms are required. One could expand $\phi(m, \dot{m})$ as a power series in m and $\dot{m}$, but this would not work for nonanalytic ϕ [see, e.g., Eq. (12)]. Instead, we use the Hamiltonian structure close to the critical point. We no longer assume $Y(m, \dot{m})$ to be linear, and split $Y(m, \dot{m})$ into the $\dot{m}$-independent part $Y(m, 0) \equiv -V'(m)$ and a $\dot{m}$-dependent part $Y(m, \dot{m}) - Y(m, 0) \equiv \dot{m} g(m, \dot{m})$. We define the control parameter u_0 as $u_0 = \partial Y / \partial \dot{m}(0,0)$. We take $u_0 \propto \varepsilon$ with ε a small parameter. To perform a consistent small-ε expansion of Eq. (4), we assume $\nabla \phi = O(\varepsilon)$, since quadratic terms in $\nabla \phi$ have to balance the contribution in $\varepsilon \partial_{\dot{m}} \phi$ coming from the term $Y \partial_{\dot{m}} \phi$. Truncating Eq. (4) at order ε^2, one recovers

Eq. (5), where the full $(m, \dot{m})$-dependence of the coefficients is kept. At order ϵ, Eq. (5) reduces to

$$\dot{m}\partial_m \phi - V'(m)\partial_{\dot{m}}\phi = 0. \tag{7}$$

The general solution of Eq. (7) reads

$$\phi(m, \dot{m}) = f(H(m, \dot{m})) + f_0, \tag{8}$$

with

$$H(m, \dot{m}) = \frac{\dot{m}^2}{2} + V(m), \tag{9}$$

and where f is, at this stage, an arbitrary function, satisfying for convenience $f(0) = 0$, and the constant f_0 ensures that the minimal value of $\phi(m, \dot{m})$ is zero. The minimum value of $V(m)$ is set to $V = 0$, so that $H \geq 0$. $H(m, \dot{m})$ is a Hamiltonian describing the $(m, \dot{m})$ dynamics at order ε as $dm/dt = \partial H/\partial \dot{m}$, $d\dot{m}/dt = -\partial H/\partial m$, and the corresponding trajectories are iso-ϕ lines. Contributions of order ε^2 to Eq. (5) yield a condition determining the derivative $f'(H)$ [43],

$$f'(H) = -\frac{\int_{m_1}^{m_2} dm\dot{m}(m, H)g(m, \dot{m}(m, H))}{\int_{m_1}^{m_2} \frac{dm}{\dot{m}(m,H)} \nabla^T H \cdot \mathbf{D} \cdot \nabla H}, \tag{10}$$

where m_1 and m_2 are such that $V(m_1) = V(m_2) = H$ and $V(m) \leq H$ for $m_1 \leq m \leq m_2$; $\dot{m}(m, H)$ is determined from Eq. (9). Note that a related method has been used to determine nonequilibrium potentials in dissipative dynamical systems [52–54].

Equations (8) and (10) provide a convenient description of a mean-field phase transition to a state with temporal oscillations. The function $f(H)$ plays a role similar to the Landau free energy at equilibrium. Let us denote as H^* the value of H which minimizes $f(H)$. The case $H^* = 0$ corresponds to usual time-independent phases, either paramagnetic or ferromagnetic depending on whether $V(m)$ is minimum for $m = 0$ or $m \neq 0$, respectively. The case $H^* > 0$, instead, corresponds to the onset of spontaneous oscillations, where $(m, \dot{m})$ follow a limit cycle in the deterministic limit $N \to \infty$. Hence, H^* may be considered as the formal order parameter of the transition to an oscillating state. Note that although the system exhibits macroscopic temporal oscillations, the probability distribution $P_N(m, \dot{m})$ is time independent (in the long-time limit), because it describes an infinite ensemble of systems oscillating at the same frequency, but with uniformly distributed phases.

In the simple yet generic case where $V(m) = \frac{1}{2}v_0 m^2$ and $g(m, \dot{m}) = \alpha_0\varepsilon - \alpha_1 m^2 - \alpha_2 m\dot{m} - \alpha_3\dot{m}^2$, $f(H)$ takes for small H the generic form

$$f(H) = -\varepsilon aH + bH^2, \tag{11}$$

where a and b can be expressed in terms of the parameters α_i [43]. The case $\varepsilon < 0$ corresponds to a time-independent phase ($H^* = 0$), while $\varepsilon > 0$ corresponds to an oscillating phase, with $H^* = \varepsilon a/2b > 0$. Thus, one finds a continuous phase transition to temporal oscillations, with an elliptic limit cycle whose size scales as $\varepsilon^{1/2}$, i.e., $m \sim \dot{m} \sim \varepsilon^{1/2}$, or more precisely $\langle m^2 \rangle \sim \langle \dot{m}^2 \rangle \sim \varepsilon$. The two observables $\langle m^2 \rangle$ and $\langle \dot{m}^2 \rangle$ constitute the practically measurable order parameters, respectively, characterizing the paramagnetic-ferromagnetic phase transition and the onset of spontaneous oscillations. From the expression (9) of the Hamiltonian H, the oscillation period τ is given in the case $V(m) = \frac{1}{2}v_0 m^2$ by $\tau = 2\pi/\sqrt{v_0}$ and, thus, is independent of ε. Yet, the scaling with ε of the different observables may differ from the results given above. Close to a tricritical point where the paramagnetic, ferromagnetic, and oscillating phases meet, one rather finds $V(m) = \frac{1}{4}v_1 m^4$ (see explicit example below). In this case, $f(H)$ takes the nonanalytic form

$$f(H) = -\varepsilon aH + cH^{3/2}, \tag{12}$$

from Eq. (10) [43], and the scaling of H^* is now $H^* \sim \varepsilon^2$ instead of $H^* \sim \varepsilon$. As $V(m)$ is proportional to m^4, m and $\dot{m}$ have different scalings with ε: $m \sim \varepsilon^{1/2}$, while $\dot{m} \sim \varepsilon$. The limit cycle is no longer elliptic, but it flattens. This actually corresponds to a period that diverges as $\tau \sim \varepsilon^{-1/2}$.

The small fluctuations of m and $\dot{m}$ around their zero average value in the paramagnetic phase $\varepsilon < 0$ can be characterized by generalized susceptibilities $\chi_m = N\langle m^2 \rangle$ and $\chi_{\dot{m}} = N\langle \dot{m}^2 \rangle$, taking into account that $\langle m^2 \rangle \sim \langle \dot{m}^2 \rangle \sim N^{-1}$ in the paramagnetic phase. When approaching the phase transition to a limit cycle ($\varepsilon \to 0^-$), both generalized susceptibilities $\chi_{\dot{m}}$ and χ_m diverge as $|\varepsilon|^{-1}$. At the critical point ($\varepsilon = 0$), one finds a different scaling of fluctuations with N: $\langle \dot{m}^2 \rangle \sim \langle m^2 \rangle \sim N^{-1/2}$. As for the finite-size fluctuations of H, we obtain that in the paramagnetic phase, $\mathrm{var}(H) \sim N^{-2}$, whereas in the oscillating phase, $\mathrm{var}(H) \sim N^{-1}$.

The rate function is a key tool for determining which solution is the macroscopically observed one when two or more solutions are present in the deterministic description. This is the case, e.g., when $f(H) = aH - bH^2 + cH^3$, with a, b, $c > 0$. Both $H^* = 0$ and $H^* = (b + \sqrt{b^2 - 3ac}/3c) > 0$ are local minima of $f(H)$, corresponding to two solutions of the deterministic equations. The macroscopically observed solution is the one with the lowest $f(H)$. Thus, varying parameters, one observes a discontinuous transition from a paramagnetic phase ($H^* = 0$) to a limit cycle phase ($H^* > 0$). An explicit example is given below.

A fine characterization of the phase transition to an oscillating state is obtained by considering the statistics of the overlap $q_{ab} = N^{-1}\sum_{i=1}^{N} s_i^a s_i^b$ between two spin

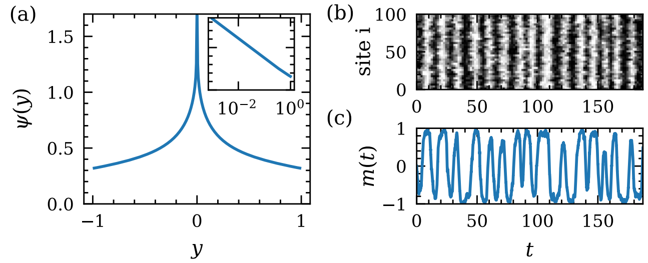

FIG. 1. (a) The scaling function $\psi(y)$ of the overlap distribution in the oscillating phase. The inset corresponds to a logarithmic x axis. (b) Evolution of the N spins with time ($s_i = +1$ in white, $s_i = -1$ in black) obtained using Monte Carlo simulations of the specific model described below, for $(T_c - T)/T_c = 0.3$, $\mu = 2$, and $N = 100$. (c) The corresponding $m(t) = N^{-1} \sum s_i$ vs time t.

configurations $\{s_i^a\}$ and $\{s_i^b\}$. Identical (opposite) configurations have an overlap $q_{ab} = 1$ ($q_{ab} = -1$), while $q_{ab} = 0$ for uncorrelated configurations. The overlap distribution $P(q)$ can be evaluated for $N \to \infty$ [43]. For $V(m) = \frac{1}{2} v_0 m^2$, we obtain for $\epsilon > 0$ (oscillating phase) the scaling form $P(q) = q_\epsilon^{-1} \psi(q/q_\epsilon)$, with $q_\epsilon = \epsilon a / b v_0$ [a and b are introduced in Eq. (11)]; the scaling function $\psi(y)$ is plotted in Fig. 1(a) (see Ref. [43] for its explicit expression). $P(q)$ has a logarithmic divergence in $q = 0$, and has a continuous support, a property usually considered as a hallmark of continuous replica symmetry breaking in disordered systems [39]. As in the latter, the presence of a nontrivial overlap distribution can be traced back to an average over many pure states [39,55].

As an explicit model, we introduce a generalization of the kinetic mean-field Ising model with ferromagnetic interactions (see, also, related models with two spin populations [10,40] or with feedback control [11]). The model involves $2N$ microscopic variables: N spins $s_i = \pm 1$ and N fields $h_i = \pm 1$. We define the magnetization $m = N^{-1} \sum_{i=1}^N s_i$ and the average field $h = N^{-1} \sum_{i=1}^N h_i$. The stochastic dynamics consists in randomly flipping a single spin s_i or a single field h_i. The flipping rates W_s and W_h depend only on m and h, $W_{s,h} = [1 + \exp(\beta \Delta E_{s,h})]^{-1}$, with $\beta = T^{-1}$ the inverse temperature and $\Delta E_{s,h}$ the variation of $E_{s,h}$ when flipping a spin s_i or a field h_i, where $E_s = -N[(J_1/2)m^2 + (J_2/2)h^2 + mh]$ and $E_h = E_s + \mu N h m$. Detailed balance is broken as soon as $\mu \neq 0$. The fluctuating derivative $\dot{m}$ determined from Eq. (1) reads $\dot{m} = -m + \tanh[\beta(J_1 m + h)]$.

Depending on (T, μ) values, the model exhibits a paramagnetic (high T), ferromagnetic (low T, low μ), or oscillating (low T, high μ) behavior. We restrict the study to $J_1 > -J_2$. An example of a phase diagram is shown in Fig. 2(a) for $J_1 = 1.4$ and $J_2 = -0.5$. The boundary of the ferromagnetic phase is obtained from the deterministic equations [43]. Other lines are obtained using the perturbative framework introduced in Eqs. (10) and (8) [43]. The three phases meet at a tricritical point (T_c, μ_c), with

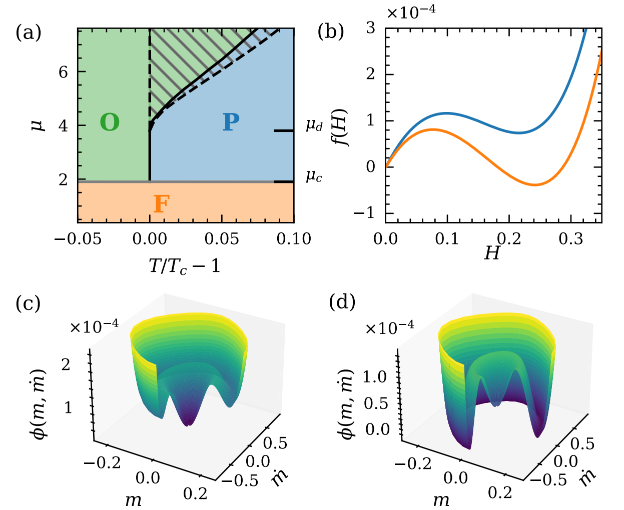

FIG. 2. (a) Phase diagram of the spin model in the $(-\varepsilon, \mu)$ plane, with $\varepsilon = (T_c - T)/T_c$, displaying the paramagnetic (P), ferromagnetic (F), and oscillating (O) phases ($J_1 = 1.4$, $J_2 = -0.5$). P and O phases coexist in the hatched area. (b) $f(H)$ for $\mu = 5.7$ and $\varepsilon = -3.5 \times 10^{-2}$ (top curve), $\varepsilon = -3.0 \times 10^{-2}$ (bottom curve). (c),(d) Corresponding rate function $\phi(m, \dot{m})$.

$T_c = (J_1 + J_2)/2$ and $\mu_c = 1 + [(J_1 - J_2)^2/4]$. For $\mu_c < \mu < \mu_d$, where $\mu_d = 1 - (J_1/J_2)$, a continuous transition from paramagnetic to oscillating states (with an elliptic limit cycle) is observed. An example of the oscillations of the N spins s_i with time and $m(t) = N^{-1} \sum_i s_i$, obtained from Monte-Carlo simulations in the oscillating phase, is plotted in Figs. 1(b) and 1(c). The rate function numerically obtained from Eq. (10) is well described by Eq. (11), with a reduced control parameter $\varepsilon = (T_c - T)/T_c$. Close to the tricritical point ($\mu \gtrsim \mu_c$), an elongated limit cycle is observed, with $m \sim \varepsilon^{1/2}$ and $\dot{m} \sim \varepsilon$. Here, the rate function is, instead, well described by the nonanalytic form of $f(H)$ obtained in Eq. (12) (the value of c is given in [43]). For $\mu > \mu_d$, a discontinuous transition from paramagnetic to oscillating states is observed. In the hatched area of Fig. 2(a), both the paramagnetic ($H^* = 0$) and limit cycle ($H^* > 0$) states are local minima of $f(H)$. The most stable solution at large but finite N is then determined as the global minimum of $f(H)$, see Fig. 2(b). It discontinuously changes from $H^* = 0$ (paramagnetic state) to $H^* > 0$ (oscillating state) when crossing the full line inside the hatched area of Fig. 2(a). The rate function $\phi(m, \dot{m})$ is plotted in Figs. 2(c) and 2(d) for the paramagnetic and oscillating states, respectively. The metastable (oscillating or paramagnetic) states are also visible. Note that the validity of the perturbative framework is limited to small $(T_c - T)/T_c$ and to either $\mu_c < \mu < \mu_d$ or small $(\mu - \mu_d)/\mu_d > 0$. A detailed study of this model, including a description of the transition between ferromagnetic and limit cycle states, will be reported elsewhere [55].

To sum up, we have shown how the Landau theory of phase transitions can be extended to describe phase transitions to an oscillating phase in nonequilibrium spin models. While previous nonequilibrium generalizations of the Landau free energy were only based on magnetization and did not address spontaneous oscillations [36–38], we defined a generalized Landau free energy as the rate function $\phi(m, \dot{m})$ associated with the joint distribution of the magnetization m and its smoothed stochastic derivative $\dot{m}$ defined in Eq. (1). The order parameter of the Landau theory is an effective Hamiltonian H, whose nonzero value indicates the presence of oscillations. The expression of $H(m, \dot{m})$ and of the nonequilibrium Landau free energy $f(H)$ can be determined explicitly from the stochastic spin dynamics. The expansion of $f(H)$ is singular close to a tricritical point where paramagnetic, ferromagnetic, and oscillating phases meet. Beyond spontaneous breaking of time translation invariance, the oscillating phase is characterized by an overlap distribution reminiscent of continuous replica symmetry breaking, although no disorder is present. Consistently with previous works [4,26–33], we also recover that the entropy production density becomes nonzero in the oscillating phase [43]. Future work will notably aim at characterizing the transition to oscillating states in finite-dimensional systems using renormalization group methods.

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