# Dispersive $\pi\pi \to K\bar{K}$ Amplitude and Giant $CP$ Violation in $B$ to Three Light-Meson Decays at LHCb

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The LHCb collaboration has recently reported the largest $CP$ violation effect from a single amplitude, as well as other giant $CP$ asymmetries in several $B$-meson decays into three charmless light mesons. It is also claimed that this is predominantly due to $\pi\pi \to K\bar{K}$ rescattering in the final state, particularly in the 1 to 1.5 GeV region. In these analyses the $\pi\pi \to K\bar{K}$ amplitude is by default estimated from the $\pi\pi$ elastic scattering amplitude and does not describe the existing $\pi\pi \to K\bar{K}$ scattering data. Here we show how the recent model-independent dispersive analysis of $\pi\pi \to K\bar{K}$ data can be easily implemented in the LHCb formalism. This leads to a more accurate description of the asymmetry, while being consistent with the measured scattering amplitude and confirming the prominent role of hadronic final state interactions, paving the way for more elaborated analyses.

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In a series of recent works the LHCb Collaboration has reported the observation of direct $CP$ symmetry violation (CPV) in charged $B$-meson charmless decays into three pseudoscalar mesons. The relevance of these processes is that the observation of CPV requires the interference between a "weak" phase, which changes sign for opposite $CP$ states, with a $CP$ invariant "strong" phase. While the source of the first one is well understood from the Cabibbo-Kobayashi-Maskawa matrix [1,2] of the standard model, and can be easily calculated in standard perturbation theory, the latter is much more troublesome due to its strong character. Moreover, it is long thought that it can be produced from short-distance quark-level contributions and/or long-distance hadronic final state interactions (FSI) [3–5]. The relevance of CPV in three-body decays (see Ref. [6] for a recent review) is that it can be studied, not only from the total or integrated charge asymmetry, which is a single number, but from the phase-space distribution of the decay, which is a function that depends on two energy variables and is much richer in structure. Moreover, the rescattering of final state hadrons is dominated by resonances that can yield huge variations throughout the phase-space distributions. The energy dependence of these distributions may allow disentangling different sources of strong phases in CPV.

In particular, CPV both in the local and integrated phase-space asymmetries between the opposite charge $B^\pm \to K^\pm\pi^+\pi^-$ and $B^\pm \to K^\pm K^+K^-$ decays was first observed at LHCb in 2013 [7], followed by the observation of the corresponding asymmetries in $B^\pm \to \pi^\pm\pi^+\pi^-$ and $B^\pm \to \pi^\pm K^+K^-$ [8]. These $B \to 3M$ analyses were soon superseded with larger statistical samples in 2014 [9]. Whereas integrated asymmetries came up of the order of 2 to 12%, it was shown that local asymmetries could be very large, when looking at localized regions in the Dalitz plots. The collaboration suggested that FSI may be a determinant factor for this giant $CP$ violation. In particular, asymmetries became very large when the Dalitz plot was projected on the invariant mass of the opposite-charged final mesons, and in the 1 to 1.5 GeV range, which was associated with the inelastic isoscalar $S$-wave $\pi^+\pi^- \leftrightarrow K^+K^-$ FSI. Even more accurate CPV results have just been presented [10], still supporting the relevance of FSI, which could also be important for CPV in charm decays [11].

It is only very recently that the LHCb has performed the full amplitude analyses of their run I data on $B^\pm \to \pi^\pm K^+K^-$ [12] and $B^+ \to \pi^+\pi^+\pi^-$ [13,14]. Their most striking feature is that, for $B^\pm \to \pi^\pm K^+K^-$, the collaboration claims $\pi\pi \to K\bar{K}$ $S$-wave rescattering to have "the largest CP asymmetry reported to date for a single amplitude of $(-66 \pm 4 \pm 2)\%$." For $B^+ \to \pi^+\pi^+\pi^-$ a similarly large value ~45%, is also found.

The inelastic FSI framework for CPV dates back to Wolfenstein and Suzuki in [3–5]. The LHCb amplitude analyses [12–14] used a very appealing particularization to $B \to 3M$ in [15,16] (Other models are also used for $B \to 3\pi$ in [13,14]). This model is relevant in the 1 to 1.5 GeV region, where final-state multiplicity is low and the $CPT$ constraint is more enlightening. It also assumes that only two particles rescatter whereas the third is a spectator. In this formulation, the $\pi^+\pi^- \leftrightarrow K^+K^-$ complex isoscalar partial $S$ wave should be described by its modulus and phase $\delta_{\pi\pi K\bar{K}}$. However, in the [15,16] formalism and its implementation by LHCb [12–14], or modification by [17,18], the $\pi\pi \to K\bar{K}$ interaction is not used. Instead, it is assumed that $K\bar{K}$ and $\pi\pi$ are the only available states and, in addition, the phase is crudely estimated as $\delta_{\pi\pi KK} \sim 2\delta_{\pi\pi\pi\pi}$ whereas its elasticity is obtained from that of $\pi\pi$ scattering. Of course, in this way they could use the model-independent dispersive analysis of $\pi\pi$ scattering data in [19]. It is true that meson-meson scattering experiments are plagued with systematic errors and have been usually described with crude models (see Refs. [20,21] for reviews). Model-independent parameterizations can only be obtained through dispersive methods, whose relevance has been repeatedly emphasized in the context of heavy particle hadronic decays [22,23]. However, as we will show below this estimate does not reproduce the $\pi\pi \to K\bar{K}$ data. Moreover, it violates Watson's theorem [24], which implies that at $KK$ threshold, and for partial waves with given total angular momentum and isospin, $\delta_{\pi\pi KK} = \delta_{\pi\pi\pi\pi}$, without that factor of 2. Furthermore, the poorly known $\pi\pi$ elasticity and the factor of 2 amplifying the already large $\delta_{\pi\pi\pi\pi}$ error gives rise to huge uncertainties in the description of the asymmetry FSI. Despite this treatment may provide a hint of the relevance of FSI, it definitely calls for an implementation using the realistic $\pi\pi \to K\bar{K}$ amplitude, consistent with data and fundamental constraints.

Fortunately, a dispersively constrained $\pi\pi \to K\bar{K}$ data analysis has become recently available [21,25]. It provides precise and model-independent parameterizations of phases and moduli for several partial waves, including the isoscalar $S$ wave. Here we show how to implement easily this dispersive parameterization within the formalism presently used by LHCb, proposed in [15,16], and how it improves dramatically the accuracy of the FSI contribution to these CPV asymmetries. Moreover, it unveils hadronic structures that were masked in the uncertainties, while providing a sound support for the FSI prominent role in these giant $CP$ violations. Implementing these amplitudes in future LHCb analyses will provide much more precise descriptions and may allow us to understand further hadronic details otherwise swamped by the huge uncertainties of the present estimates.

Let us briefly recall the FSI formalism in [15,16], with simplified notation and assuming $CPT$ conservation. Consider the $\mathcal{A}^- = \langle\lambda|\mathcal{H}_W|h\rangle$ decay amplitudes of a meson $h$ into a hadron state $\lambda$ and its $CP$ conjugated process $\mathcal{A}^+ = \langle\bar{\lambda}|\mathcal{H}_W|\bar{h}\rangle$. Here $\mathcal{H}_W$ is the electroweak Hamiltonian. Customarily, we write $\mathcal{A}^\pm = A_\lambda + B_\lambda e^{\pm i\gamma}$, where $A_\lambda, B_\lambda$ are $CP$ invariant and only the weak phase $\gamma$ sign changes under $CP$. However, when the final state $\lambda$ is coupled to other physically accessible states $\lambda'$, we could consider that it has been produced directly from the source or via another intermediate state. Formally, to the lowest order effect due to FSI, we write [15,18]

$$\mathcal{A}^\pm_{\rm LO} = A_\lambda + B_\lambda e^{\pm i\gamma} + i\sum_{\lambda'} \hat{f}_{\lambda'\lambda}(A_{\lambda'} + B_{\lambda'} e^{\pm i\gamma}), \quad (1)$$

where $\hat{f}_{\lambda\lambda'}$ is the two-body scattering partial wave related to the $S$ matrix by $S_{\lambda\lambda'} = \delta_{\lambda\lambda'} + 2i\hat{f}_{\lambda\lambda'}$. The factor of 2 in front of $\hat{f}$, absent in [15], is standard for partial waves and essential [4,18] to arrive to Eq. (1). Now $A_\lambda$ and $B_\lambda$ are understood as decay amplitudes without FSI. The above expression is formal and $\lambda'$ represents the two particles that rescatter in a definite spin and isospin state. $CP$ asymmetries are then defined through $\Delta\Gamma_\lambda = \Gamma_{h\to\lambda} - \Gamma_{\bar{h}\to\bar{\lambda}}$ with, generically, $\Gamma = |\mathcal{A}_{\rm LO}|^2$.

These processes involve three final mesons, but there is strong evidence that, at least in some regions of phase space, the first two-body scattering largely dominates the FSI [26] and the other meson acts as a spectator. Four or more meson intermediate states are negligible below CM energies of 1 GeV and relatively small up to roughly 1.5 GeV, where giant CPV is observed.

Following [15], let us consider for now just the isoscalar $S$-wave $\pi\pi \leftrightarrow K\bar{K}$ rescattering, i.e., $\lambda, \lambda'$ can be $\pi\pi$ or $KK$, which is the most interesting contribution to $\Delta\Gamma_{KK(\pi\pi)}$ in $B^\pm \to K^\pm K^+K^-(K^\pm\pi^+\pi^-)$, when the two-meson invariant mass is in the 1 to 1.5 GeV range. We will add the other terms and waves later. Since $2i\hat{f}_{\pi\pi KK} = S_{\pi\pi KK} = |S_{\pi\pi KK}|\exp(i\delta_{\pi\pi KK})$, we can write

$$\Delta\Gamma_{KK} \simeq \mathcal{C}|S_{\pi\pi KK}|\cos(\delta_{\pi\pi KK} + \Phi_{KK})F(M_K^2). \quad (2)$$

Following [15] we define $\mathcal{C} = 4|K|\sin\gamma$, where $K = |K|\exp(i\Phi_{KK}) = B^*_{KK}A_{\pi\pi} - B^*_{\pi\pi}A_{KK}$, with $K_{KK} = -K_{\pi\pi}$ and $\Phi_{KK} = \Phi_{\pi\pi} + \pi$ due to $CPT$. Within this first approximation, $\mathcal{C}$ can be considered constant compared to the strong $s$ dependence of $S_{\pi\pi KK}$. The Dalitz form factor is $F(M_K^2) = (M_K^2)_{\max} - (M_K^2)_{\min}$, which are obtained from kinematics. The amplitude symmetrization in the two like-sign kaons is neglected as low-mass regions for each neutral $KK$ pair are very separated in phase space. $CPT$ implies [15] that these rescattering contributions satisfy $\Delta\Gamma_{KK} = -\Delta\Gamma_{\pi\pi}$.

However, two crude estimates originally made in [15] have become standard but are not needed and can be easily improved. Note they have been used in [16–18] and also in the LHCb implementation of this model in [12–14]. A possible reason for such estimates was that meson-meson scattering are plagued with systematic uncertainties and frequently analyzed with crude models. However, at the time of [15] a dispersively constrained analysis existed for $\pi\pi \to \pi\pi$ [19]. Thus, in order to use this dispersive representation, the first approximation was to assume a formalism with only two channels: $1 = \pi\pi$ and $2 = KK$, so that $S$-matrix unitarity implies

$$(S_{\lambda\lambda'}) = \begin{pmatrix} \eta e^{2i\delta_{11}} & i\sqrt{1-\eta^2}e^{i(\delta_{11}+\delta_{22})} \\ i\sqrt{1-\eta^2}e^{i(\delta_{11}+\delta_{22})} & \eta e^{2i\delta_{22}} \end{pmatrix},$$

where $\eta$ is the $\pi\pi \to \pi\pi$ elasticity. Hence, the required $S_{\pi\pi KK}$ was avoided by replacing in Eq. (2)

$$|S_{\pi\pi KK}| \to \sqrt{1-\eta^2}, \qquad (3)$$

$$\delta_{\pi\pi KK} \to \delta_{\pi\pi\pi\pi} + \delta_{KKKK} \simeq 2\delta_{\pi\pi\pi\pi}, \qquad (4)$$

where in the last step $\delta_{KKKK} \simeq \delta_{\pi\pi\pi\pi}$, was assumed, since little is known about $\delta_{KKKK}$. Finally, setting $\Phi_{KK} = 0$ as well as $\delta_{\pi\pi KK} = 0$ above 1.5 GeV, we reproduce in Fig. 1 the results of [15] for $\Delta\Gamma_{KK}$ (bottom) and $\Delta\Gamma_{\pi\pi}$ (top), projected from LHCb results [7], as a function of the two-meson invariant mass $M^2_{\rm sub} = s$. Only the normalization constant $\mathcal{C}$ is free. Note that as nicely shown in [15], due to $CPT$ symmetry, and just by changing its global sign, Eq. (2) roughly describes both asymmetries from $K\bar{K}$ threshold to $M_{\rm sub} \simeq 1.5$ GeV, i.e., $S$-wave FSI dominate the $s$ dependence in that region.

However, neither of these estimates is needed because both $\delta_{\pi\pi KK}$ and $|S_{\pi\pi KK}|$ data up to 2 GeV exist since the 1980s from the Argonne [28], Brookhaven I [29], and Brookhaven II [30] Collaborations, shown in Fig. 2. Note that to compare with data we employ the usual normalization

$$|g_0^0(s)| = \frac{\sqrt{s}}{4(q_\pi q_K)^{1/2}}|S_{\pi\pi KK}(s)|, \qquad s > 4m_K^2, \quad (5)$$

with $q_P = \sqrt{s/4 - m_P^2}$ the $P = \pi, K$ CM momenta. There we see that Eqs. (3) and (4) fail to describe both the $|S_{\pi\pi KK}|$ and $\delta_{\pi\pi KK}$ data, respectively. For these curves we use the $\delta_{\pi\pi\pi\pi}$ and $\eta$ obtained in [19], because it has become customary in the literature, although those dispersively constrained fits were updated in [31]. Had we used the latter, with smaller uncertainties, the comparison with data would be even worse. Recall also that we are subtracting $2\pi$ [32] to make $2\delta_{\pi\pi\pi\pi}$ fit in the plot. Hence, Eqs. (3) and (4) should be avoided. But then one might wonder if the claimed relevance of $\pi\pi \to K\bar{K}$ FSI in giant CPV depends crucially on such crude estimates and their large uncertainties, or if they still hold when a realistic $\pi\pi \to K\bar{K}$ parameterization is used instead.

Luckily, only very recently, but very timely, model-independent dispersive analyses of $\pi\pi \to KK$ data, using hyperbolic dispersion relations (Roy-Steiner equations), have become available in [25] and updated in [21]. These provide accurate constrained fits to data (CFD) up to 1.47 GeV, the maximum applicability of these relations, continuously matched to unconstrained fits up to 2 GeV, for both $\delta_{\pi\pi KK}$ and $|S_{\pi\pi KK}|$, shown in Fig. 2. Note that for the

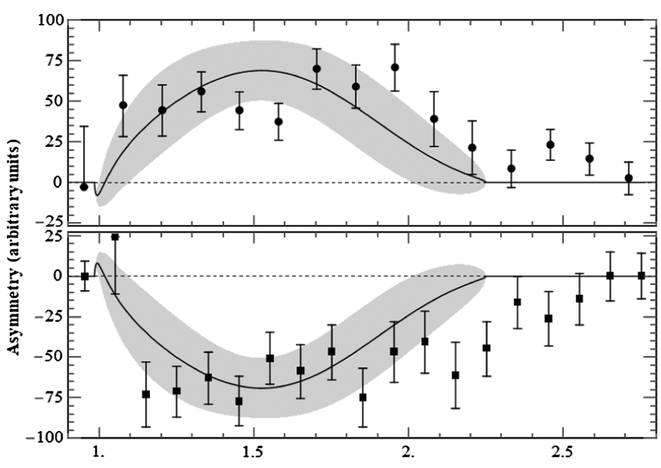


FIG. 1. $CP$ asymmetries for $B^\pm \to K^\pm\pi^+\pi^-$ (top) and $B^\pm \to K^\pm K^+K^-$ (bottom), from Eq. (2) and the estimates in Eq. (3) and (4). The plot is identical to Fig. 1 in [15]. Data from [27].

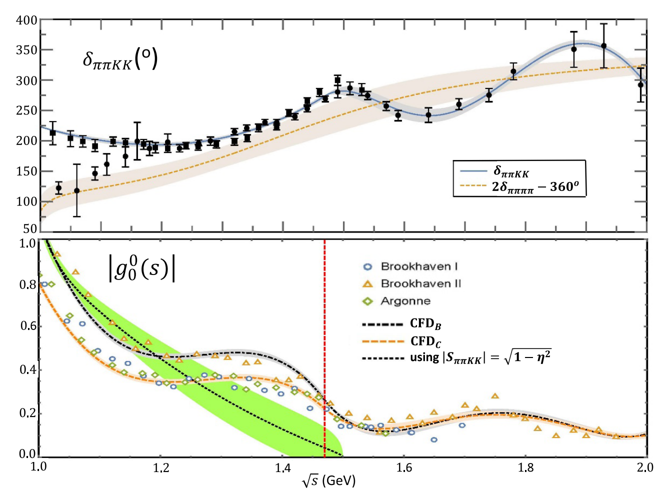


FIG. 2. Top: $\delta_{\pi\pi K\bar{K}}$ data from [28] (squares) and [29] (circles). The dashed line is the Eq. (4) estimate, although subtracting $2\pi$ to fit in the plot, and using [19] (PY) for $\delta_{\pi\pi\pi\pi}$. The continuous line is the dispersively constrained fit from [25] (PR). The five first data points of [29] below 1.2 GeV are in conflict with Watson's Theorem and dispersive analyses of $\pi\pi \to \pi\pi$ and are commonly discarded. Bottom: $|g_0^0(s)|$ data. The green band is Eq. (3) and the grey and orange bands correspond to the dispersive analysis in [25].

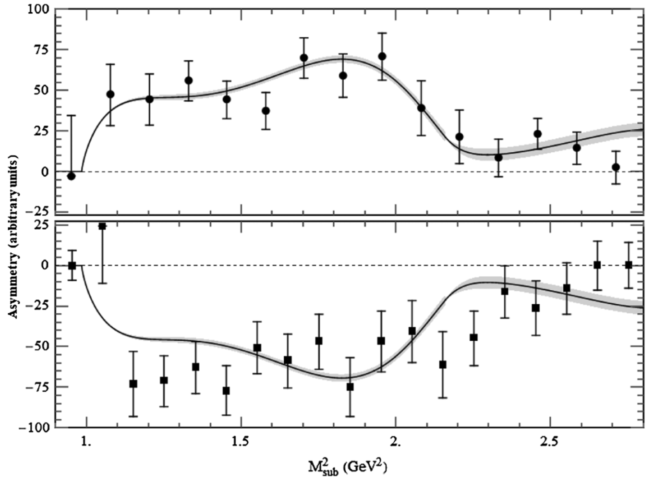


FIG. 3. As Fig. 1 but using in Eq. (2) the dispersively constrained CFD parameterization of $\pi\pi \to K\bar{K}$ data from [21,25]. Note the huge increase in precision with respect to Fig. 1 and the new patterns due to resonance interplay.

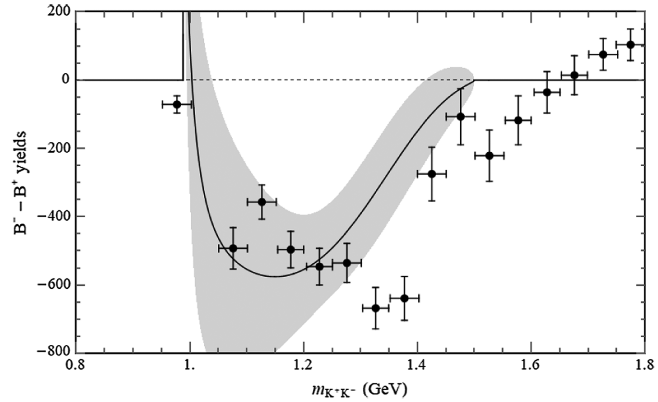


FIG. 4. Total $B^\pm \to K^\pm K^+ K^-$ asymmetry. LHCb data from the sum of Figs. 6(c) and 6(d) in [9]. Central line, using Eqs. (3) and (4), identical to [16]. We have added here the huge uncertainty in that description.

modulus there are two solutions. We will present results for the higher one since their difference up to 1.47 GeV can be reabsorbed in the normalization parameter and at the end yield very similar results.

Thus, in Fig. 3 we show the asymmetry results when the CFD $\pi\pi \to K\bar{K}$ dispersive analysis is used in Eq. (2). Our $\chi^2_{\mathrm{d.o.f.}} = 1$ and we have freed the parameter $\Phi$ finding a nonvanishing preferred value $(-34 \pm 9)°$. There is an impressive improvement in precision with respect to Fig. 1. Remarkably, it also shows peaks and dips associated with the interplay of the $f_0(980)$, $f_0(1370)$ and $f_0(1500)$ resonances [33] that were concealed in Fig. 1 within the large uncertainty from Eqs. (3) and (4).

Furthermore, the full run I LHCb data on the $B^\pm \to K^\pm K^+ K^-$ CPV asymmetry [9], were described in [16] with Eq. (2) above, but divided by $(1 + s/\Lambda^2_\lambda)(1 + s/\Lambda^2_{\lambda'})$, to mimic the mild $s$ dependence of the source term for each $\lambda$ pair, with $\Lambda_{KK} = 4$ GeV and $\Lambda_{\pi\pi} = 3$ GeV. Using then Eqs. (3) and (4), we have reproduced in Fig. 4 the central value of [16], but also adding the huge uncertainty due to such estimates. In contrast, we show in Fig. 5 the result when using the CFD phase and modulus from [25,33]. The central line follows much better the data, with dramatically smaller uncertainties, again unraveling the interplay between resonances. See Ref. [34] for details. Above 1.5 GeV this approach is not expected to be valid, due to the increasing relevance of $4\pi$ and other resonances and FSI with higher angular momenta.

All in all, these results confirm, using realistic and accurate FSI, that $\pi\pi \to K\bar{K}$ rescattering does indeed play a dominant role in the appearance and the $s$ dependence of giant CPV at LHCb in the 1 to 1.5 GeV region.

Let us now reintroduce other relevant terms, following the more complete model of [16], adopted by the LHCb analyses [12–14]. Thus, we recast Eq. (1) as

$$\begin{aligned}\mathcal{A}^\pm_{\mathrm{LO}} = {} & \sum_J (a^J_{\lambda\mathrm{NR}} + b^J_{\lambda\mathrm{NR}} e^{\pm i\gamma})/(1 + s/\Lambda^2_\lambda) \\ & + \sum_{JR} (a^R_\lambda + b^R_\lambda e^{\pm i\gamma}) F^{\mathrm{BW}}_{R\lambda} P_J(\cos\theta) \\ & + i \sum_{\lambda',J} \hat{f}^J_{\lambda'\lambda} (a^J_{\lambda'\mathrm{NR}} + b^J_{\lambda'\mathrm{NR}} e^{\pm i\gamma})/(1 + s/\Lambda^2_{\lambda'}), \qquad (6)\end{aligned}$$

where the angular momenta $J$ is explicitly separated from $\lambda$. Note that terms without FSI and other mild $s$-dependent contributions are grouped into a nonresonant (NR) part. Besides, the strong $s$ dependence of elastic scattering, $\lambda' = \lambda$, is described with usual Breit-Wigner shapes. Namely, $(1 + i\hat{f}^J_{\lambda\lambda}) A^J_{\lambda R} \to a^R_0 F^{\mathrm{BW}}_R P_J(\cos\theta)$, with $\theta$ the helicity angle between the like-sign mesons in the Gottfried-Jackson frame, and

$$F^{\mathrm{BW}}_R = \frac{1}{m^2_R - s - i m_R \Gamma_R(s)}, \qquad \Gamma_R(s) = \frac{q_\pi(s) m_R \Gamma_R}{q_\pi(m^2_R) s^{1/2}}.$$

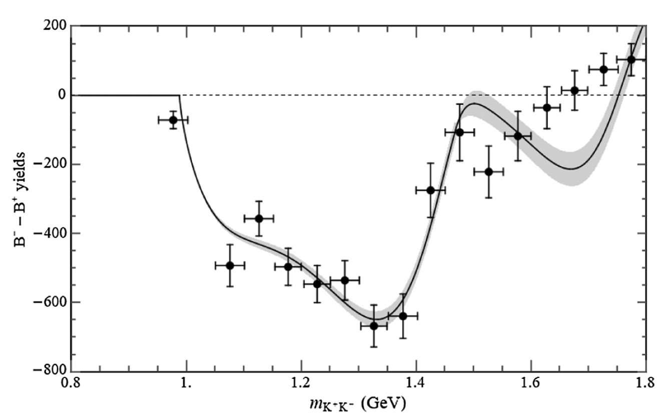


FIG. 5. As in Fig. 4 but using the dispersively constrained fit to $S_{\pi\pi KK}$ data in [25]. Note the dramatic improvement in precision and the unveiling of resonant structures.

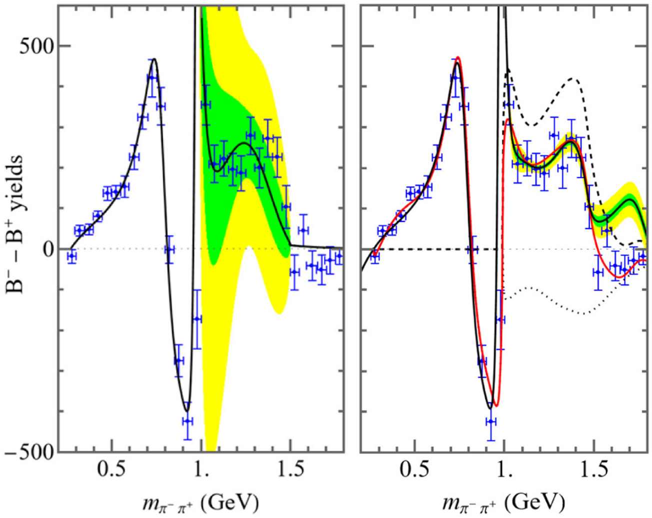


FIG. 6. $B^\pm \to K^\pm\pi^+\pi^-$ asymmetry in the $\cos(\theta) < 0$ region. Left: the central value reproduces Fig. 11 in [16]. We have added here the one (green) and three (yellow) standard deviation bands due to the crude estimates in Eq. (3) and (4). Right: Same but using the dispersive $\pi\pi \to K\bar{K}$ data analysis in [25] (black line), as proposed here. The accuracy improvement is dramatic. In addition, we show in red the full $\Delta\Gamma$ (Eq. 44 in [16]) but with the dispersive $\pi\pi \to \pi\pi$ isoscalar $S$ wave in [37] instead of just a $f_0(980)$ Breit-Wigner. The high-mass region is now well described and the too large $f_0(980)$ peak disappears, while including the $f_0(500)$ and $f_0(1370)$. In the 1–1.5 GeV region the terms containing the $\pi\pi \to KK$ amplitude (dashed) largely dominate those without it (dotted).

The two first terms in Eq. (6) correspond to a familiar isobar model, whereas inelastic FSI appear in the third term, dominated by $\hat{f}^0_{\pi\pi KK}$. The resonances to be considered depend on the process. For instance, for the $B^\pm \to K^\pm\pi^+\pi^-$ asymmetries, the energies below the $KK$ threshold become accessible. In [16] the $J = 1$, 0 waves were approximated only with the $\rho(770)$ and $f_0(980)$ resonances, respectively, by setting

$$A_{\lambda R} = a^\rho_\lambda F^{\rm BW}_\rho(s)k(s)\cos(\theta) + a^f_\lambda F^{\rm BW}_f(s), \qquad (7)$$

with $k(s) = \sqrt{1 - 4m_\pi^2/s}$ and similarly for $B_{\lambda R}$ amplitudes. Thus, on the left panel of Fig. 6 we reproduce the central value of the partial CPV asymmetry obtained in [16] with this improved model, and using Eqs. (3) and (4). Note the nice $\rho(770)$ resonant peak and dip structure around 770 MeV and the marked peak of the $f_0(980)$. However, once again we are providing the huge uncertainties that appear due to such standard crude estimates. In contrast, on the right panel we show the remarkable accuracy attained when using the dispersive $\pi\pi \to K\bar{K}$ amplitude instead. Details can be found in [34].

So far we have limited ourselves to the crude but appealing model formulated in [16], in order to show the accuracy improvement when using the recent dispersively constrained parameterizations of $\pi\pi \to K\bar{K}$ in the same kind of analyses that had been widely used before. It is enough to restore back all the instances of $2\delta_{\pi\pi\pi\pi}$ and $\sqrt{1-\eta^2}$ by, respectively, the $\delta_{\pi\pi KK}$ and $|S_{\pi\pi KK}|$ parameterizations by in [21,25] (beware of the different notation [38]).

However, the accuracy attained in $S$-wave FSI, opens an interesting outlook for further studies. Hence, reconsidering contributions neglected so far becomes even more appealing. The proponents of this model already pointed out some possible improvements, particularly the inclusion of a realistic $\pi\pi$ $S$ wave. Thus Fig. 6 shows in red the result of replacing the naive single $f_0(980)$ Breit-Wigner shape with the dispersive $\pi\pi$ data analysis in [37], which also describes the $f_0(500)$ and $f_0(1370)$. The contributions containing the $\pi\pi \to K\bar{K}$ amplitudes still dominate the 1 to 1.5 GeV region. Further waves and resonances could also be implemented in a similar way.

Our level of precision calls for a future replacement of this simple isobar model with leading order FSI corrections by the full treatment of the three-body decay, but containing the correct two-body rescattering amplitude. A first step would be to consider the all-order two-body contributions [4,17,18], although eventually it should include dispersively constrained third-particle effects following what has been done for $D$ or other heavy meson decays [39–41], but restricted to certain regions of the $B$-decay phase space. Finally, three-body contributions should be included. For ongoing efforts in these topics we refer to [26,42,43] and references therein. This Letter, therefore, paves the way for several future developments.

In summary, we have shown how to implement the recent dispersively constrained parameterizations of $\pi\pi \to K\bar{K}$ to describe final state interactions in charmless three-body $B$ decays, avoiding standard crude estimates within a popular model used by LHCb and others to describe giant $CP$ violation. This will help reducing by about an order of magnitude the uncertainty due to FSI in that model. As a result, the dominant role of inelastic final state interactions in the strong $s$ dependence measured in the 1 to 1.5 GeV region, previously based on crude estimates, is confirmed with realistic interactions. Moreover, this dramatic reduction of uncertainty when using the dispersive analyses of $\pi\pi \to K\bar{K}$ data, opens the way for a more detailed description of additional hadronic features and a precise treatment of further data. This is particularly relevant in the amplitude analysis that should be carried out for data just released by LHCb [10] or to be obtained in the near future.

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