Binary Dynamics through the Fifth Power of Spin at $O(G^2)$

Zvi Bern,[1] Dimitrios Kosmopoulos,[1] Andrés Luna,[2] Radu Roiban,[3] and Fei Teng[3]
[1]*Mani L. Bhaumik Institute for Theoretical Physics, University of California at Los Angeles, Los Angeles, California 90095, USA*
[2]*Niels Bohr International Academy, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark*
[3]*Institute for Gravitation and the Cosmos, Pennsylvania State University, University Park, Pennsylvania 16802, USA*

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We use a previously developed scattering-amplitudes-based framework for determining two-body Hamiltonians for generic binary systems with arbitrary spin S. By construction this formalism bypasses difficulties with unphysical singularities or higher-time derivatives. This framework has been previously used to obtain the exact velocity dependence of the $\mathcal{O}(G^2)$ quadratic-in-spin two-body Hamiltonian. We first evaluate the S^3 scattering angle and two-body Hamiltonian at this order in G, including not only all operators corresponding to the usual worldline operators, but also an additional set due to an interesting subtlety. We then evaluate S^4 and S^5 contributions at $O(G^2)$ which we confirm by comparing against aligned-spin results. We conjecture that a certain shift symmetry together with a constraint on the high-energy growth of the scattering amplitude specify the Wilson coefficients for the Kerr black hole to all orders in the spin and confirm that they reproduce the previously obtained results through S^4.

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Introduction.—The landmark detection of gravitational waves by the LIGO/Virgo Collaboration [1] heralds an era of remarkable discoveries in astronomy, cosmology, and perhaps even particle physics. The fundamental issues addressed in this Letter, leading to the identification of a new shift symmetry and of a subtlety in the counting of independent interactions, lay a foundation for future phenomenological applications. While current detectors are not sufficiently sensitive, the remarkable precision of forthcoming detectors [2] will demand equally precise theoretical predictions which include detailed properties of gravitational-wave sources including their spin [3].

The post-Minkowskian (PM) framework [4], which resums the velocity expansion present in the post-Newtonian approach [5–12] has been previously applied to the spinning two-body problem [13–24]. In this Letter we approach it with amplitudes methods.

Vines [17] obtained the energy-momentum tensor of a Kerr black hole at $O(G)$ with the full spin and velocity dependence and derived the corresponding two-body Hamiltonian. This stress tensor was shown [14,18] to be equivalent with the minimal amplitudes of Ref. [25], which were used [15] to recover the Hamiltonian of Ref. [17]. At $O(G^2)$, a PM spin-orbit Hamiltonian is known [13]. There has also been progress at this order on obtaining PM higher-spin interactions [14,16,18], including the complete quadratic-in-spin interactions for the inspiral phase of generic compact objects [19–21]. Recently, quartic-in-spin results have been given for binary Kerr black holes [24] at $O(G^2)$. The scattering angle for generic spinning bodies at the quadratic-in-spin level and at $O(G^3)$, including radiation effects, was recently given in [26].

High orders in the spin bring a number of subtleties. Among them is the complete categorization of all independent interactions. In the worldline-effective-field-theory formalism [6–12] this is achieved by eliminating all Lagrangian terms with higher time derivatives [8,27]. The amplitudes-based approaches using massive spinor helicity [25] can introduce unphysical singularities beyond the quartic-in-spin order [18,24]. At spin-5/2 and beyond, Compton amplitudes free of such singularities were constructed in Refs. [28,29]. With a local Lagrangian starting point, the amplitudes-based formalism of Ref. [19] bypasses these issues to all orders in spin. This formalism has been tested for quadratic in spin contributions at $O(G^2)$ in Refs. [19,20] and confirmed using the worldline [21] and worldline-quantum-field-theory [23] formalisms.

Amplitudes-based methods [30–34] established the state of the art in the PM expansion by producing the first conservative spinless two-body Hamiltonian at $O(G^3)$ and $O(G^4)$ [35], with various aspects confirmed in a number of studies [36]. Such methods led to new results that include spin [14,16,18,24,37] and tidal effects [38]. They also led to the discovery of new structures such as the double-copy relation between gauge and gravity theories [39] and a conjecture that a single scalar function—the eikonal phase—determines both spinning and spinless classical observables [19].

Here we use the amplitudes-based field-theory formalism of Ref. [19] and a Lagrangian containing classes of operators through quintic order in spin to derive two-body interaction Hamiltonian up to $O(S^5)$ for black holes and more general compact objects. To obtain a complete description through a given power of spin one would need to systematically add all operators that give independent contributions to the physical observables up to power of spin of interest.

The Hamiltonian at $O(G)$ for binary Kerr black holes is known to all orders in spin [17]. Reference [24] obtained a binary Hamiltonian at $O(G^2)$ through S^4 from amplitudes expected to describe Kerr black holes. Explicit results through fifth order in spin, a symmetry of the $O(G)$ Hamiltonian of Ref. [17] and an additional assumption lead us to formulate a conjecture for the structure of the two-body Hamiltonian for Kerr black holes to all orders in spin and determine the Hamiltonian at S^5. We consider a variety of independent interactions sufficient to illustrate the features of this conjecture, and leave for the future a systematic study of all possible interactions.

Review of formalism.—In the classical limit of a scattering process, the momentum transfer $\boldsymbol{q}$ is much smaller than the rest mass, $|\boldsymbol{q}| \ll m$, while the rest-frame spin is large, $\boldsymbol{q}\cdot\boldsymbol{S}/m \sim O(1)$. Products of classical spin tensors are related [19] to symmetric products of Lorentz generators,

$$\begin{aligned}&\varepsilon(p_1)\{M^{a_1b_1}M^{a_2b_2}\ldots M^{a_jb_j}\}\varepsilon(p_2)\\&\quad=S(p_1)^{a_1b_1}S(p_1)^{a_2b_2}\cdots S(p_1)^{a_jb_j}\varepsilon(p_1)\cdot\varepsilon(p_2)+\cdots,\quad(1)\end{aligned}$$

where $\{\ldots\}$ indicates the symmetric product, $\varepsilon(p_1)$ and $\varepsilon(p_2)$ are boosted spin coherent states describing the incoming and outgoing polarization tensors of a higher-spin particle, and the ellipsis stand for subleading terms in the classical limit. Antisymmetric combinations of M^{ab} are simplified using the Lorentz algebra. The spin tensor is obtained by boosting the one in the rest frame, so it respects the covariant spin supplementary condition (SSC) $p_aS^{ab}=0$. The classical spin vector follows from $S^{ab}\equiv-\epsilon^{abcd}p_cS_d/m$, which follows by boosting the analogous rest-frame relation.

We work in a field-theory framework with a Lagrangian that simply gives a covariantization of all the spin-induced multipole moments. Causality-based no-go theorems for the quantum consistency of higher-spin interactions [40,41] are avoided by interpreting it as an effective theory, valid only at sufficiently large impact parameter, i.e., only in the classical regime. See Ref. [42] for a connection between resolvability of the time delay and the range of validity of an EFT. We first describe operators that have on-shell three-point vertices. Following Ref. [19] we separate the Lagrangian into a minimal and nonminimal part, $\mathcal{L}=\mathcal{L}_{\text{min}}+\mathcal{L}_{\text{nonmin}}$. The former is

$$\mathcal{L}_{\text{min}}=-R+\frac{1}{2}\eta^{ab}\nabla_a\phi_s\nabla_b\phi_s-\frac{1}{2}m^2\phi_s\phi_s,\qquad(2)$$

where we use only tangent-space indices. We take the higher-spin field ϕ_s to be in a real representation of the Lorentz group. We do not require it to be transverse, so this representation is reducible and contains spins ranging from 0 to s [19]. The covariant derivative is $\nabla_c\phi_s\equiv e^\mu_c[\partial_\mu\phi_s+(i/2)\omega_{\mu ab}M^{ab}\phi_s]$, where e^μ_c is the (inverse) vierbein, $\omega_{\mu ab}$ is the spin connection, and M^{ab} are Lorentz generators in this representation.

In the nonminimal Lagrangian, we consider two classes of linear-in-curvature operators and also selected curvature-square operators, $\mathcal{L}_{\text{nonmin}}=\mathcal{L}_C+\mathcal{L}_H+\mathcal{L}_{R^2}$. More generally, there are infinite sequences of additional operators to consider, including those with different index contractions or higher powers of the Riemann tensor. In general, the coefficients of these operators need to be matched to either theoretically or experimentally determined values, with coefficients being particularly simple for black holes. The first family of linear-in-curvature operators is

$$\begin{aligned}\mathcal{L}_C=&\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\frac{C_{\text{ES}^{2n}}}{m^{2n}}\nabla_{f_{2n}}\ldots\nabla_{f_3}R_{af_1bf_2}\\&\times\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2}\ldots\mathbb{S}^{f_{2n})}\nabla^b\phi_s\\&-\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{C_{\text{BS}^{2n+1}}}{m^{2n+1}}\nabla_{f_{2n+1}}\ldots\nabla_{f_3}\tilde{R}_{(a|f_1|b)f_2}\\&\times\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2}\ldots\mathbb{S}^{f_{2n+1})}\nabla^b\phi_s,\end{aligned}\qquad(3)$$

where $\mathbb{S}^a\equiv[(-i)/2m]\epsilon^{abcd}M_{cd}\nabla_b$ is the Pauli-Lubanski vector, and $\tilde{R}_{abcd}\equiv\frac{1}{2}\epsilon_{abij}R^{ij}_{cd}$ is the dual Riemann tensor. The operators in Eq. (3) are in one-to-one correspondence with the nonminimal operators in Ref. [9].

The second family of linear-in-curvature operators we include is given by [43],

$$\begin{aligned}\mathcal{L}_H=&-\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(2n)!(2n+1)}\frac{H_{2n}}{m^{2n-2}}\nabla_{f_{2n}}\ldots\nabla_{f_3}R^{(a}{}_{f_1}{}^{b)}{}_{f_2}\\&\times\phi_sM_a^{(f_1}M_b^{f_2}\mathbb{S}^{f_3}\ldots\mathbb{S}^{f_{2n})}\phi_s\\&+\sum_{n=1}^{\infty}\frac{(-1)^nn}{(2n+1)!(n+1)}\frac{H_{2n+1}}{m^{2n-1}}\nabla_{f_{2n+1}}\ldots\nabla_{f_3}\tilde{R}^{(a}{}_{f_1}{}^{b)}{}_{f_2}\\&\times\phi_sM_a^{(f_1}M_b^{f_2}\mathbb{S}^{f_3}\ldots\mathbb{S}^{f_{2n+1})}\phi_s.\end{aligned}\qquad(4)$$

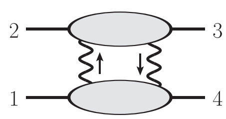
The normalization is chosen such that, upon using external spin tensors satisfying the covariant SSC, the three-point amplitudes depend only on $C_{2n}\equiv C_{\text{ES}^{2n}}+H_{2n}$ and $C_{2n+1}\equiv C_{\text{BS}^{2n+1}}+H_{2n+1}$. Comparison with Ref. [17] fixes $C_n=1$ for all $n\geq 2$ for a Kerr black hole. Because it relies on the SSC rather than the equations of motion, the equality of the three-point on-shell matrix elements of the operators in $\mathcal{L}_C$ and $\mathcal{L}_H$ *does not imply* that their difference is a local higher-curvature operator. As we will see, $\mathcal{L}_C$ and $\mathcal{L}_H$ contribute differently to the classical gravitational Compton amplitude, so both need to be included in an EFT.

Operators with higher powers of the Riemann tensor and its derivatives first contribute at S^4 and can also encode tidal effects. Our conjectured structure of the two-body Hamiltonian of Kerr black holes allows such terms at S^4 and requires that they be present starting at S^5 so as to cancel poor high-energy behavior. Leaving the former for future studies, to explore our conjectured QFT definition of the Kerr black hole it is sufficient to include

$$\begin{aligned}\mathcal{L}_{R^2} = &\frac{1}{1800m^7}(E_1 R_{af_1bf_2}\nabla_{f_5}\tilde{R}_{cf_3df_4} + E_2\nabla_{f_5}R_{af_1bf_2}\tilde{R}_{cf_3df_4})\nabla^{(a}\nabla^{c)}\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2}\mathbb{S}^{f_3}\mathbb{S}^{f_4}\mathbb{S}^{f_5)}\nabla^{(b}\nabla^{d)}\phi_s\\ &+\frac{1}{1800m^5}(E_3 R_{af_1bf_2}\nabla_{f_5}\tilde{R}_{cf_3df_4} + E_4\nabla_{f_5}R_{af_1bf_2}\tilde{R}_{cf_3df_4})\nabla^{c}\phi_s M^{a(f_1}M^{|b|f_2}\mathbb{S}^{f_3}\mathbb{S}^{f_4}\mathbb{S}^{f_5)}\nabla^{d}\phi_s\\ &+\frac{1}{1800m^7}(2E_5 R_{eabf_1}\nabla_{f_2}\tilde{R}^{e}{}_{cdf_3} + E_6 R_{aebf_1}\nabla^{e}\tilde{R}_{cf_2df_3} + E_7\nabla_{f_2}R_{eabf_1}\tilde{R}^{e}{}_{cdf_3})\nabla^{(a}\nabla^{c)}\phi_s\mathbb{S}^{m}\mathbb{S}_{m}\mathbb{S}^{(f_1}\mathbb{S}^{f_2}\mathbb{S}^{f_3)}\nabla^{(b}\nabla^{d)}\phi_s,\end{aligned} \tag{5}$$

where $E_1, \ldots, E_7$ are Wilson coefficients. For generic compact objects one would need to include all R^2 operators with independent matrix elements under the SSC.

Having specified the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{min}} + \mathcal{L}_{\text{nonmin}}$, the four-point Compton amplitude is straightforwardly obtained using Feynman rules. The generalized unitarity method [44] then gives the one-loop integrand with four external higher-spin states. The relevant generalized cut at $O(G^2)$ is

$\text{Cut}_{\text{t-channel}} =$ 

, (6)

where the blobs represent on-shell gravitational Compton amplitudes. By adjusting terms that vanish on shell, the Compton amplitudes can be chosen to satisfy generalized gauge invariance, so the physical-state projectors are manifestly Lorentz invariant and independent of reference momenta [45]. Apart from inclusion of higher powers of the spin vector, the construction of the integrand follows the discussion in Ref. [19] so we will not detail it here. We then use FIRE [46] as well as Forde's method [47] to extract the coefficients of the scalar triangle integrals which determine the classical amplitude.

Results.—In general the amplitude is a sum of scalar box, triangle, and bubble-integral contributions. The box integrals correspond to iteration of lower-order terms and carry no new information. The bubble integrals contain purely quantum information and are hence dropped. The new classical information at this order is encoded in the triangle integrals and their coefficients [32,33], whose structure is

$$\mathcal{M}^{\triangle+\nabla} = \frac{2\pi^2 G^2 \varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3}{\sqrt{-q^2}}\sum_n\sum_i \alpha^{(n,i)}\mathcal{O}^{(n,i)}. \tag{7}$$

Here n is the number of spin vectors and the index i labels their independent contractions. For brevity, we include here explicitly only the $S_1^n S_2^0$ terms with $n = 2$ through $n = 5$. The relevant operators $\mathcal{O}^{(n,i)}$ are given in Table I, while Supplemental Material contains the full amplitudes [48]. We use the shorthand notation $\mathcal{E}_j = -i\epsilon^{\mu\nu\rho\tau}u_{1\mu}u_{2\nu}q_\rho a_{j\tau}$, as well as $u_i^\mu = p_i^\mu/m_i$, $a_i^\mu = S_i^\mu/m_i$ and $\sigma = u_1\cdot u_2$.

We parametrize the coefficients $\alpha^{(n,i)}$ in Eq. (7) as

$$\begin{aligned}\alpha^{(2,i)} &= \frac{m_1^2 m_2^2}{16(-1+\sigma^2)^2}(\gamma^{(2,i)}m_1 + \delta^{(2,i)}m_2),\\ \alpha^{(3,i)} &= \frac{m_1^2 m_2^2\sigma}{8(-1+\sigma^2)^2}(\gamma^{(3,i)}m_1 + 2\delta^{(3,i)}m_2),\\ \alpha^{(4,i)} &= \frac{m_1^2 m_2^2}{1536(-1+\sigma^2)^3}\left(\gamma^{(4,i)}m_1 + \frac{8}{5}\delta^{(4,i)}m_2\right),\\ \alpha^{(5,i)} &= \frac{m_1^2 m_2^2\sigma}{768(-1+\sigma^2)^3}\left(\gamma^{(5,i)}m_1 + \frac{1}{75}\delta^{(5,i)}m_2\right),\end{aligned} \tag{8}$$

where $\gamma^{(k,i)}$ and $\delta^{(k,i)}$ are polynomials in σ^2. They correspond to the probe limits $m_2 \ll m_1$ and $m_1 \ll m_2$, respectively. We list the coefficients $\gamma^{(k,i)}$ in Table II in terms of combinations $Z_{k,j}$ of the C_n Wilson coefficients, which we collect in Table III.

We parametrize the polynomials $\delta^{(k,i)}$ that govern the limit $m_1 \ll m_2$ as

$$\delta^{(k,i)} = \sum_{\ell=0}^{5}\delta_\ell^{(k,i)}\sigma^{2\ell}. \tag{9}$$

TABLE I. The independent $S_1^2 S_2^0$ and $S_1^4 S_2^0$ structures are given in the table. The $S_1^3 S_2^0$ and $S_1^5 S_2^0$ structures follow from these via $\mathcal{O}^{(3,i)} = \mathcal{E}_1\mathcal{O}^{(2,i)}$ and $\mathcal{O}^{(5,i)} = \mathcal{E}_1\mathcal{O}^{(4,i)}$.

	i		i		i	
$\mathcal{O}^{(2,i)}$	1	$\mathcal{E}_1^2$	2	$q^2(u_2\cdot a_1)^2$	3	$(q\cdot a_1)^2$
$\mathcal{O}^{(4,i)}$	1	$\mathcal{E}_1^4$	2	$q^2(u_2\cdot a_1)^2\mathcal{E}_1^2$	3	$q^4(u_2\cdot a_1)^4$
	4	$(q\cdot a_1)^2\mathcal{E}_1^2$	5	$q^2(q\cdot a_1)^2(u_2\cdot a_1)^2$	6	$(q\cdot a_1)^4$

TABLE II. The $\gamma^{(m,i)}$ polynomials for $S_1^m S_2^0$ where the $Z_{i,j}$ are defined in Table III. The second column gives the value of i in $\gamma^{(m,i)}$.

$\gamma^{(2,i)}$	1	$7+23C_2-Z_{2,1}\sigma^2(102-95\sigma^2)$,
	2	$5-11C_2+5Z_{2,1}\sigma^2(6-7\sigma^2)$,
	3	$12Z_{2,2}(\sigma^2-1)^2(5\sigma^2-1)$,
$\gamma^{(3,i)}$	1	$Z_{3,1}(5-9\sigma^2)$,
	2	$Z_{3,1}(7\sigma^2-3)$,
	3	$4Z_{3,2}(\sigma^2-1)(5\sigma^2-3)$,
$\gamma^{(4,i)}$	1	$44C_3+59Z_{4,2}-Z_{4,1}\sigma^2(250-239\sigma^2)$,
	2	$72C_3-78Z_{4,2}+Z_{4,1}\sigma^2(276-294\sigma^2)$,
	3	$28C_3-9Z_{4,2}+7Z_{4,1}\sigma^2(2-3\sigma^2)$,
	4	$12Z_{4,3}(1-\sigma^2)(23-102\sigma^2+95\sigma^4)$,
	5	$12Z_{4,3}(\sigma^2-1)(11-30\sigma^2+35\sigma^4)$,
	6	$24Z_{4,4}(\sigma^2-1)^3(5\sigma^2-1)$,
$\gamma^{(5,i)}$	1	$Z_{5,1}(7-13\sigma^2)$,
	2	$2Z_{5,1}(11\sigma^2-5)$,
	3	$Z_{5,1}(3\sigma^2-1)$,
	4	$12Z_{5,2}(\sigma^2-1)(9\sigma^2-5)$,
	5	$12Z_{5,2}(\sigma^2-1)(3-7\sigma^2)$,
	6	$8Z_{5,3}(\sigma^2-1)^2(3-5\sigma^2)$,

The coefficients $\delta_\ell^{(k,i)}$ determining $\delta^{(2,i)}$ and $\delta^{(3,i)}$ are given in Table IV. We note that while $\delta_\ell^{(2,i)}$ depends only on the combination C_2 of Wilson coefficients, $\delta_\ell^{(3,i)}$ depends separately on C_2 and H_2. The coefficients of the polynomials $\delta^{(4,i)}$ and $\delta^{(5,i)}$ are given in Table V and depend separately on C_n, H_n, and E_n.

For Kerr black holes, the coefficients $C_n = 1$ set to zero all the $\gamma^{(k,i)}$ coefficients in the second column of Tables II and III. The remaining operators in Table I with nonzero $\gamma^{(k,i)}$ coefficients have, up to quantum-suppressed terms $p_i \cdot q = \pm q^2/2 \to 0$, the shift symmetry

$$a_i^\mu \to a_i^\mu + \xi_i q^\mu / q^2, \qquad i = 1, 2, \tag{10}$$

where ξ_i are arbitrary constants and q^2 was included so the classical q scaling is uniform. Inspection of the all-orders-in-spin 1PM scattering amplitude of two Kerr black holes of Ref. [17] reveals that it exhibits this symmetry and is broken by inclusion of general spin-induced multipole moments. We therefore conjecturally *define* the scattering amplitude of two Kerr black holes as the amplitude which realizes the symmetry (10). This definition fixes the Wilson coefficients; at 1PM order it recovers those of Ref. [17]. It will be interesting to see if this definition holds at higher orders in G.

This definition is consistent with the vanishing of $\delta^{(2,3)}$ for $C_2 = 1$. Requiring that $\delta^{(3,3)}$ vanishes further sets $H_2 = 1$, which is also the value that leads to a Compton amplitude with good high-energy properties [49–51]. Reference [52] fixes $H_2 = 1$ for general bodies, not just for the Kerr black hole, by requiring that the form of the equation of motion is invariant under the change of SSC. See Ref. [7] for an alternative perspective. In contrast, string theory predicts state-dependent values for H_2, perhaps due to the spectrum containing more than a single massive higher-spin state [50].

Realization of the shift symmetry (10) at this order requires that $\delta^{(4,4)}$, $\delta^{(4,5)}$, and $\delta^{(4,6)}$ vanish. Demanding this fixes $H_4 = 0$, which recovers the amplitude obtained in Ref. [24] and, notably, leaves H_3 undetermined. The proposed symmetry (10) and the resulting values for Wilson coefficients are consistent with the $S_1^m S_2^{4-m}$ amplitude included in Supplemental Material [48].

Requiring the presence of the symmetry (10), or that $\delta^{(5,4)}$, $\delta^{(5,5)}$, and $\delta^{(5,6)}$ vanish, determines $H_3 = 3/2$ and shows that H_5 is degenerate with R^2 terms at this order. It, however, leaves undetermined certain R^2 Wilson

TABLE III. Useful combinations of Wilson coefficients.

$Z_{2,1} = C_2 + 1$,	$Z_{2,2} = C_2 - 1$,
$Z_{3,1} = 3C_2 + C_3$,	$Z_{3,2} = C_2 - C_3$,
$Z_{4,1} = 3C_2^2 + 4C_3 + C_4$,	$Z_{4,3} = C_2^2 - C_4$,
$Z_{4,2} = 3C_2^2 + C_4$,	$Z_{4,4} = 3C_2^2 - 4C_3 + C_4$,
$Z_{5,1} = 10C_2C_3 + 5C_4 + C_5$,	$Z_{5,2} = 2C_2C_3 - C_4 - C_5$, $Z_{5,3} = 2C_2C_3 - 3C_4 + C_5$,

TABLE IV. Coefficients of the polynomials $\delta^{(2,i)}$ and $\delta^{(3,i)}$. The second column gives the value of i in $\gamma^{(m,i)}$.

$\delta_0^{(2,i)}$	1	$8Z_{2,1}$,
	2	$4(3-C_2)$,
	3	$-4Z_{2,2}$,
$\delta_1^{(2,i)}$	1	$-(68+52C_2)$,
	2	$-12Z_{2,1}$,
	3	$68Z_{2,2}$,
$\delta_2^{(2,i)}$	1	$60Z_{2,1}$,
	2	0,
	3	$-124Z_{2,2}$,
$\delta_3^{(2,i)}$	1	0,
	2	0,
	3	$60Z_{2,2}$,
$\delta_0^{(3,i)}$	1	$3(H_2-2)H_2-(C_2-8)C_2$,
	2	$C_2(4C_2-5)+2C_3-5(H_2-2)H_2$,
	3	$(5-2C_2)C_2-2C_3+(H_2-2)H_2$,
$\delta_1^{(3,i)}$	1	$C_2(2C_2-13)-2C_3-3(H_2-2)H_2$,
	2	$5(C_2-H_2)(2-C_2-H_2)$,
	3	$2[C_2(3C_2-8)+4C_3-(H_2-2)H_2]$,
$\delta_2^{(3,i)}$	1	0,
	2	0,
	3	$(11-4C_2)C_2-6C_3+(H_2-2)H_2$,

TABLE V. The coefficients of the $\delta^{(4,i)}$ and $\delta^{(5,i)}$ polynomials for the $S_1^4 S_2^0$ and $S_1^5 S_2^0$ amplitudes. Only nonzero coefficients of $\delta^{(5,i)}$ are given in the table.

$\delta_0^{(4,i)}$	1	$8[45C_2^2 + 5C_3 - 5C_2(C_3 + 9H_2) + 5(H_2 - 1)H_3 - H_4]$,
	2	$-10[36C_2^2 - 7C_3 - 25C_2C_3 - 4C_4 - 48C_2H_2 + 9(H_2 - 1)H_3] - 4H_4$,
	3	$4H_4 - 10C_4 + 15C_2(9C_2 - 11H_2)$,
	4	$10[4C_4 - 5C_3 + C_2(36C_2 - 11C_3 - 24H_2) - 5H_3(H_2 - 1)] - 4H_4$,
	5	$10[C_2(9C_2 + 10C_3 - 15H_2) + 4H_2H_3 - 2(C_3 + C_4 + 2H_3)] + 8H_4$,
	6	$135C_2^2 - 5C_2(8C_3 + 9H_2) - 10[4C_3 + C_4 + 4(H_2 - 1)H_3] + 4H_4$,
$\delta_1^{(4,i)}$	1	$-10[120C_2^2 + 51C_3 + 4C_4 - C_2(19C_3 + 108H_2) + 19(H_2 - 1)H_3 - 2H_4]$,
	2	$2[615C_2^2 + 85C_3 - 50C_4 - 5C_2(49C_3 + 153H_2) + 45(H_2 - 1)H_3 + 8H_4]$,
	3	$50[C_3 + (H_2 - 1)H_3] - 165C_2^2 - 5C_2(10C_3 - 33H_2) - 4H_4$,
	4	$2[65C_3 - 825C_2^2 + 50C_4 + 5C_2(43C_3 + 99H_2) + 35(H_2 - 1)H_3 + 4H_4]$,
	5	$10[15C_2^2 - 5C_3 - 11C_2C_3 + 16C_4 - 15C_2H_2 - 23(H_2 - 1)H_3] - 28H_4$,
	6	$345C_2H_2 - 825C_2^2 + 80[6C_3 + (H_2 - 1)H_3] - 24H_4$,
$\delta_2^{(4,i)}$	1	$5[243C_2^2 + 78C_3 - 14C_2C_3 + 14C_4 - 201C_2H_2 + 30(H_2 - 1)H_3] - 12H_4$,
	5	$-2[525C_2^2 - 55C_3 + C_2(55C_3 - 525H_2) - 175(H_2 - 1)H_3 + 6H_4]$,
	3	$50[(C_2 - 1)C_3 + H_3 - H_2H_3]$,
	4	$10[273C_2^2 - 33C_3 - 31C_2C_3 - 44C_4 - 165C_2H_2 + (H_2 - 1)H_3] - 4H_4$,
	5	$-10[69C_2^2 - 51C_3 + 43C_2C_3 + 14C_4 - 75C_2H_2 - 69(H_2 - 1)H_3] + 32H_4$,
	6	$2[-640C_3 + 30C_4 + 5C_2(195C_2 + 32C_3 - 105H_2) + 24H_4]$,
$\delta_3^{(4,i)}$	1	$5[16C_3 - C_2(57C_2 + 16C_3 - 57H_2)]$,
	2	$350[(C_2 - 1)C_3 + H_3 - H_2H_3]$,
	4	$-10[183C_2^2 - 47C_3 + 23C_2C_3 - 30C_4 - 129C_2H_2 + 3(H_2 - 1)H_3]$,
	5	$10[45C_2^2 - 79C_3 + 79C_2C_3 - 45C_2H_2 - 85(H_2 - 1)H_3] - 12H_4$,
	6	$-10[C_2(225C_2 + 56C_3 - 153H_2) + 8H_2H_3 + 4(-34C_3 + 2C_4 - 2H_3 + H_4)]$,
$\delta_4^{(4,i)}$	4	$10[-22C_3 + C_2(39C_2 + 22C_3 - 39H_2)]$,
	5	$-350[(C_2 - 1)C_3 - H_3(H_2 - 1)]$,
	6	$5[3(-40C_3 + 2C_4 + C_2(85C_2 + 24C_3 - 71H_2)) + 8(H_2 - 1)H_3] + 12H_4$,
$\delta_5^{(4,i)}$	6	$5[16C_3 - C_2(57C_2 + 16C_3 - 57H_2)]$,
$\delta_0^{(5,i)}$	1	$50[39C_4 - 22C_3^2 - 72C_3H_2 - 9C_4H_2 + 24C_2(7C_3 - 2H_3) + 6H_3^2 + 21(H_2 - 1)H_4] - 69H_5 + 2(7E_5 + 4E_6 - 7E_7 - 2E_2 + E_4 + 8E_3 + 2E_1)$,
	2	$300C_2(2H_3 - 13C_3) + 25[128C_3^2 - 57C_4 + 30C_5 + 180C_3H_2 + 12C_4H_2 + 8H_3^2 - 66(H_2 - 1)H_4] + 37H_5 + 14E_5 - 14E_7 - E_2 - 31E_4 - 13E_3 + E_1$,
	3	$50[14C_3^2 - 96C_2C_3 + 66C_3H_2 + 9C_4H_2 + 54C_2H_3 - 6H_2H_4 + 6(C_4 + C_5 + H_4)] + 106H_5 - 8E_6 + 3E_2 - 33E_4 - 29E_3 - 3E_1$,
	4	$240C_5 - 5325C_4 + 1050H_4 + 50[42C_3H_2 - 56C_3^2 + 18C_4H_2 + 6C_2(25C_3 - 8H_3) + 16H_3^2 - 21H_2H_4] + 527H_5 + 18E_5 + 14E_6 - 34E_7 - 11E_2 + 19E_4 + 47E_3 + 3E_1$,
	5	$660(5C_4 - 6C_5 - 5H_4) + 50[88C_3^2 - 66C_3H_2 - 111C_4H_2 - 24H_3^2 + 6C_2(23H_3 - 17C_3) + 66H_2H_4] + 702H_5 + 4E_5 - 2E_6 - 20E_7 - E_2 - 61E_4 - 43E_3 - E_1$,
	6	$20[30H_4 - 285C_4 - 48C_5 + 5(18C_3H_2 - 8C_3^2 + 24C_4H_2 + 9C_2(5C_3 - 2H_3) + 8H_3^2 - 6H_2H_4)] + 596H_5 + 2[2E_5 + 3E_6 - 10E_7 - 2E_2 - 7(2E_4 + E_3) + E_1]$,
$\delta_1^{(5,i)}$	1	$-75[228C_2C_3 + 39C_4 + 6C_5 - 76C_3H_2 - 4C_4H_2 - 72C_2H_3 + 8H_3^2 + 14(H_2 - 1)H_4] + 69H_5 - 42E_5 - 24E_6 + 42E_7 + 11E_2 + 5E_4 - 11(3E_3 + E_1)$,
	2	$50[33C_4 + 6C_5 + 48C_3H_2 + 72C_4H_2 - 36C_2(C_3 - 3H_3) - 44H_3^2 - 36(H_2 - 1)H_4] - 212H_5 - 8(7E_5 + 2E_6 - 7E_7 - E_2 - 10E_4 - 3E_3 + E_1)$,
	3	$-75[9C_4 - 84C_2C_3 + 2C_5 + 44C_3H_2 - 4C_4H_2 + 40C_2H_3 + 8H_3^2 + 10(H_2 - 1)H_4] - 281H_5 - 14E_5 + 8E_6 + 14E_7 - 3E_2 + 75E_4 + 57E_3 + 3E_1$,
	4	$60(250C_4 + 52C_5 - 35H_4) + 50[56C_3^2 - 3H_2(6C_3 + 29C_4 - 14H_4) - 48H_3^2 + 6C_2(35H_3 - 93C_3)] - 1054H_5 - 92E_5 - 66E_6 + 124E_7 + 39E_2 - 69E_4 - 203E_3 - 17E_1$,
	5	$5[3060C_3H_2 - 880C_3^2 + 60C_2(9C_3 - 34H_3) + 3(328C_5 - 5C_4(169 - 200H_2) + 80H_3^2 - 730(H_2 - 1)H_4)] - 1579H_5 - 50E_5 + 82E_7 + 9(E_2 + 19E_4 - 2E_6) + 67E_3 - 5E_1$,
	6	$20[80C_3^2 + 1095C_4 + 84C_5 - 120C_3H_2 - 510C_4H_2 - 160H_3^2 + 60C_2(9H_3 - 19C_3) + 90(H_2 - 1)H_4] - 1788H_5 - 8(5E_5 + 5E_6 - 11E_7 - 2E_2 - 17E_4 - 5E_3 + 2E_1)$,

(Table continued)

TABLE V. *(Continued)*

$\delta_2^{(5,i)}$	1	$50[22C_3^2+21C_4(H_2-1)-42C_3H_2+42C_2(2C_3-H_3)]+28E_5+16E_6-28E_7-7E_2-7E_4+17E_3+7E_1,$
	2	$25[153C_4+4(147C_2C_3-32C_3^2-69C_3H_2-57C_4H_2-78C_2H_3+26H_3^2)+138H_2H_4-6(C_5+23H_4)+7H_5]$ $+42E_5+16E_6-42E_7-7E_2-49E_4-11E_3+7E_1,$
	3	$-175[4C_3^2+6C_4(H_2-1)-4H_3^2+6H_4-6H_2H_4-H_5]+14(E_5-E_7-3E_4-2E_3),$
	4	$15(70H_4-815C_4-224C_5)+50[56C_3^2-3H_2(30C_3-40C_4+7H_4)+6C_2(87C_3-38H_3)+32H_3^2]$ $+527H_5+5(26E_5+18E_6+53E_3+5E_1)-146E_7-9(5E_2-9E_4),$
	5	$10[15C_4(91-89H_2)-440C_3^2-156C_5-2070C_3H_2+90C_2(19C_3-3H_3)+200(H_3^2+6(H_2-1)H_4)]$ $+1052H_5+88E_5+42E_6-104E_7-15E_2-159E_4-5E_3+13E_1,$
	6	$4[75C_4(46H_2-91)-120C_5+150C_2(65C_3-32H_3)+50(20H_3^2+9H_4-9H_2(2C_3+H_4))+447H_5]$ $+12(8E_5+7E_6-12E_7-2E_2-20E_4-3E_3+3E_1),$
$\delta_3^{(5,i)}$	4	$-50[56C_3^2+51C_4(H_2-1)-66C_3H_2+6C_2(19C_3-11H_3)]-56E_5-38E_6+56E_7+17E_2-31E_4-109E_3-11E_1,$
	5	$25(176C_3^2-588C_2C_3-171C_4+24C_5+348C_3H_2+156C_4H_2+240C_2H_3-80H_3^2-174(H_2-1)H_4-7H_5)$ $-42E_5-22E_6+42E_7+7E_2+49E_4-19E_3-7E_1,$
	6	$20[585C_4-12C_5-30H_4-10(138C_2C_3+8C_3^2-36C_3H_2+33C_4H_2-66C_2H_3+8H_3^2-3H_2H_4)]$ $-596H_5-8(11E_5+9E_6-13E_7-2E_2-23E_4-E_3+4E_1),$
$\delta_4^{(5,i)}$	6	$100[8C_3^2+6C_4(H_2-1)-30C_3H_2+3C_2(23C_3-10H_3)]+2(14E_5+11E_6-14E_7-2E_2-26E_4+E_3+5E_1),$

coefficients. The combination appearing in the amplitude can be fixed by requiring that, as at lower powers of the spin, the amplitude does not grow faster than the spin-independent part at large σ, where we take $\mathcal{E}_{1,2}\sim\sigma$ because of its momentum dependence. Equivalently, one may require that the classical part of the one-loop amplitude does not grow faster at high energies than the tree-level amplitude. The polynomials $\delta^{(5,i)}$ are then uniquely fixed, given below together with $\gamma^{(5,i)}$,

$$\gamma_{\rm Kerr}^{(5,1)}=16(7-13\sigma^2),\quad \frac{1}{75}\delta_{\rm Kerr}^{(5,1)}=24(1-4\sigma^2),$$
$$\gamma_{\rm Kerr}^{(5,2)}=32(11\sigma^2-5),\quad \frac{1}{75}\delta_{\rm Kerr}^{(5,2)}=48(2+\sigma^2),$$
$$\gamma_{\rm Kerr}^{(5,3)}=16(3\sigma^2-1),\quad \frac{1}{75}\delta_{\rm Kerr}^{(5,3)}=8(12-16\sigma^2+7\sigma^4).\quad (11)$$

These results are collected in an attached *Mathematica* text file [48]. It is interesting to understand if the symmetry (10) and the high-energy scaling are sufficient to determine all Wilson coefficients for all powers of the spin at $O(G^2)$ and perhaps beyond when supplemented with other information such as tree-level matching [17].

Reference [19] related directly the amplitude coefficients and the coefficients of a set of spin structures in the position-space Hamiltonian. Table VI includes those that depend only on $\boldsymbol{S}_1$. We express the position-space Hamiltonian in terms of 2, 9, 18, 43, 86 $\boldsymbol{S}^n$ structures, keeping both $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ for $n=1$, 2, 3, 4, 5, respectively. These structures and their coefficients are included in *Mathematica*-readable files in Supplemental Material [48]. Using this Hamiltonian, which depends only on canonical variables, the conservative part of any bound or unbound physical observable can be determined straightforwardly by solving Hamilton's equations.

We compared the scattering angle through S^4 in the aligned-spin limit with the results of Refs. [6,16] for Kerr black holes and found complete agreement. The explicit expressions used in this comparison are given in Supplemental Material [48,53]. We also found that our scattering angle for generic bodies is consistent with a generalization of Ref. [54] that departs from Kerr geometry [55]. Interestingly, the coefficients of R^2 operators in the framework of Ref. [54] are related to quadratic combinations of our C_n coefficients.

TABLE VI. The Hamiltonian spin structures for the first five orders in $\boldsymbol{S}_1$.

$(1/r^2)\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^4)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2$	$(1/r^2)\boldsymbol{S}_1^2$
$(1/r^2)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2$	$(1/r^4)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^6)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2\boldsymbol{L}\cdot\boldsymbol{S}_1$
$(1/r^4)\boldsymbol{S}_1^2\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^4)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^4$	$(1/r^6)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2$
$(1/r^8)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^4$	$(1/r^4)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2\boldsymbol{S}_1^2$	$(1/r^6)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2\boldsymbol{S}_1^2$
$(1/r^4)\boldsymbol{S}_1^4$	$(1/r^6)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^4\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^6)\boldsymbol{S}_1^4\boldsymbol{L}\cdot\boldsymbol{S}_1$
$(1/r^{10})(\boldsymbol{r}\cdot\boldsymbol{S}_1)^4\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^6)(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2\boldsymbol{S}_1^2\boldsymbol{L}\cdot\boldsymbol{S}_1$	$(1/r^8)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2\boldsymbol{S}_1^2\boldsymbol{L}\cdot\boldsymbol{S}_1$
$(1/r^8)(\boldsymbol{r}\cdot\boldsymbol{S}_1)^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)^2\boldsymbol{L}\cdot\boldsymbol{S}_1$		

Finally, let us comment on a special class of linear-in-curvature operators which also include factors of $\nabla_a \phi_s M^{ab}$, which may be thought of as the off-shell covariantization of the $p_a S^{ab}$. An example is $(D_2/m^2) R_{abcd} \nabla_i \phi_s M^{ai} M^{cd} \nabla^b \phi_s$ at S^2. The three-point matrix elements of all operators containing $\nabla_a \phi_s M^{ab}$ vanish in the classical limit, being proportional to the SSC $p_a S^{ab} = 0$. The higher-point matrix elements of these operators are more subtle, and indeed explicit calculations reveal that these operators contribute nontrivially to classical observables. Therefore, covariant operators that give the same three-point matrix elements upon using SSC do not necessarily give equal four- or higher-point matrix elements, nor is it guaranteed that the difference can be absorbed into curvature-square operators. Interestingly, all contributions linear in D_n to the classical amplitude are also proportional to $(H_2 - 1)$, which vanishes if $H_2 = 1$; this is, however, no longer the case for the nonlinear dependence on D_n.

Conclusions.—In this Letter we constructed the $O(G^2)$ 2PM two-body Hamiltonian for general compact objects, including Kerr black holes. We did so by extracting it from a variety of new amplitudes computed in the field-theory approach of Ref. [19]. Our explicit results are included in Supplemental Material [48]. Here we comment on two new and unexpected features that we identified and require further investigation.

We encountered a larger number of operators with independent Wilson coefficients at each order in spin; they include all those of the worldline approach, and others that either have fixed coefficients or do not have an obvious counterpart in the worldline approach [55]. Effectively, every gauge-invariant structure of the four-dimensional classical Compton corresponds to an independent Wilson coefficient. For linear-in-curvature operators we find that apart from C_i which agree with the worldline perspective, the Wilson coefficients H_i also have independent contribution, so through $O(S^4)$ there are three additional coefficients. We demonstrated this point by explicitly computing amplitudes up to $O(S^5)$. It is important to understand the origin of this additional freedom in our formalism, for example, whether it is a consequence of the unconstrained nature of the higher-spin field we use, and whether it corresponds to astrophysical phenomena beyond the current worldline description. We expect that a categorization of all independent higher-spin interactions in both the worldline and field-theory approaches together with a systematic comparison of results for observables will help resolve these issues.

Based on our explicit results and the observation of a spin shift symmetry, we also conjectured that certain spin-dependent structures characterize the Kerr-black-hole interactions to all orders in spin, at least through $O(G^2)$. We proposed this together with the requirement that the amplitude grows no worse than the spin-independent part of $\mathcal{M}$ at high energies as a field-theory *definition* of the Kerr black hole limit. It would be important to understand the physical interpretation of the shift symmetry and whether these constraints properly single out an effective field theory that describes the Kerr black hole of general relativity and study their consequences.

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