Learning Logical Pauli Noise in Quantum Error Correction

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The characterization of quantum devices is crucial for their practical implementation but can be costly in experimental effort and classical postprocessing. Therefore, it is desirable to measure only the information that is relevant for specific applications and develop protocols that require little additional effort. In this Letter, we focus on the characterization of quantum computers in the context of stabilizer quantum error correction. For arbitrary stabilizer codes, subsystem codes, and data syndrome codes, we prove that the logical error channel induced by Pauli noise can be estimated from syndrome data under minimal conditions. More precisely, for any such code, we show that the estimation is possible as long as the code can correct the noise.

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For any quantum device, it is desirable to characterize both its individual components as well as their interplay [1,2]. For the characterization of single quantum gates, protocols such as quantum process tomography (e.g., Ref. [3]) or gate set tomography [4–6] can be used. To characterize the interplay of multiple components, randomized benchmarking [7,8] as well as crosstalk detection [9] and estimation [10,11] protocols are available. The general goals are (i) to build trust in the correct functioning of the device, (ii) to be able to reduce the errors on the hardware level and improve the software calibration, and (iii) to compare different devices and platforms in a fair way.

However, such characterization protocols can be quite resource-intensive, requiring many experimental runs of the device, and such protocols' output can be challenging to interpret. Therefore, it has become a pressing issue to obtain easy-to-use information, such as Pauli error rates directly [11–14], ideally using only data that is easy to obtain. The estimation of Pauli noise is also practically interesting because randomized compiling can be used to project the actual noise onto Pauli noise [15,16]. This has explicitly been discussed in the context of quantum error correction (QEC) [17].

In the context of QEC, it has been suggested to reduce the experimental effort of characterization by extracting information from the syndrome data, which is usually collected during error correction anyway [18–26]. Such an approach is complementary to the standard benchmarking before operation. It has the additional advantage of benchmarking all components in the context of the targeted application and making it easier to detect crosstalk. Indeed, syndrome data has been used to calibrate decoders and observe signatures of crosstalk in experiments on the [4,1,2] code [27], the repetition code [28], and the surface code [29]. Finally, estimation based on syndrome data is the only method of characterization that is not invasive, in the sense that the encoded logical information is not perturbed by the measurements. Thus, it is at least in principle suited for estimation of noise in a time-dependent environment [23,30].

For general stabilizer codes, however, the theoretical foundation of such schemes is currently lacking. Since the syndrome measurements must preserve the encoded state, it is not *a priori* clear that they should even contain sufficient information about the noise to be useful for QEC. For example, as shown in our previous work [26], a complete Pauli channel can only be estimated from syndrome data if it is known that the Pauli errors are not correlated across too many qubits, quantified by the *pure* distance. This limit on correlations can be quite strict, as can be seen for the toric code, which has a pure distance of $d = 4$ independent of system size. Hence, this assumption is violated by natural noise processes such as error propagation in the stabilizer measurements, which can introduce data errors on all participating qubits.

In this Letter, we show that the estimation of error rates is possible under much more practical conditions if one focuses only on information that is actually relevant for QEC. It is not necessary to distinguish between logically equivalent errors. Thus, it suffices to estimate the logical noise channel instead of the physical one. At least for phenomenological Pauli noise models, we prove that the situation is as good as one could reasonably hope: as long as the noise affecting a stabilizer code can be corrected by it, one can also estimate the logical noise channel from the corresponding syndrome measurements.

The proof is based on our general framework [26], but extended to consider the logical instead of the physical channel. Similar to randomized benchmarking, we consider the problem in Fourier space [12]. This representation corresponds to a description of the logical channel in terms of moments instead of probabilities. Exploiting a weak assumption of limited correlations, we can further simplify the description by switching from regular moments to a set of canonical moments. Both the logical channel and the syndrome measurements can be represented by linear equations on a small set of canonical moments. By considering the ranks of these two linear systems, we then show that the syndrome measurements determine the logical channel. Computing the ranks boils down to counting a specific subset of logical operators of the code, which we solve by employing a recent generalization of the cleaning lemma [31] of QEC.

Stabilizer codes.—Let us quickly recap the most important features of stabilizer codes for our purposes. A more thorough introduction can, e.g., be found in the books [32,33]. A stabilizer code is described by a commuting subgroup $\mathscr{S} \subseteq \mathcal{P}^n$ of the n-qubit Pauli group, called "stabilizer group." It must fulfill $-I \notin \mathscr{S}$. The "codespace" is then the simultaneous $+1$ eigenspace of all the stabilizers. As is usual in the context of QEC, we disregard phases and view $\mathscr{S}$ as a subgroup of the "effective Pauli group" $\boldsymbol{P}^n \coloneqq \mathcal{P}^n/\{\pm 1, \pm i\}$. This is an Abelian group, but the relevant commutation relations of $\mathcal{P}^n$ can be encoded in the "bicharacter" $\langle \cdot, \cdot \rangle$ on $\boldsymbol{P}^n$, given by

$$\langle a, e \rangle \coloneqq \begin{cases} +1, & a \text{ and } e \text{ commute in } \mathcal{P}^n \\ -1, & a \text{ and } e \text{ anticommute in } \mathcal{P}^n \end{cases}. \quad (1)$$

By definition, all elements of $\mathscr{S}$ act trivially on the encoded states. We can also consider Pauli operators that map the codespace to itself, but do not necessarily act as the identity. These form the set $\mathcal{L} \subseteq \boldsymbol{P}^n$ of "logical operators." It can be shown that $\mathcal{L}$ is exactly the set of Pauli operators that commute with all stabilizers. Formally, we can express this as the "annihilator" $\mathscr{S}^\perp$ of $\mathscr{S}$ in $\boldsymbol{P}^n$ under the above bicharacter, i.e.,

$$\mathcal{L} \coloneqq \mathscr{S}^\perp \coloneqq \{l \in \boldsymbol{P}^n \colon \langle s, l \rangle = +1 \quad \forall \; s \in \mathscr{S}\}. \quad (2)$$

In particular, we have $\mathscr{S} \subseteq \mathcal{L}$ since each stabilizer is itself a logical operator that implements the logical identity. If a logical operator (other than a stabilizer) occurs as an error, this cannot be detected and the encoded state is corrupted. The distance d of a code is defined as the minimal weight of an element of $\mathcal{L} \backslash \mathscr{S}$. This measures the error correction capabilities of the code. We call a set of qubits $R \subseteq \{1, \ldots, n\}$ "correctable" if it only supports trivial logical operators. This definition is inspired by the discussions in Refs. [34,35]. In particular, if $|R| < d$, then R is correctable. This is however generally not an equivalence, and there can be many correctable regions of size much larger than d. For example, any rectangular region of side length at most $d-1$ on the $d \times d$ toric code is correctable, but contains more than d qubits.

We will focus on phenomenological Pauli noise models and thus do not take into account the details of error propagation inside the measurement circuits. We can then consider rounds of error correction, and between two rounds a new Pauli error occurs. These Pauli errors are described by a quantum channel P, which is given by a probability distribution

$$P\colon \boldsymbol{P}^n \mapsto [0, 1]. \quad (3)$$

Later we will also impose some locality assumptions on this channel.

Standard error correction using a stabilizer code proceeds as follows: in each round, a set of generators $g_1, \ldots, g_m \in \mathscr{S}$ is measured. Ideally, the state lies in the codespace and thus all measurements return $+1$. However, if an error $e \in \boldsymbol{P}^n$ occurred beforehand, the outcome of the measurement of g_i is $\langle g_i, e \rangle = \pm 1$. The collection of measurement outcomes of all generators is called the "syndrome" $S(e)$ of an error e. Based on the syndrome, a decoder tries to guess the error that occurred, and applies it as a correction r. Since errors that only differ by stabilizers are logically equivalent, the ideal decoding strategy for a given syndrome S is to return a maximum likelihood estimate of the form

$$r = \underset{e \in \boldsymbol{P}^n \colon S(e) = S}{\arg\max} \sum_{s \in \mathscr{S}} P(es). \quad (4)$$

Thus, full knowledge of the physical channel P is not necessary for optimal decoding. Instead, it is sufficient to know the "logical channel" P_L, which we define by averaging P over cosets of $\mathscr{S}$:

$$P_L\colon \boldsymbol{P}^n \to [0, 1], \qquad P_L(e) = \frac{1}{|\mathscr{S}|} \sum_{s \in \mathscr{S}} P(es). \quad (5)$$

We note that often the term "logical channel" is defined to be a map acting only on the logical information, conditioned on each syndrome (e.g., Refs. [36–38]). That is, if the code encodes k qubits there is one distribution on $\boldsymbol{P}^k$ for each syndrome. However, this definition depends on the choice of correction for each syndrome since the state needs to be mapped back to the codespace. Here, we define the logical channel in a decoder-independent way. In particular, we only consider "predecoding" noise, i.e., the noise before any potential decoding operation. In other words, our definition, Eq. (5), can be viewed as a lift of the logical channels for each syndrome from $\boldsymbol{P}^k$ to $\boldsymbol{P}^n$, resulting in a distribution P_L, which is constant on cosets of $\mathscr{S}$. In particular, P_L contains all the same information as the logical channels in the usual sense. In standard error correction, it is assumed that the logical channel is known, and the task is to find a good decoding for each syndrome. Here, however, we will consider a "reverse" problem: given (an estimate of) the syndrome statistics, can we (uniquely) obtain the logical channel P_L? Perhaps surprisingly, we will show that this is possible as long as the noise affecting the code is correctable in a certain sense.

Moments.—To tackle this estimation problem, we will first switch our description of P via a Fourier transform. The Fourier transform $\mathcal{F}[f]$ of a function $f\colon \boldsymbol{P}^n \to \mathbb{R}$ is defined as

$$\mathcal{F}[f]\colon \boldsymbol{P}^n \to \mathbb{R}, \qquad \mathcal{F}[f](a) = \sum_{e\in\boldsymbol{P}^n} \langle a, e\rangle f(e). \quad (6)$$

This is also sometimes called "Walsh-Hadmard transform" [12]. From the definition, we see that for any stabilizer $s \in \mathscr{S}$, $\mathcal{F}[P](s)$ is exactly the expectation of s in repeated rounds of error correction. It can thus be computed from the measured syndrome statistics. In analogy, we denote $E = \mathcal{F}[P]$ and call this the set of moments, i.e., there is one moment $E(a)$ for each $a \in \boldsymbol{P}$. One should however keep in mind that only the moments corresponding to stabilizers can be measured without destroying the encoded information. Since the Fourier transform is an invertible transformation, with inverse given by

$$\mathcal{F}^{-1}[f](e) = \frac{1}{|\boldsymbol{P}^n|} \sum_{a\in\boldsymbol{P}^n} \langle a, e\rangle f(a), \quad (7)$$

knowing all moments E is equivalent to knowing the complete error distribution P.

Since we are only interested in learning the logical channel, only a subset of all moments needs to be estimated. These are exactly the moments corresponding to logical operators. To see why this is the case, let us first introduce the convolution on $\boldsymbol{P}^n$. For two functions $f, g\colon \boldsymbol{P}^n \to \mathbb{R}$, their convolution is defined by

$$(f * g)(e) = \sum_{e'\in\boldsymbol{P}} f(e')g(ee'). \quad (8)$$

As expected, it can be shown that convolutions transform into products under Fourier transform:

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]. \quad (9)$$

The logical channel P_L, defined in Eq. (5), can be written as the convolution of the physical channel P with the uniform probability distribution over stabilizers $U_{\mathcal{S}}$,

$$P_L = P * U_{\mathscr{S}}. \quad (10)$$

It is well known that $\mathcal{F}[U_{\mathscr{S}}] = \Phi_{\mathscr{S}^\perp} = \Phi_{\mathcal{L}}$, where $\Phi_{\mathcal{L}}$ is the indicator function of $\mathcal{L}$ [39]. Therefore the logical channel can be characterized in Fourier space by the moments

$$E_{\mathcal{L}} \coloneqq E \cdot \Phi_{\mathcal{L}}. \quad (11)$$

This is a special instance of the averaging versus subsampling duality explained in [39]. To summarize the above discussion, the logical channel is fully characterized by the moments corresponding to logical operators. The estimation problem can then be phrased as follows: given the moments $E_{\mathscr{S}}$ of all stabilizers, compute the moments $E_{\mathcal{L}}$ of all logical operators.

Correctable noise.—The above estimation problem cannot be solved for arbitrary channels P, since in general the moments are independent of each other. Here, our assumption of limited correlations becomes important.

To formalize this assumption, consider a set of "supports" $\Gamma \subseteq 2^{\{1,\ldots,n\}}$, where $2^{\{1,\ldots,n\}}$ denotes the powerset of $\{1,\ldots,n\}$. These supports are allowed to overlap with each other. We assume that on each support $\gamma \in \Gamma$, there acts an independent Pauli channel $P_\gamma\colon \boldsymbol{P}^\gamma \to [0, 1]$. Thus, the noise is correlated across each support, but not between different supports. If the supports are small, any high weight error must arise as a combination of independent lower weight errors. This is the scenario where error correction has a chance to improve the fidelity. On the other hand, if the supports are too large, error correction usually fails. Thus, we assume that the noise is "correctable" in the following sense.

Definition 1.—A Pauli channel P described by a set of supports $\Gamma \subseteq 2^{\{1,\ldots,n\}}$ is called "correctable" if the following two conditions are fulfilled: (1) for all $\gamma_1, \gamma_2 \in \Gamma$, the union $\gamma_1 \cup \gamma_2$ is a correctable region; (2) $P_\gamma(I) > \frac{1}{2}$ for all $\gamma \in \Gamma$.

We see from the definition of distance that the first condition is fulfilled, in particular if $|\gamma| \le \lfloor (d-1)/2 \rfloor$ for all $\gamma \in \Gamma$. The second condition simply states that the error rates should not be too large. It guarantees that all moments are positive, i.e., $E(a) > 0$ for all $a \in \boldsymbol{P}^n$. We would like to emphasize that our definition of correctable noise requires quite a weak condition: actual QEC requires the noise level to be below some code-dependent threshold, which is always lower than the one imposed by our condition (2). Definition 1 is also distinct from the Knill-Laflamme condition [40], [[32], Theorem 10.1], which is usually applied to a subnormalized part of the full error channel.

Since the multiplication of independent Pauli random variables corresponds to a convolution of their probability distributions, the full channel P can be written as a convolution of the independent local channels:

$$P = *_{\gamma\in\Gamma} P_\gamma. \quad (12)$$

In this notation, we set $P_\gamma(e) = 0$ if $\mathrm{supp}(e) \not\subseteq \gamma$. In order to better capture this structure in Fourier space, we can introduce a set of "canonical moments" F (which we called "transformed moments" before [26]). For $a, b \in \boldsymbol{P}^n$, let us write $b \le a$ if b is a substring of a. Then we define the canonical moments as

$$F\colon \boldsymbol{P}^n \to \mathbb{R}, \qquad F(a) = \prod_{b\in\boldsymbol{P}^n:\ b\leq a} E(b)^{\mu(b,a)}, \tag{13}$$

where μ is the Möbius function defined by

$$\mu(b,a) = \begin{cases} (-1)^{|a|-|b|}, & b \leq a \\ 0, & \text{otherwise} \end{cases}, \tag{14}$$

which is well known in combinatorics [41]. The Möbius function is defined in such a way that in Eq. (13), we divide out that part of the moment $E(a)$ that is already described by substrings $b \leq a$, without "double counting" any substring. Essentially, while the regular moments E also capture correlations across all subsets of their support, the canonical moments only capture correlations across their full support. The advantage is that a small set of canonical moments is sufficient to fully describe the channel. In particular, the following two facts about canonical moments are shown in the Supplemental Material [42]. First of all, we only need to consider the canonical moments that lie completely inside a channel support γ, since $F(a) = 1$ if $\mathrm{supp}(a) \not\subseteq \gamma$ for all $\gamma \in \Gamma$. The set of such canonical moments is $F_{\Gamma'} = [F(a)]_{a\in\Gamma'}$, where

$$\Gamma' = \{a \in \boldsymbol{P}^n\colon\ \exists \gamma \in \Gamma \text{ such that } \mathrm{supp}(a) \subseteq \gamma\}. \tag{15}$$

Furthermore, the regular moments E are obtained from the canonical moments F by

$$E(a) = \prod_{b\leq a} F(b). \tag{16}$$

Identifiability.—Since the moments $E_{\mathscr{S}}$ can be obtained from the syndrome measurements, and the channel is fully described by the canonical moments $F_{\Gamma'}$, estimation of the physical channel boils down to solving the system of equations

$$E(s) = \prod_{a\in\Gamma', a\subseteq s} F(a). \tag{17}$$

For correctable noise, all moments are positive. Then, Eq. (17) can be transformed into a system of linear equations by taking logarithms. This system can be expressed by the coefficient matrix $D_{\mathscr{S}}$, whose rows are labeled by stabilizers and whose columns are labeled by elements of Γ', with entries

$$D_{\mathscr{S}}[s,a] = \begin{cases} 1, & a \subseteq s \\ 0, & \text{otherwise} \end{cases}. \tag{18}$$

As we have proven before [26], a unique solution exists if the range of correlations of the error channel P is smaller than the pure distance of the code. Correctable noise generally does not fulfill this strict condition. Thus, the system is underdetermined and the physical channel P cannot be estimated just from the syndrome measurements.

We are, however, only interested in estimating the logical channel, Eq. (5), which contains less information. As derived above, Eq. (11), it suffices to consider the moments $E_{\mathcal{L}}$. The question is now whether the moments $E_{\mathcal{L}}$ can be computed from the measured moments $E_{\mathscr{S}}$, i.e., whether the corresponding equations of the form Eq. (17) are linearly dependent after taking logarithms. In other words, the logical channel can be uniquely estimated from the syndrome measurements if

$$\mathrm{rank}(D_{\mathscr{S}}) = \mathrm{rank}(D_{\mathcal{L}}). \tag{19}$$

This condition is equivalent to $\mathrm{rank}(D_{\mathscr{S}}^T D_{\mathscr{S}}) = \mathrm{rank}(D_{\mathcal{L}}^T D_{\mathcal{L}})$. We will prove this by showing the even stronger statement

$$D_{\mathscr{S}}^T D_{\mathscr{S}} \propto D_{\mathcal{L}}^T D_{\mathcal{L}}. \tag{20}$$

First, note that $D_{\mathscr{S}}^T D_{\mathscr{S}}$ can be easily computed from its definition,

$$D_{\mathscr{S}}^T D_{\mathscr{S}}[a,b] = |\{s \in \mathscr{S}\colon\ a \leq s \text{ and } b \leq s\}|. \tag{21}$$

The analogous statement holds for $D_{\mathcal{L}}$. By rewriting Eq. (20) in terms of individual entries, we see that the logical channel can be uniquely estimated from the syndrome statistics if for all $a, b \in \Gamma'$,

$$|\{s \in \mathscr{S}\colon\ a, b \leq s\}| = c|\{l \in \mathscr{S}^{\perp}\colon\ a, b \leq l\}|, \tag{22}$$

where c is a constant independent of a, b. This is a counting problem that depends only on global properties of the stabilizers and logical operators, but not on their specific form. To solve this counting problem, we will employ the well-known cleaning lemma, which was first stated by Bravyi and Terhal [34]. Informally, this lemma states that any correctable region can be cleaned from logical operators.

Lemma 1.—Let R be a correctable region. Then any coset $[l] \in \mathcal{L}/\mathscr{S}$ of logical operators has a representative $l \in \mathcal{L}$ that has no support on R, i.e., $\mathrm{supp}(l) \cap R = \emptyset$.

Using this lemma, we can prove Eq. (22). For all $a, b \in \Gamma'$ we have

$$\begin{aligned}
&|\{l \in \mathcal{L}\colon\ a \leq l \text{ and } b \leq l\}| \\
&\quad = \sum_{l\in\mathcal{L}} [a \leq l \text{ and } b \leq l] \\
&\quad = \sum_{[l]\in(\mathcal{L}/\mathscr{S})} \sum_{s\in\mathscr{S}} [a \leq ls \text{ and } b \leq ls] \\
&\quad = \sum_{[l]\in(\mathcal{L}/\mathscr{S})} \sum_{s\in\mathscr{S}} [a \leq s \text{ and } b \leq s] \\
&\quad = |\mathcal{L}/\mathscr{S}| \cdot |\{s \in \mathscr{S}: a \leq s \text{ and } b \leq s\}|.
\end{aligned}$$

In the second equality, we split the total sum into smaller sums over logically equivalent subsets of logical operators. Then, the third equality follows from the cleaning lemma:

since a and b correspond to canonical moments, they must be fully contained in some supports $\gamma_a, \gamma_b \in \Gamma$. For correctable noise, $\gamma_a \cup \gamma_b$ is a correctable region. Thus, if the union of the supports of a and b is fully contained in $\gamma_a \cup \gamma_b$, it must also be a correctable region. By the cleaning lemma, we can choose the representative l of the coset $[l]$ such that it acts trivially on that region. Then, a is a substring of ls if and only if it is a substring of s, and the same holds for b. This finishes the proof of Eq. (22).

We can summarize the discussion of the main text in the following theorem.

Theorem 1.—A Pauli channel P can be estimated up to logical equivalence from the syndrome measurements of a stabilizer code if P is correctable in the sense of Definition 1.

Note that while we focused on stabilizer codes with perfect measurements for simplicity, several generalizations of this result are possible. Measurement errors can be incorporated using the framework of quantum data syndrome codes [53]. Furthermore, we can also consider subsystem codes [54], which generalize stabilizer codes by allowing for some noncommuting measurements. A full account of these generalizations, including all proofs that are omitted in the main text, is given in the Supplemental Material [42]. The main theorem presented there might also be interesting in contexts other than QEC.

Conclusion.—We have shown that the measurements performed during QEC contain enough information to estimate a large class of phenomenological Pauli noise models up to logical equivalence. Informally, as long as the code can correct the noise, it can also be estimated from the syndrome measurements. This result opens up new characterization possibilities since the previous results have focused only on estimating physical channels. Our result applies to data syndrome codes and general subsystem codes, which encompass most codes in the literature.

While the focus of this Letter is on the fundamental identifiability of Pauli noise from syndrome data in the setting of general subsystem codes, our proofs also suggest a concrete estimation scheme. Since it is sufficient to consider as many equations as there are free parameters in the polynomial system, Eq. (17), this system can in principle be solved in polynomial time in the code size. The sample complexity, however, depends on the conditioning of this system, and hence on the specifics of the code. We note, however, that for, e.g., topological codes, estimation is expected to be possible from local subregions of the code, which implies an efficient sampling complexity [26]. In order to work out these ideas, a specific analysis of concrete codes is required, which is ongoing research.

The focus of this Letter is on phenomenological noise models. For quantum communication or storage, this might be a reasonable assumption. In the context of fault-tolerant quantum computing, however, full circuit level noise models are more realistic than phenomenological ones, which introduces additional complications already for decoding in the first place. A common approach to this problem is to consider approximate noise models. For example, a minimum-weight perfect matching decoder maps the actual noise to a simplified graph with weighted edges [23,55]. Here, our results apply directly, and the edge weights can be estimated up to logical equivalence by solving our equation system, Eq. (17).

The situation is less clear if one is interested in more details than such an effective noise model provides. In this case, one might attempt to transfer our results using a cutoff for late errors, following Delfosse *et al.* [56], or using a mapping from circuit noise to subsystem codes, as given in Refs. [43–45]. We think that our work can serve as a basis for many possible research questions on characterization in the context of QEC.

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