Geometrical Theory of Electromagnetic Nonreciprocity

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Recent advances in electromagnetic nonreciprocity raise the question of how to engineer the nonreciprocal electromagnetic response with geometrical approaches. In this Letter, we examine this problem by introducing generalized electromagnetic continua consisting structured points, which carry extra degrees of freedom over coordinate transformation. We show that general nonreciprocal media have a unique time-varying Riemannian metric structure with local spinning components. It is demonstrated that the nonreciprocity can be alternatively identified as the torsion tensor of a Riemann-Cartan space, which could provide analytic expressions for the magneto-optical effect and the axionic magnetoelectric coupling. Our theory not only gives a deeper insight into the fundamental understanding of electromagnetic nonreciprocity but also provides a practical principle to geometrically design nonreciprocal devices through frame transformation.

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Introduction.—The idea of studying gravitational effects with analogue curved spacetimes in the laboratory is a century-old classical topic in physics [1,2]. Such successive extension has led to a comprehensive understanding of extreme gravity phenomena such as Hawking radiation [3], Unruh effect [4] and Lorentz violation [5] in acoustics [6], Bose-Einstein condensations [7], laser pulses [8], or dielectric media [9–11]. In recent years, the analogy between light propagation in gravitational fields and in engineered optical media [12–15] was reversely applied to design novel optical devices from invisible cloaking, extreme plasmonics [16] to topology engineering [17] with the method of coordinate transformation (CT) [18–20]. Despite significant advances in nonlinearity [21], time-domain [22], non-Hermitian [23], and fully covariant formulations for noninertial observer [24], however, the current framework of transformation optics, which roots in the well-established Maxwell-Lorentz relation of the curved vacuum [13–15], has been mostly dedicated to reciprocal systems.

Reciprocity is an internal symmetry of light waves [25], which imposes fundamental constraints on transmission, reflection and emission [26–29]. In nonreciprocal systems, the transmission of waves becomes asymmetric when sources and receivers are interchanged [30,31], and the in-gap topological states can be delocalized in metamaterials with nonreciprocal hopping [32]. It enables scattering-free unidirectional photonic devices, of fundamental importance for applications in signal processing, energy harvesting and thermal-emission control. In general, nonreciprocal responses require broken time reversal symmetry [33], which is exclusively achieved through the gyrotropic magneto-optical effect of ferrites, ferromagnets, metals, or semiconductors in an external magnetic field [33]. Recently, there is an increasing interest to implement linear nonreciprocity with new paradigms including multiferroices and topological materials with axionic magnetoelectric coupling [34,35], and time-varying materials based on linear [15] or angular [36–40] mechanical momentum biasing and time modulation [30,31,41,42]. The latter has initiated a broad range of nonmagnet, compact nonreciprocal devices at optical frequencies, as well as bring novel concepts such as time interface [43], temporal Kramers-Kronig relation [44], and time-bandwidth limit breaking [45,46]. Prior to this work, nonetheless, there still lacks a general theory of transformation optics to engineer nonreciprocal responses.

In this Letter, we develop a minimal extension of transformation optics to unify different implementations of linear nonreciprocity without linear momentum bias in the continuum limit, by extending transformation media to generalized continua with inner deformable degrees of freedom (d.o.f.). Independent from CT, a unique time-varying Riemannian metric structure is introduced to characterize the local spinning of material points inside the nonreciprocal materials. From the active viewpoint, rotating directors specifying the inner structure build up an effective tetrad whose nontrivial geometry models an equivalent Riemann-Cartan geometry theory for nonreciprocity. The dual geometry, which captures the universal microscopic feature of nonreciprocity, allows designing novel nonreciprocal devices by tailoring the inner d.o.f. of material points via frame transformation. Our work unveils the geometrical origin of electromagnetic nonreciprocity beyond the limitation of the Maxwell-Lorentz relation, further widening the scope of possibilities to manipulate the fully vectorial nature of light.

Electromagnetic nonreciprocity.—In covariant electrodynamics, the generic Lagrangian of a loss-free local linear medium is $\mathcal{L} = (1/4)\chi^{ijkl}F_{ij}F_{kl}$, which defines the covariant constitutive equation [47]

$$\mathcal{H}^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}, \tag{1}$$

where the field tensor $F_{kl} = (\mathbf{E}/c, \mathbf{B})$ and the excitation tensor $\mathcal{H}^{ij} = (-c\mathbf{D}, \mathbf{H})$. The constitutive tensor χ^{ijkl}, which is antisymmetric in pairs ij and kl separately, has 36 independent components. The $(1+3)$-decomposition of Eq. (1) is equivalent to the usual bi-anisotropic constitutive relation [48]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \epsilon_0 \boldsymbol{\epsilon} & (\boldsymbol{\chi} + i\boldsymbol{\kappa})/c \\ (\boldsymbol{\chi} - i\boldsymbol{\kappa})^T/c & \mu_0 \boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix},$$

where the permittivity $\boldsymbol{\epsilon}$ and the permeability $\boldsymbol{\mu}$ are complex tensors, while the Tellegen term $\boldsymbol{\chi}$ and the chiral tensor $\boldsymbol{\kappa}$ are real by definition. In general, the antisymmetrically imaginary parts of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, which describe the gyro-electric or magnetic effects arising from spin precession at microwave and electron cyclotron orbital motion at optical frequencies in a magnetic field, and $\boldsymbol{\chi}$, which describes the magnetoelectric coupling due to orbital magnetization, are responsible for the nonreciprocal electromagnetic response of matters.

On the other hand, the constitutive tensor of a curved vacuum described by the Maxwell-Lorentz relation only contains the principle part of χ^{ijkl}: $\chi^{ijkl} = Z_0\sqrt{-g}(g^{ik}g^{jl} - g^{il}g^{jk})$, with $Z_0 = \mu_0 c$ denoting the vacuum impedance, g^{ij} the inverse of g_{ij} and $g = \det(g_{ij})$ [13,20]. In vector notation, it takes the form

$$\boldsymbol{\epsilon} = \boldsymbol{\mu} = -\frac{\sqrt{-g}}{g_{00}}[g^{ab}], \qquad \boldsymbol{\chi} = \frac{[g_{0a}]}{g_{00}}, \tag{2}$$

where g^{ab} describes the symmetric part of the electronic polarization (magnetization) which was usually assumed can be generated by coordinate transformation, while g_{0a} describes the antisymmetric nonreciprocal magnetoelectric coupling due to frame dragging. The geometric origin of other nonreciprocal constitutive terms remains unexplored. This problem is of paramount importance for materials at rest in laboratory which are assumed to be described by static spacetimes with vanishing g_{0a}. Here, we develop a geometrical theory which unifies the magneto-optical effect, axion magnetoelectric coupling and the time-varying Riemannian system carrying local spinning components.

Transformation optics with structured points.—In conventional transformation optics, the CT $\boldsymbol{x} \to \boldsymbol{x}'(\boldsymbol{x})$ creates a curved Riemannian space with the metric $\boldsymbol{g}' = \boldsymbol{J}\boldsymbol{g}\boldsymbol{J}^T$ where the constitutive tensors transform as tensor densities $\{\boldsymbol{\epsilon}', \boldsymbol{\mu}'\} = \det(\boldsymbol{J})\boldsymbol{J}\{\boldsymbol{\epsilon}, \boldsymbol{\mu}\}\boldsymbol{J}^T$ [18]. Here, $\boldsymbol{J}^a{}_\alpha = \partial x^a/\partial x'^\alpha$ is the Jacobian matrix, $\boldsymbol{g}_{ab} = \delta_{ab}$ is the flat Euclidean metric. From the perspective of classical field theory, the transformed spaces are modeled as continua comprised of infinitesimally sized geometrical points, where the geometrical information is encoded in the displacement vector $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}' - \boldsymbol{x}$. However, the three translational d.o.f. of $\boldsymbol{u}$ are less than the six independent components of a symmetric metric, indicating that the transformed medium is only a subset of the general Riemannian medium. To overcome this limitation, an anholonomic extension of transformation optics was proposed to deal with the chiral medium [49,50], where the tetrad decomposition of the metric is utilized to characterize the twisting of the local structure [15]. This is indicative of the fact that the points are endowed with tensorial physical quantities χ^{ijkl}, which originate from the structural anisotropy or inhomogeneity in molecular or artificially engineered materials.

We now elaborate on our generalized theory of transformation optics for complex electromagnetic materials which contain chiral [50] and nonreciprocal media as special cases by breaking space-inversion and time-reversal symmetries, respectively. To incorporate inner structures into transformation optics, we generalize the electromagnetic continuum to be a collection of structured points with inner deformable d.o.f. In analogy to the mechanical microcontinuum theory [51], each point is in itself a deformable medium. To reconcile the concept of deformable points (which implies finite size) with the continuum hypothesis, each material point is represented by a geometric point P and three direction vectors $\boldsymbol{e}_\alpha$, $(\alpha = 1, 2, 3)$ attached to P. Here, the position vector of P labels its spatial coordinate $\boldsymbol{x}$ and $\boldsymbol{e}_\alpha$ describe the relative positions of the inner structure contained in the material point, which can deform arbitrarily. In essence, this picture is compatible with the concept of metamaterials, where the metamolecules, represented by geometrical points in the homogenized continuum limit, can be deformed independently from adjacent points. Compared to their detailed structural geometries, we are interested to the spatial transformation of the point microconstituents. To describe the intrinsic deformation of these material points, we introduce the frame deformation for the directors by $\boldsymbol{e}'_\alpha = \boldsymbol{e}'_\alpha(\boldsymbol{x}, t, \boldsymbol{e}_\alpha)$ in addition to CT at each P [51]. Because the points are considered to be infinitesimally small, we only consider the linear approximation on $\boldsymbol{e}_\alpha$,

$$\boldsymbol{e}'_\alpha = \boldsymbol{F}^a{}_\alpha \boldsymbol{e}_a. \tag{3}$$

Here, the frame transformation $\boldsymbol{F} = [F^a{}_\alpha(\boldsymbol{x}, t)]$ is a nonsingular matrix with positive determinant $\det(\boldsymbol{F}) > 0$. In general, $\boldsymbol{F}$ has nine extra d.o.f. over CT, which account for the local rotation and stretch of points. To fully characterize the deformation, it requires to consider the transformation of both the spatial coordinates and directors of the material points. To be specific, we introduce the total transformation of the point P by

$$\boldsymbol{M} = \boldsymbol{F}\boldsymbol{J}, \tag{4}$$

where $\boldsymbol{F}$ and $\boldsymbol{J}$ denote the transformation matrices of frame deformation and coordinate transformation, respectively. Without loss of generality, we choose $\boldsymbol{e}_\alpha$ as the local basis (tetrad fields) which span the tangent space for all tensorial quantities. Therefore, Eq. (3) can be interpreted as the active transformation of basis vectors. Replacing $\boldsymbol{J}$ with $\boldsymbol{M}$, we obtain the generalized transformation media

$$\{\boldsymbol{\varepsilon}', \boldsymbol{\mu}'\} = \det(\boldsymbol{M})\boldsymbol{M}\{\boldsymbol{\varepsilon}, \boldsymbol{\mu}\}\boldsymbol{M}^T. \tag{5}$$

Equation (5) is the first main result of this work, which is a direct generalization of usual transformation optics. It allows to create complex media for the full control of light with the composite transformation: First, a CT defines a transformation medium to engineer light rays. Subsequently, the frame deformation introduce further local manipulation for the polarization of light. The final transformed medium is described by a spatial metric tensor

$$\boldsymbol{g}' = \boldsymbol{M}\boldsymbol{g}\boldsymbol{M}^T. \tag{6}$$

By making use the matrix's polar decomposition, the frame deformation can be decomposed into

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{S}, \quad \text{with} \quad \boldsymbol{S}^2 = \boldsymbol{F}^T\boldsymbol{F}, \tag{7}$$

where the orthogonal matrix $\boldsymbol{R}$ and the symmetric matrix $\boldsymbol{S}$ describe the local rotation and stretch of the material points, respectively. For simplicity, we only consider frame rotation where the material particles can be regarded as points carrying oriented rigid triads. A full description of the continuum requires to specific the orientation ϕ^i of each point besides the displacement field u_i. In principle, we have the deformation measure distortion β_{ij} and contortion κ_{ijk} [52,53],

$$\beta_{ij} = \partial_i u_j - \omega_{ij}, \qquad \kappa_{ijk} = -\partial_i \phi_{jk}, \tag{8}$$

where the bivector $\omega_{ij} = -\omega_{ji} = \frac{1}{2}\epsilon_{ijk}\phi^k$ is dual to ϕ^i, and $\kappa_{ijk} = -\kappa_{ikj}$. For vanishing frame rotation, the points and their relative distances completely determine the geometry, and the distortion reduces to the strain $\epsilon_{ij} = \frac{1}{2}(\beta_{ij} + \beta_{ji})$, which measures the difference $g_{ij} = \delta_{ij} + \epsilon_{ij}$. With nonvanishing ω_{ij}, the strain becomes asymmetric, and the antisymmetric part gives to the relative rotation: $\beta_{[ij]} = \frac{1}{2}(\beta_{ij} - \beta_{ji}) = \omega_{ij} - R_{ij}$, where $R_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i)$ denotes the frame rotation associated with CT [54]. Figure 1(a) schematically depicts the oriented continuum with purely local rotation. Nematic liquid crystals provide an exemplification of the oriented media where the optical anisotropy associated with the rodlike molecules define the directors [50].

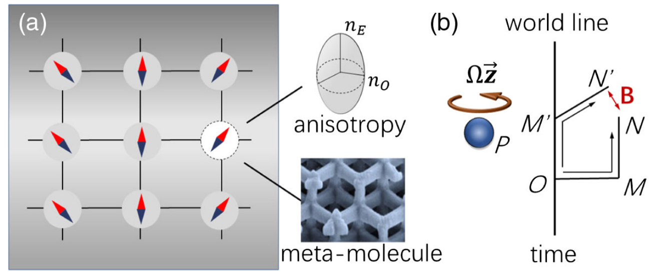

FIG. 1. (a) Schematics of the continuum comprised of structured points. (b) Torsion is introduced by the Burgers circuit associated with rotating directors.

Electrodynamics of extended spinning objects.—Before applying our theory to bulk nonreciprocal materials, we first study the electrodynamics of isolated spinning neutral objects. It has been shown that mechanical rotation leads to several chirality-dependent scattering phenomena, such as the frequency shift for circularly polarized light in the rotational Doppler effect [36,37] and the polarization rotation in the rotational photon drag effect [38,39]. Here, we assume the object, which has scalar ϵ and μ, is spinning along z axis with a constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$. Unlike previous works [39,40], we consider the spinning object as a meta-molecule of a metamaterial in the infinitesimal limit. The time-reversal symmetry breaking requires taking into account the effect of relativistic frame dragging, which was not considered in [50]. Using Lorentz boost from the object rest frame rotating at instantaneous velocity $\mathbf{v} = (-\Omega y, \Omega x, 0)^{\mathrm{T}}$ to the lab system, the constitutive relations at low velocity approximation $v/c \ll 1$ are given by [33]

$$\begin{aligned}\mathbf{D} &= \epsilon_0\epsilon\mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c^2} \times \mathbf{H}, \\ \mathbf{B} &= \mu_0\mu\mathbf{H} - (\epsilon\mu - 1)\frac{\mathbf{v}}{c^2} \times \mathbf{E}.\end{aligned} \tag{9}$$

The steady instantaneous velocity satisfies $\nabla \cdot \mathbf{v} = 0$, $\nabla \times \mathbf{v} = 2\boldsymbol{\Omega}$. Plugging (9) into Maxwell's equations and after some vector algebra, we obtain [53]

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \frac{\epsilon\mu - 1}{c\epsilon}\mathcal{K}(\mathbf{H}) \tag{10}$$

to first order of v/c, where

$$\mathcal{K}(\mathbf{H}) = \nabla(\mathbf{v} \cdot \mathbf{H}) - 2\mathbf{H} \times \boldsymbol{\Omega} - (\mathbf{H} \cdot \nabla)\mathbf{v}. \tag{11}$$

For simplicity, we consider the electromagnetic field propagating along the z direction. From (9), $\mathbf{E}$ and $\mathbf{H}$ have distant-dependent longitudinal components of order $\mathcal{O}(v/c)$. Under these considerations, the last term in (10) is negligible of order $\mathcal{O}(v/c)$. Taking the curl of (10) and dropping terms due to Coriolis and centrifugal forces of order $\mathcal{O}(\mathbf{\Omega}^2)$ yield [53]

$$\nabla^2\mathbf{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + 2\frac{\epsilon\mu - 1}{c^2}\mathbf{\Omega}\times\frac{\partial\mathbf{E}}{\partial t} = 0. \tag{12}$$

Equation (12) is identical in form to the wave equation in a magneto-optical material described by

$$\mathbf{D} = \epsilon_0\epsilon\mathbf{E} + i\frac{1}{\omega}\mathbf{\Omega}\times\mathbf{E}, \tag{13}$$

where the rotation vector plays the role of a pseudo-magnetic field. Proceeding as above for the magnetic field gives rise to the gyromagnetic constitutive relation $\mathbf{B} = \mu_0\boldsymbol{\mu}\mathbf{H} - i\omega^{-1}\mathbf{\Omega}\times\mathbf{H}$. Notably, Eq. (12) is consistent with the quantum mechanics analysis for certain rotating plasmonic nanoparticles whose polarizability takes the same form as static magnetized particles in an external magnetic field [40] and the homogenized model of the metamaterial with spinning inclusions [55]. This effect illustrates that the equivalence between the mechanical rotation and the magnetic field applies not only to mechanics effects of extended objects in the Einstein–de Haas effect [56] and the reversed Barnett effect [57], but also for the constitutive relations of matters consisting of spinning components.

Nonreciprocity from time-varying Riemannian metric.—The gyrotropic response (13) suggests that spinning objects can be regarded as material points of the nonreciprocal medium in the infinitesimal limit. Applying this idea to an electromagnetic continuum with each point carrying a spinning d.o.f. leads to a Riemannian geometry description for the gyrotropic medium. In the spirit of transformation optics, we consider a thought experiment to create a nonreciprocal medium from a reciprocal medium, such as a dielectric with static structural inhomogeneity. It could be an anisotropic medium with $\boldsymbol{\varepsilon}_{ab}(\boldsymbol{\mu}_{ab})$ or a metamaterial with reciprocal homogenized constitutive parameters. At each point, we set up a local orthonormal frame $\boldsymbol{e}_a(x)$ which aligns with the global Cartesian axes at the initial moment. To break time reversal symmetry, we consider the inner structure carrying its local frame spins with angular frequency $\mathbf{\Omega}$ around a rotating axis, which relates with the Cartesian basis by $\boldsymbol{e}_\alpha(\boldsymbol{r},t) = R^a_\alpha(\boldsymbol{r},t)\boldsymbol{e}_a$. It is reasonable to assume that the local frames at neighboring points connect smoothly. According to Eq. (5), the proposed nonreciprocal medium is described by time-varying real symmetric parameters

$$\{\boldsymbol{\varepsilon}'(t), \boldsymbol{\mu}'(t)\} = \det(\boldsymbol{R})\boldsymbol{R}\{\boldsymbol{\varepsilon}, \boldsymbol{\mu}\}\boldsymbol{R}^T. \tag{14}$$

In this regard, the transformed space is a Riemannian space with a time-varying metric tensor $\boldsymbol{g}'(t) = \boldsymbol{R}\boldsymbol{g}\boldsymbol{R}^T$, where the associated spinning d.o.f. produce the imaginary antisymmetric parts of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. Equation (14) is another main result of this work.

Electromagnetism in arbitrary noninertial frame and axionic magnetoelectric coupling.—Thus far we have proposed nonreciprocal transformation media with spinning components, we now express the Maxwell's equations in the local frame spanned by $\boldsymbol{e}_\alpha$. Because $\boldsymbol{e}_\alpha$ are not coordinate basis in general, they have nonvanishing Lie brackets: $[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = C^\gamma{}_{\alpha\beta}\boldsymbol{e}_\gamma$ where the antisymmetric structure constant $C^\gamma{}_{\alpha\beta} = e^\gamma{}_a(\partial_\alpha e^a{}_\beta - \partial_\beta e^a{}_\alpha)$ [47]. The nonvanishing $C^\gamma{}_{\alpha\beta}$ characterizes the closed failure of order $O(\epsilon^2)$ of the infinitesimal parallelogram along the integral curves of $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_\beta$. In the general nonorthonormal basis, the affine connection of the Riemannian space is given by $\Gamma^\gamma{}_{\alpha\beta} = \{^{\gamma}_{\alpha\beta}\} + H^\gamma{}_{\alpha\beta}$ [47], where $\{^{\gamma}_{\alpha\beta}\}$ is the Christoffel symbol and $H^\gamma{}_{\alpha\beta} = \frac{1}{2}(C^\gamma{}_{\alpha\beta} + C_\alpha{}^\gamma{}_\beta + C_\beta{}^\gamma{}_\alpha)$. By replacing ordinary partial derivatives by the covariant ones $\partial_\alpha \to \nabla_\alpha$, the Maxwell's equations in an arbitrary anholonomic frame read

$$\tilde{\nabla}_\beta\mathcal{H}^{\alpha\beta} - \frac{1}{2}C_{\beta\gamma}{}^\alpha\mathcal{H}^{\gamma\beta} + \frac{1}{2}C_{\beta\gamma}{}^\gamma\mathcal{H}^{\alpha\beta} = \mathcal{J}^\alpha, \tag{15}$$

$$\tilde{\nabla}_{[\alpha}F_{\beta\gamma]} - C_{[\alpha\beta}{}^\delta F_{\gamma]\delta} = 0, \tag{16}$$

where $\tilde{\nabla}_\alpha$ is the covariant derivative associated with $\{^{\gamma}_{\alpha\beta}\}$, $[\ldots]$ denotes the antisymmetrization of tensors. Note that in a coordinate basis the C terms vanish, Eqs. (15) and (16) collapse into the usual covariant Maxwell's equations in the Riemannian spacetime. In an orthonormal frame $g_{\alpha\beta}$ are constant, $\Gamma^\gamma{}_{\alpha\beta}$ is antisymmetric in its lower indices due to the vanishing Christoffel symbol, and above equations become the anholonomic Maxwell's equations in moving frames [47]. Traditionally, the noncoordinate basis is used to construct the proper reference frame for local accelerating observers with $C^\gamma_{\alpha\beta}$ representing the noninertial effect. As mentioned earlier, however, we adopt an active perspective where it specifies the real geometrical structure from the orientation of the local anisotropy at each material point during the local spatial transformation. As depicted in Fig. 1(b), the active viewpoint coincides with the non-Riemannian geometry theory of the dislocation continuum, where the spinning of the local atomic lattice axes defines the antisymmetric connection and the dislocation density measures the torsion tensor of the effective Riemann-Cartan space [58–61]. By identifying $C^\gamma_{\alpha\beta}$ as the torsion tensor, Eqs. (15) and (16) recover the Maxwell's equations in a Riemann-Cartan space [62]. The above program akin to the usual minimal coupling procedure to treat wave equations in a curved space. However, Eq. (16) is not compatible with the U(1) gauge invariance [63,64]. To keep the gauge invariance, we choose the semiminimal coupling where the definition of the field tensor is the same as in the Riemannian space, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Thereby, the homogeneous Gauss-Faraday equation is irrelevant to torsion, $\partial_{[\alpha} F_{\beta\gamma]} = 0$.

Equation (15) contains the field strengths as well as the geometrical information of the spacetime where the additional constitutive terms are interpreted as the torsion tensor. As a simplest example, we consider $\boldsymbol{e}_{\hat{0}} = \boldsymbol{e}_0, \boldsymbol{e}_{\hat{3}} = \boldsymbol{e}_3$ and the frame transformation

$$\begin{pmatrix} \boldsymbol{e}_{\hat{1}} \\ \boldsymbol{e}_{\hat{2}} \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \tag{17}$$

where the hat emphasizes it is a local orthonormal basis. Obviously, the only nonvanishing structural constants are $C^2_{01} = C^1_{20} = \Omega$. Without loss of generality, Eq. (15) can be written as

$$\partial_\alpha \mathcal{H}^{\alpha\beta} = \mathcal{J}^\beta, \tag{18}$$

where

$$\mathcal{H}^{\alpha\beta} = \sqrt{-g} F^{\alpha\beta} + \frac{1}{2} x^\alpha C^{\beta\gamma\delta} F_{\gamma\delta}. \tag{19}$$

Equation (18) is identical to the material form of the Maxwell equation in Minkowski spacetime, which allows interpreting the torsional space as a transparent medium described by (19) in a flat system. This result, obtained by a purely geometrical approach based on the nonminimal coupling between torsion and electromagnetic field, is consistent with the quantum computation of the vacuum polarization effect in a torsional space [64,65]. By introducing a pseudoscalar $\theta(x)$, the torsion addition in (19) can be expressed as an axion term corresponding to the axion Lagrangian $\Delta\mathcal{L} = \frac{1}{2}\theta\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\delta\gamma} = -2\theta \mathbf{E}\cdot\mathbf{B}$, where $\epsilon^{\alpha\beta\gamma\delta}\theta = x^\alpha C^{\beta\gamma\delta}$ [35,66,67]. The axion angle depends linearly on the spacetime coordinate: $\theta(x) = \Omega_\mu x^\mu$. For the rotating frame, the spatial components of $C^{ijk}(i,j,k = 1,2,3)$ vanishes identically and θ depends only on the spatial position $\theta(\boldsymbol{r}) = \boldsymbol{\Omega}\cdot\boldsymbol{x}$. Therefore, the electrodynamics of the torsional medium describes the electromagnetic response of Weyl semimetals where torsion measures the separation of Weyl nodes [67]. It is known that this type of magnetoelectric coupling can be reduced to the magneto-optical response described by Eq. (13) [33]. Consequently, torsion is also interpreted as the gyrotropic vector of magneto-optical materials.

Asymmetric energy-momentum tensor.—Lastly, we consider the symmetry of the electromagnetic energy-momentum tensor $\Theta^{\alpha\beta}$. By definition, the Minkowski energy-momentum tensor of a conventional transformation medium, $\Theta^{\alpha\beta} = (2/\sqrt{-g})\delta\mathcal{L}/\delta g_{\alpha\beta}$, is symmetric. However, direct evaluation of $\Theta^{\alpha\beta}$ from Eq. (19) gives rise to the gauge-dependent form

$$\Theta^{\alpha\beta} = -F^{\alpha\rho} F^\beta{}_\rho + g^{\alpha\beta} F^{\rho\sigma} F_{\rho\sigma} + \frac{\Omega^\beta}{4} \epsilon^{\alpha\rho\gamma\lambda} A_\rho F_{\gamma\lambda}. \tag{20}$$

It is interesting that torsion does not enter the energy density $E = \Theta^{00}$. Moreover, the torsion term renders the Maxwell stress tensor asymmetric, $\Theta^{ij} \neq \Theta^{ji} (i,j = 1,2,3)$. To avoid the gauge dependence, we adopt the canonical form $\Theta^{ij} = E^i D^j + H^i B^j - \frac{1}{2}\delta^{ij}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$. From the conservation of total angular momentum, the antisymmetric part of Θ^{ij} gives rise to the rate of change of the spin angular momentum

$$\partial_k \mathcal{S}^{ijk} = 2\Theta^{[ij]} = \frac{1}{2}\epsilon^{ijk}\Omega_k |\mathbf{E}|^2, \tag{21}$$

where the spin angular momentum density $\mathcal{S}^{ijk} = \epsilon^{ijk} E^j A^k$. For infinitely bulk media, the asymmetry of Θ^{ij} reflects the presence of the internal torque for material points, which produces the Faraday rotation.

Concluding remarks.—We emphasize that the analysis on the electrodynamics of torsional spaces is not applicable to gyromagnetic media because of the lack of a covariant formalism for gyromagnetic response. There might require a dual electromagnetism theory parallel to axion electrodynamics [68,69]. In addition, while our work focuses on nanophotonics, our theory may be extended to analyze electromagnetic phenomena at large scales such as exploring the spacetime structure with cosmic microwave background radiation [70].

In summary, we have extended the theory of transformation optics to generalized electromagnetic continua consisting of structured material points. Our theory provides a unified geometrical description for typical linear nonreciprocal media regardless of their different physical origins. Geometrically, the nonreciprocal medium is interpreted as either a time-varying Riemannian space with spinning components or, equally, a static Riemann-Cartan space where torsion is mimicked by the spinning material points. Our theory is a direct generalization of covariant electrodynamics in the Riemannian spacetime. Together with engineering light rays with CT, the developed formalism based on frame transformation provides a practical strategy to design novel nonreciprocal electromagnetic devices by controlling the full vectorial d.o.f. of light. Our theory may be of interest to systematically create torsional spaces to manipulate the spin of other classical waves such as elastic and acoustic waves [71].

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