Learning Many-Body Hamiltonians with Heisenberg-Limited Scaling

Hsin-Yuan Huang,[1] Yu Tong,[1,2] Di Fang,[2,3] and Yuan Su[4]
[1]Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA
[2]Department of Mathematics, University of California, Berkeley, California 94720, USA
[3]Simons Institute for the Theory of Computing, University of California, Berkeley, California 94720, USA
[4]Microsoft Quantum, Redmond, Washington 98052, USA

(Received 3 November 2022; accepted 18 April 2023; published 16 May 2023)

Learning a many-body Hamiltonian from its dynamics is a fundamental problem in physics. In this Letter, we propose the first algorithm to achieve the Heisenberg limit for learning an interacting N-qubit local Hamiltonian. After a total evolution time of $\mathcal{O}(\epsilon^{-1})$, the proposed algorithm can efficiently estimate any parameter in the N-qubit Hamiltonian to ϵ error with high probability. Our algorithm uses ideas from quantum simulation to decouple the unknown N-qubit Hamiltonian H into noninteracting patches and learns H using a quantum-enhanced divide-and-conquer approach. The proposed algorithm is robust against state preparation and measurement error, does not require eigenstates or thermal states, and only uses polylog(ϵ^{-1}) experiments. In contrast, the best existing algorithms require $\mathcal{O}(\epsilon^{-2})$ experiments and total evolution time. We prove a matching lower bound to establish the asymptotic optimality of our algorithm.

DOI: 10.1103/PhysRevLett.130.200403

Learning an unknown Hamiltonian H from its dynamics $U(t) = e^{-iHt}$ is an important problem that arises in quantum sensing and metrology [1–9], quantum device engineering [10–15], and quantum many-body physics [16–23]. In quantum sensing and metrology, the Hamiltonian H encodes signals that we want to capture. A more efficient method to learn H implies the ability to extract these signals faster, which could lead to substantial improvement in many applications, such as microscopy, magnetic field sensors, and positioning systems. In quantum computing, learning the unknown Hamiltonian H is crucial for calibrating and engineering the quantum device to design quantum computers with lower error rates. In quantum many-body physics, the unknown Hamiltonian H characterizes the physical system of interest. Obtaining knowledge of H is hence crucial to understanding microscopic physics. A central goal in these applications is to find the most efficient approach for learning H.

In this Letter, we focus on the task of learning many-body Hamiltonians describing a quantum system with a large number of constituents. For concreteness, we consider an N-qubit system with geometrically local interactions. Given any unknown N-qubit geometrically local Hamiltonian H, we can represent H as

$$H = \sum_{a=1}^{M} \lambda_a E_a. \tag{1}$$

Here, $\lambda_1, \ldots, \lambda_M$ are the unknown parameters and $S = \{E_1, \ldots, E_M\} \subseteq \{I, X, Y, Z\}^{\otimes N}$ is a subset of N-qubit Pauli operators. Each Pauli operator E_a acts nontrivially on at most $k = \mathcal{O}(1)$ qubits, and each qubit is acted on by $\mathcal{O}(1)$ of the Pauli operators in S. Many-body Hamiltonians with nearest-neighbor interactions on one-dimensional chains, two-dimensional square lattices, and three-dimensional cubic lattices are all special cases of geometrically local Hamiltonians. The goal is to learn the parameters λ_a in the unknown Hamiltonian H. In previous works on learning many-body Hamiltonians [24–35], in order to reach an ϵ precision in estimating the parameters λ_a, the number of experiments and the total time required to evolve the system have a scaling of at least ϵ^{-2}. However, the ϵ^{-2} precision scaling is likely not the best-possible scaling for learning an unknown many-body Hamiltonian H.

In quantum sensing and metrology, the scaling of ϵ^{-2} for learning an unknown parameter to ϵ error is known as the standard quantum limit. For simple classes of Hamiltonians, such as a single-qubit Hamiltonian $H = \omega Z$ with unknown parameter ω, one can surpass the standard quantum limit using quantum-enhanced protocols [1,3,7,36–38]. The true limit set by the basic principles of quantum mechanics is known as the Heisenberg limit, which suggests a scaling of ϵ^{-1}. There are two well-known approaches for achieving the Heisenberg limit for learning $H = \omega Z$. The first approach [3–5] considers evolving a highly entangled state over $\ell = \mathcal{O}(\epsilon^{-1})$ qubits of the system under ℓ parallel Hamiltonian evolutions $(e^{-iHt})^{\otimes \ell}$ with $t = \mathcal{O}(1)$. The second approach [1,39,40] considers long-time coherent evolution $e^{-i\omega t Z}$ with $t = \mathcal{O}(\epsilon^{-1})$ over a single qubit. While the first approach was proposed earlier, the second

approach has the advantage of requiring only a single qubit without entanglement.

The ϵ^{-1} scaling underlying the two approaches corresponds to the "total evolution time." If a protocol uses J experiments, where the jth experiment uses the unknown Hamiltonian evolution $e^{-iHt_{j,1}}, \ldots, e^{-iHt_{j,K_j}}$, then the total evolution time is defined as

$$T \triangleq \sum_{j=1}^{J} \sum_{k=1}^{K_j} t_{j,k}. \tag{2}$$

In the first approach, each experiment uses $\mathcal{O}(\epsilon^{-1})$ constant time Hamiltonian evolutions in parallel, while the second approach uses $\mathcal{O}(\epsilon^{-1})$ constant time Hamiltonian evolutions sequentially resulting in a single long-time evolution. Both quantum sensing approaches result in a total evolution time of $\mathcal{O}(\epsilon^{-1})$.

These quantum-enhanced approaches could be applied to noninteracting systems as studied in multiparameter quantum sensing [41–44]. However, they are challenging to apply in interacting systems with a large system size N and many unknown parameters. The difficulty stems from the many-body interactions in the Hamiltonian H. As time t becomes larger, the entanglement growth in e^{-itH} will cause all the unknown parameters in H to tangle with one another. The many-body entanglement can be seen as a form of decoherence, which kills the quantum enhancement. To prevent the system from becoming too entangled, prior work on learning many-body Hamiltonians focuses on a short-time t, which loses the quantum enhancement and obtains, at best, an ϵ^{-2} scaling.

In this Letter, we propose the first learning algorithm to achieve the Heisenberg limit for learning interacting many-body Hamiltonian. Figure 1 illustrates our algorithm. We prove the following performance guarantee.

Theorem 1: There is an algorithm robust to state preparation and measurement error [45] that achieves the following: For any unknown N-qubit geometrically local Hamiltonian $H = \sum_{a=1}^{M} \lambda_a E_a$ with $|\lambda_a| \le 1$, after a total evolution time $T = \mathcal{O}(\epsilon^{-1} \log(\delta^{-1}))$, the learning algorithm can obtain estimates $\hat{\lambda}_a$ from the experiments, such that $\Pr[|\hat{\lambda}_a - \lambda_a| \le \epsilon] \ge 1 - \delta$ for all $a \in \{1, \ldots, M\}$.

In quantum sensing and metrology, one often considers the standard deviation of the estimate. We can show that to ensure the standard deviation $\sqrt{\mathbb{E}[|\hat{\lambda}_a - \lambda_a|^2]} \le \epsilon$, we only need a total evolution time of $T = \mathcal{O}(\epsilon^{-1})$. This is because each estimate $\hat{\lambda}_a$ comes from a linear combination of $\mathcal{O}(1)$ eigenvalue estimates through a Hadamard transform, as shown in [[51], Eq. (25)]. Each eigenvalue estimate has standard deviation at most $\mathcal{O}(\epsilon)$ as guaranteed by [[40], Theorem I.1]. Consequently, their linear combination $\hat{\lambda}_a$ also has a standard deviation that scales as $\mathcal{O}(\epsilon)$. Hence, our algorithm saturates the Heisenberg limit in terms of the standard deviation. Our algorithm has the additional advantage of not requiring eigenstates or thermal states of the Hamiltonian H. Each of our experiments consists of the preparation of a noisy all-zero state $|0^N\rangle$, the evolution under the Hamiltonian H interleaved with single-qubit Clifford gates, and a noisy Z-basis measurement. The total number of experiments is only $\mathcal{O}(\text{polylog}(\epsilon^{-1}))$, which is significantly smaller than $\Theta(\epsilon^{-1})$. After running the experiments, the classical computational time to estimate all parameters is only $\mathcal{O}(N\text{polylog}(\epsilon^{-1}))$. Detailed statements can be found in [[51], Theorems 13 and 21]. We note that our result generalizes to all low-intersection Hamiltonians as given in [[51], Definition 2].

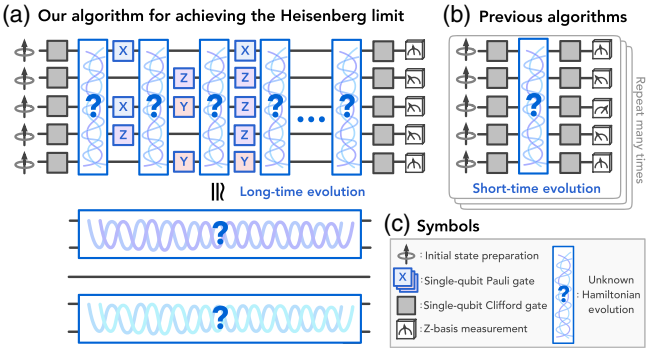

FIG. 1. Algorithms for learning many-body Hamiltonians. (a) Our algorithm for achieving the Heisenberg limit ϵ^{-1}. We perform long-time coherent evolutions interleaved with random Pauli operators. The effective Hamiltonian is decoupled into noninteracting patches and can be efficiently learned. The algorithm only needs $\mathcal{O}(\text{polylog}(\epsilon^{-1}))$ experiments and a total evolution time of $\mathcal{O}(\epsilon^{-1})$. (b) Previous algorithms for achieving the standard quantum limit ϵ^{-2}. Previous methods [26,29,33,34] repeatedly run a short-time evolution for many times. One needs $\mathcal{O}(\epsilon^{-2})$ experiments and a total evolution time of $\mathcal{O}(\epsilon^{-2})$. (c) Symbols: The symbols used in (a), (b). The unknown Hamiltonian evolution is $U(t) = e^{-iHt}$.

To establish the optimality of the proposed algorithm, we prove a matching lower bound.

Theorem 2: Suppose there is a learning algorithm robust to state preparation and measurement error that achieves the following. For any unknown N-qubit geometrically local Hamiltonian $H = \sum_{a=1}^{M} \lambda_a E_a$ with $|\lambda_a| \le 1$, after a total evolution time T, the learning algorithm can obtain estimates $\hat{\lambda}_a$ from the experiments, such that $\Pr[|\hat{\lambda}_a - \lambda_a| \le \epsilon] \ge 1 - \delta$ for all $a \in \{1, \ldots, M\}$. Then, $T = \Omega(\epsilon^{-1} \log(\delta^{-1}))$.

Thus, there is no algorithm that can perform asymptotically better than the one given in Theorem 1. Moreover, the lower bound can be seen as an algorithmic proof of the Heisenberg limit with the failure probability δ taken into account. It holds not only for algorithms with a fixed set of experiments but also for adaptive experiments that use information from previous experiment outcomes, following the setup in [52–55].

In the following, we provide the ideas for designing the proposed learning algorithm and establishing the proof of the main results. All parts except for the last are devoted to Theorem 1. The last part is Theorem 2.

Reshaping an unknown Hamiltonian.—A key technique used throughout the design of our learning algorithm is the idea of reshaping an unknown Hamiltonian using Hamiltonian simulation techniques. Recall that given a set of Hamiltonians $H_1, \ldots, H_K$ and the ability to implement the unitaries $e^{-itH_1}, \ldots, e^{-itH_K}$, many Hamiltonian simulation techniques allow one to approximately implement the unitary $e^{-it\sum_{k=1}^{K} H_k}$. Note that these approximation formulas are valid for unitaries and no knowledge of the underlying Hamiltonian is required. As such, they are applicable to the learning problem considered here.

For example, a randomized Hamiltonian simulation algorithm known as qDRIFT [56–58] considers an approximation (as a quantum channel) given by

$$e^{-it\sum_{k=1}^{K} H_k} \approx e^{-i(t/r)H_{k_r}} \ldots e^{-i(t/r)H_{k_1}}, \tag{3}$$

where r is an integer that sets the approximation error, $k_1, \ldots, k_r$ are independent random variables sampled according to some probability distribution over $\{1, \ldots, K\}$. Alternatively, we can also use the second-order Trotterization method [59–61] in our algorithm to reduce the asymptotic scaling of the number of Clifford gates required. Higher-order Trotterizations are not used because they require evolving backward in time.

Now, consider the unknown N-qubit Hamiltonian H that we hope to learn. We want to reshape it into the following Hamiltonian to facilitate learning:

$$\tilde{H} \triangleq \sum_{k=1}^{K} w_k H_k, \tag{4}$$

where $H_k \triangleq w_k U_k H U_k^\dagger$ and U_k is a unitary for each $k = 1, 2, \ldots, K$. The weights $w_k \geq 0$. Any choice of unitaries U_k and weights w_k can be used. Later, to achieve the Heisenberg limit, we will choose specific U_k and w_k to ensure $\tilde{H}$ disentangles the many-body system into noninteracting patches involving few qubits and has known eigenvectors irrespective of what H is. Our choice for each U_k will be a tensor product of Pauli operators. Using either qDRIFT or Trotterization, we only need to implement e^{-iH_k}, which can be done through $e^{-iH_k t} = U_k e^{-i(w_k t)H} U_k^\dagger$. To be more specific, we can implement $e^{-iH_K t}$ by first applying the unitary $U_k^\dagger$, letting the system evolve for time $w_k t$, and then applying U_k. Using Hamiltonian simulation techniques, we can evolve under the N-qubit unitary $e^{-it\tilde{H}}$. This reshaping technique is related to experimental approaches for engineering Hamiltonians through pulse sequences or strong fields [62–68]. Similar ideas have also been used to project H into the quantum Zeno subspace [69,70]. The reshaping will lead to a small approximation error, which we discuss later (for a detailed discussion, see [[51], Sec. IV and VI]).

Learning a few-qubit Hamiltonian.—We now show how the Hamiltonian reshaping technique is useful in learning Hamiltonians. We begin with a simple question: how can one learn a few-qubit Hamiltonian on $\mathcal{O}(1)$ qubits with Heisenberg-limited precision scaling? If we naively apply quantum process tomography [54,71–77] to learn the unknown Hamiltonian, we would have an ϵ^{-2} dependence in the number of measurements needed, where ϵ is the desired precision of the Hamiltonian parameters. Current methods with a Heisenberg-limited scaling typically require the Hamiltonian to be of a simple form, e.g., $H = \lambda X$ [1–9,39,40]. Therefore we need to consider a different method.

We show that for a few-qubit Hamiltonian we can learn all the parameters involved using $\mathcal{O}(\epsilon^{-1}\log(\delta^{-1}))$ total evolution time, and $\mathcal{O}(\mathrm{polylog}(\epsilon^{-1})\log(\delta^{-1}))$ number of experiments. As an example, let us consider an arbitrary two-qubit Hamiltonian,

$$H = \sum_{P,P' \in \{I,X,Y,Z\}} \lambda_{PP'} P \otimes P', \tag{5}$$

with $|\lambda_{PP'}| \leq 1$. Suppose we want to estimate the parameter λ_{XZ}. Then we can consider reshaping the unknown Hamiltonian H using $U_1 = I$, $U_2 = X_1$, $U_3 = Z_2$, $U_4 = X_1 Z_2$, and $w_1 = w_2 = w_3 = w_4 = \frac{1}{4}$. The new unknown Hamiltonian, after reshaping, is given by

$$\begin{aligned}\tilde{H} &\triangleq \frac{1}{4}(H + X_1 H X_1 + Z_2 H Z_2 + X_1 Z_2 H X_1 Z_2) \\ &= \lambda_{XZ} X_1 Z_2 + \lambda_{XI} X_1 + \lambda_{IZ} Z_2.\end{aligned} \tag{6}$$

The second equality is because the linear combination over the four terms eliminates all Pauli terms in H that do not have I or X on the first qubit and I or Z on the second qubit.

This new unknown Hamiltonian $\tilde{H}$ gives us one crucial advantage: we have access to its eigenstates. This is because in $\tilde{H}$, for each qubit, there is only one (nonidentity) Pauli operator associated with it. The eigenbasis for the new unknown Hamiltonian $\tilde{H}$ is always given by $\{|+\rangle|0\rangle, |+\rangle|1\rangle, |-\rangle|0\rangle, |-\rangle|1\rangle\}$ regardless of the values of the unknown coefficients. We can use this information, together with the robust phase estimation algorithm in [40], to estimate the differences between pairs of eigenvalues, which in turn yield the parameters $\lambda_{XZ}, \lambda_{XI}, \lambda_{IZ}$ through a Hadamard transform. The procedure for applying random Pauli operators and obtaining parameters from eigenvalue estimation is described in detail in [[51], Sec. II.B and III.B]. By using different choices of $U_1, \ldots, U_4$ to reshape H, we can get all the parameters $\lambda_{PP'}$ in the two-qubit

Hamiltonian H. The same idea generalizes to arbitrary Hamiltonians on $\mathcal{O}(1)$ qubits.

Learning a many-qubit Hamiltonian through divide and conquer.—If we want to learn a many-qubit Hamiltonian by directly applying the above method, the total evolution time will scale exponentially with the number of qubits. Here, we present a divide-and-conquer approach to address this problem. To illustrate the proposed approach, let us consider a simple example of an inhomogeneous Heisenberg model on N qubit with a Hamiltonian given by

$$H = \sum_{\alpha=1}^{N-1} (\lambda_x^\alpha X_\alpha X_{\alpha+1} + \lambda_y^\alpha Y_\alpha Y_{\alpha+1} + \lambda_z^\alpha Z_\alpha Z_{\alpha+1}), \quad (7)$$

where $\lambda_x^\alpha, \lambda_y^\alpha, \lambda_z^\alpha$ are the unknown parameters, and $X_\alpha, Y_\alpha, Z_\alpha$ are the Pauli operators acting on qubit α. Suppose we want to learn the parameter λ_x^1 on the first two qubits. In order to achieve this, we reshape the unknown Hamiltonian H with $U_1 = I$, $U_2 = X_3$, $U_3 = Y_3$, $U_4 = Z_3$, and $w_1 = w_2 = w_3 = w_4 = \frac{1}{4}$. The new unknown Hamiltonian after the reshaping is given by

$$\begin{aligned}\tilde{H} &= \frac{1}{4}(H + X_3 H X_3 + Y_3 H Y_3 + Z_3 H Z_3) \\ &= \tilde{H}_{1,2} + \tilde{H}_{\geq 4}, \end{aligned} \quad (8)$$

where

$$\tilde{H}_{1,2} = \lambda_x^{1,2} X_1 X_2 + \lambda_x^{1,2} Y_1 Y_2 + \lambda_x^{1,2} Z_1 Z_2 \quad (9)$$

and $\tilde{H}_{\geq 4}$ only contains terms acting on qubits $4, 5, \ldots, N$. The second equality in Eq. (8) holds for the following reason: for each Pauli operator $P \in \{I, X, Y, Z\}^{\otimes N}$, if it acts nontrivially on the third qubit, then we can show that

$$\frac{1}{4}(P + X_3 P X_3 + Y_3 P Y_3 + Z_3 P Z_3) = 0. \quad (10)$$

On the other hand, for Pauli operator P that acts as identity on the third qubit, we can show that

$$\frac{1}{4}(P + X_3 P X_3 + Y_3 P Y_3 + Z_3 P Z_3) = P. \quad (11)$$

Therefore from Eq. (8), after the reshaping, the new Hamiltonian $\tilde{H}$ does not generate entanglement between qubits 1,2 and the rest of the system, and these two qubits evolve under the Hamiltonian $\tilde{H}_{1,2}$. This enables us to apply the learning algorithm for few-qubit Hamiltonians to $\tilde{H}_{1,2}$ to estimate λ_x^1.

We can apply the above idea to learn every parameter in the Hamiltonian with a number of experiments that scales linearly in the system size N rather than exponential in N. We show that one could do better than linear scaling by a parallelization technique. In particular, we discuss how one could learn all the parameters $\lambda_x^1, \lambda_x^4, \lambda_x^7, \cdots$ in parallel. Consider reshaping the unknown N-qubit Hamiltonian H given in Eq. (7) using $U_1 = I$, $U_2 = X_3 X_6 X_9 \ldots$, $U_3 = Y_3 Y_6 Y_9 \ldots$, $U_4 = Z_3 Z_6 Z_9 \ldots$, and $w_1 = w_2 = w_3 = w_4 = 1/4$. Then the new Hamiltonian under reshaping is given by $\tilde{H} = \tilde{H}_{1,2} + \tilde{H}_{4,5} + \tilde{H}_{7,8} + \cdots$, where $\tilde{H}_{\alpha,\alpha+1}$ is supported on qubits α and $\alpha+1$ for all $\alpha = 1, 4, 7, \ldots$. Using a reshaping based on four unitaries $U_1, \ldots, U_4$, we have turned the unknown N-qubit interacting Hamiltonian H into a new Hamiltonian $\tilde{H}$ with many noninteracting patches of two qubits. Each two-qubit patch is now evolving independently from the others. This decoupling enables us to estimate the parameters in parallel using the algorithm for learning few-qubit Hamiltonians.

This divide-and-conquer method works for any local Hamiltonian defined in Eq. (1). For this more general class of Hamiltonians, we determine how the reshaping is done by performing a coloring over its cluster interaction graph (a graph consisting of clusters of qubits that are acted on by a Pauli term in the Hamiltonian) [[51], Lemma 5]. The coloring enables us to choose qubits, on which we apply random I, X, Y, Z operators to decouple clusters of the same color from each other, thus enabling parallel estimation of the parameters associated with these clusters. For details, see [[51], Sec. I.B, II, and V]. A complete description of our algorithm for general local Hamiltonians can be found in [[51], Algorithm 2]. The cost of the algorithm is summarized in [[51], Theorems 13 and 21] (for the randomization and Trotterization approaches, respectively).

Characterizing approximation error in reshaping Hamiltonians.—The estimation error of the proposed learning algorithm depends on the quantum measurement error as well as the approximation error when we reshape the unknown Hamiltonian into other forms. One way to analyze the approximation error is through the error analysis considered in [56] if we use qDRIFT to reshape or in [78] when using the second-order Trotter formula. However, these analyses are concerned with the error in the worst-case scenario over all possible input states and all observables. For the learning task given here, it leads to an overestimation of the approximation error as some key properties of the problem are not incorporated.

Consider the example of learning an inhomogeneous Heisenberg model on N qubits given in the previous section. To evolve under the N-qubit Hamiltonian reshaped $\tilde{H}$ for time t, the analysis in [56] shows that the approximation error of qDRIFT with r steps is given by $\mathcal{O}(N^2 t^2/r)$. Here, $\tilde{H}$ is decoupled into many two-qubit patches that do not interact with each other, which prevents errors from propagating across the entire N-qubit system. We are interested only in the accuracy of evolving each patch, and the error from elsewhere in the system should not affect estimations of local observables. Similar considerations have been used to improve the error analysis of Hamiltonian simulation

methods based on observable and initial state information [79–83]. In our case, a tighter analysis [84] using these facts shows that the approximation error is given by $\mathcal{O}(t^2/r)$ without an N dependence. We give the improved analysis for reshaping Hamiltonians using the randomization approach in [[51], Sec. IV]. The improved analysis for using the second-order Trotter formula is given in [[51], Sec. VI].

Establishing a matching lower bound.—We prove a matching lower bound of $T = \Omega(\epsilon^{-1}\log(\delta^{-1}))$ on the total evolution time T [86]. The optimality with respect to the ϵ dependence is obtained by the Heisenberg limit. However, the optimality with respect to the failure probability δ has not been proven in the literature. We consider any learning algorithm that can run new experiments based adaptively on the outcomes of previous experiments. In order to handle adaptivity, we consider the rooted tree representation of the learning algorithm [53,55], and consider the task of distinguishing between two distinct Hamiltonians $H_\pm = \pm\epsilon Z$.

We begin by considering how well one could use a single experiment to distinguish $H_\pm$, which is characterized by the total variation (TV) distance between the probability distribution over experimental outcomes under $H_\pm$. We characterize the TV distance in a single experiment. Then we consider an induction over every subtree of the learning algorithm to establish the TV distance over multiple experiments. A central technique is to control how each additional experiment improves one's ability to distinguish $H_\pm$. The proof of the lower bound is given in [[51], Sec. VII].

Discussion.—Our work shows that the Heisenberg limit can be achieved in the task of learning a large class of many-body local Hamiltonians with many unknown parameters. On the theoretical side, the central open question is whether our result can be extended to learning other classes of many-body Hamiltonians. For example, in an N-qubit Hamiltonian with all-to-all two-body interactions, our techniques achieve the Heisenberg limit with a quadratic dependence on system size N by learning all pairwise interactions one by one. This gives rise to the following question: can we achieve a scaling of $T = \mathcal{O}(\epsilon^{-1}\log(\delta^{-1}))$ for N-qubit Hamiltonians with all-to-all interactions? In addition to the above example, can we achieve the Heisenberg limit for learning fermionic or bosonic many-body Hamiltonians? Answering these questions is important for applications such as reconstructing the structure of large molecules or learning the interactions in an exotic quantum material. Even more ambitiously, can one achieve the above scaling for learning the unknown parameters in an arbitrary N-qubit Hamiltonian without any structure? On the practical side, the central question is how to achieve the Heisenberg limit with minimal controllable quantum operations. For example, could one achieve the scaling $T = \mathcal{O}(\epsilon^{-1}\log(\delta^{-1}))$ for learning N-qubit local Hamiltonian H in a restricted model where we cannot interleave the unknown Hamiltonian evolution with single-qubit gates and can only control state preparation and measurement? Understanding these questions will be crucial for physically achieving the Heisenberg limit in learning many-body Hamiltonians.

The authors thank Matthias Caro, Richard Kueng, Lin Lin, Jarrod McClean, Praneeth Netrapalli, and John Preskill for valuable input and inspiring discussions. H.-Y. H. is supported by a Google Ph.D. fellowship and a MediaTek Research Young Scholarship. Y. T. is supported in part by the U.S. Department of Energy Office of Science (DE-SC0019374), Office of Advanced Scientific Computing Research (DE-SC0020290), Office of High Energy Physics (DE-ACO2-07CH11359), and under the Quantum System Accelerator project. Work supported by DE-SC0020290 is supported by the DOE QuantISED program through the theory consortium "Intersections of QIS and Theoretical Particle Physics" at Fermilab. The Institute for Quantum Information and Matter is a NSF Physics Frontiers Center. D. F. is supported by NSF Quantum Leap Challenge Institute (QLCI) program under Grant No. OMA-2016245, NSF DMS-2208416, and a grant from the Simons Foundation under Grant No. 825053.

H.-Y. H. and Y. T. contributed equally to this work.

[1] M. de Burgh and S. D. Bartlett, Quantum methods for clock synchronization: Beating the standard quantum limit without entanglement, Phys. Rev. A **72**, 042301 (2005).
[2] A. Valencia, G. Scarcelli, and Y. Shih, Distant clock synchronization using entangled photon pairs, Appl. Phys. Lett. **85**, 2655 (2004).
[3] D. Leibfried, M. D. Barrett, T. Schaetz, J. Britton, J. Chiaverini, W. M. Itano, J. D. Jost, C. Langer, and D. J. Wineland, Toward Heisenberg-limited spectroscopy with multiparticle entangled states, Science **304**, 1476 (2004).
[4] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Optimal frequency measurements with maximally correlated states, Phys. Rev. A **54**, R4649 (1996).
[5] H. Lee, P. Kok, and J. P. Dowling, A quantum Rosetta stone for interferometry, J. Mod. Opt. **49**, 2325 (2002).
[6] K. McKenzie, D. A. Shaddock, D. E. McClelland, B. C. Buchler, and P. K. Lam, Experimental Demonstration of a Squeezing-Enhanced Power-Recycled Michelson Interferometer for Gravitational Wave Detection, Phys. Rev. Lett. **88**, 231102 (2002).
[7] M. J. Holland and K. Burnett, Interferometric Detection of Optical Phase Shifts at the Heisenberg Limit, Phys. Rev. Lett. **71**, 1355 (1993).
[8] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, Phys. Rev. A **46**, R6797 (1992).
[9] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D **23**, 1693 (1981).
[10] N. Boulant, T. F. Havel, M. A. Pravia, and D. G. Cory, Robust method for estimating the Lindblad operators of a dissipative quantum process from measurements of the

density operator at multiple time points, Phys. Rev. A **67**, 042322 (2003).
[11] L. Innocenti, L. Banchi, A. Ferraro, S. Bose, and M. Paternostro, Supervised learning of time-independent Hamiltonians for gate design, New J. Phys. **22**, 065001 (2020).
[12] E. Ben Av, Y. Shapira, N. Akerman, and R. Ozeri, Direct reconstruction of the quantum-master-equation dynamics of a trapped-ion qubit, Phys. Rev. A **101**, 062305 (2020).
[13] M. D. Shulman, S. P. Harvey, J. M. Nichol, S. D. Bartlett, A. C. Doherty, V. Umansky, and A. Yacoby, Suppressing qubit dephasing using real-time Hamiltonian estimation, Nat. Commun. **5**, 5156 (2014).
[14] S. Sheldon, E. Magesan, J. M. Chow, and J. M. Gambetta, Procedure for systematically tuning up cross-talk in the cross-resonance gate, Phys. Rev. A **93**, 060302(R) (2016).
[15] N. Sundaresan, I. Lauer, E. Pritchett, E. Magesan, P. Jurcevic, and J. M. Gambetta, Reducing unitary and spectator errors in cross resonance with optimized rotary echoes, PRX Quantum **1**, 020318 (2020).
[16] N. Wiebe, C. Granade, C. Ferrie, and D. Cory, Quantum Hamiltonian learning using imperfect quantum resources, Phys. Rev. A **89**, 042314 (2014).
[17] N. Wiebe, C. Granade, C. Ferrie, and D. G. Cory, Hamiltonian Learning and Certification Using Quantum Resources, Phys. Rev. Lett. **112**, 190501 (2014).
[18] G. Verdon, J. Marks, S. Nanda, S. Leichenauer, and J. Hidary, Quantum hamiltonian-based models and the variational quantum thermalizer algorithm, arXiv:1910.02071.
[19] D. Burgarth and A. Ajoy, Evolution-Free Hamiltonian Parameter Estimation through Zeeman Markers, Phys. Rev. Lett. **119**, 030402 (2017).
[20] J. Wang, S. Paesani, R. Santagati, S. Knauer, A. A. Gentile, N. Wiebe, M. Petruzzella, J. L. O'Brien, J. G. Rarity, A. Laing *et al.*, Experimental quantum Hamiltonian learning, Nat. Phys. **13**, 551 (2017).
[21] H. Y. Kwon, H. G. Yoon, C. Lee, G. Chen, K. Liu, A. K. Schmid, Y. Z. Wu, J. W. Choi, and C. Won, Magnetic Hamiltonian parameter estimation using deep learning techniques, Sci. Adv. **6**, eabb0872 (2020).
[22] D. Wang, S. Wei, A. Yuan, F. Tian, K. Cao, Q. Zhao, Y. Zhang, C. Zhou, X. Song, D. Xue, and S. Yang, Machine learning magnetic parameters from spin configurations, Adv. Sci. **7**, 2000566 (2020).
[23] H.-Y. Huang, R. Kueng, and J. Preskill, Predicting many properties of a quantum system from very few measurements, Nat. Phys. **16**, 1050 (2020).
[24] Z. Li, L. Zou, and T. H. Hsieh, Hamiltonian Tomography via Quantum Quench, Phys. Rev. Lett. **124**, 160502 (2020).
[25] L. Che, C. Wei, Y. Huang, D. Zhao, S. Xue, X. Nie, J. Li, D. Lu, and T. Xin, Learning quantum Hamiltonians from single-qubit measurements, Phys. Rev. Res. **3**, 023246 (2021).
[26] J. Haah, R. Kothari, and E. Tang, Optimal learning of quantum Hamiltonians from high-temperature gibbs states, arXiv:2108.04842.
[27] W. Yu, J. Sun, Z. Han, and X. Yuan, Practical and efficient Hamiltonian learning, arXiv:2201.00190.
[28] D. Hangleiter, I. Roth, J. Eisert, and P. Roushan, Precise Hamiltonian identification of a superconducting quantum processor, arXiv:2108.08319.
[29] D. S. Franca, L. A. Markovich, V. Dobrovitski, A. H. Werner, and J. Borregaard, Efficient and robust estimation of many-qubit Hamiltonians, arXiv:2205.09567.
[30] A. Zubida, E. Yitzhaki, N. H. Lindner, and E. Bairey, Optimal short-time measurements for Hamiltonian learning, arXiv:2108.08824.
[31] E. Bairey, I. Arad, and N. H. Lindner, Learning a Local Hamiltonian from Local Measurements, Phys. Rev. Lett. **122**, 020504 (2019).
[32] C. E. Granade, C. Ferrie, N. Wiebe, and D. G. Cory, Robust online Hamiltonian learning, New J. Phys. **14**, 103013 (2012).
[33] A. Gu, L. Cincio, and P. J. Coles, Practical black box Hamiltonian learning, arXiv:2206.15464.
[34] F. Wilde, A. Kshetrimayum, I. Roth, D. Hangleiter, R. Sweke, and J. Eisert, Scalably learning quantum many-body Hamiltonians from dynamical data (2022).
[35] S. Krastanov, S. Zhou, S. T. Flammia, and L. Jiang, Stochastic estimation of dynamical variables, Quantum Sci. Technol. **4**, 035003 (2019).
[36] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).
[37] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, Achieving the Heisenberg limit in quantum metrology using quantum error correction, Nat. Commun. **9**, 78 (2017).
[38] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).
[39] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Entanglement-free Heisenberg-limited phase estimation, Nature (London) **450**, 393 (2007).
[40] S. Kimmel, G. H. Low, and T. J. Yoder, Robust calibration of a universal single-qubit gate set via robust phase estimation, Phys. Rev. A **92**, 062315 (2015).
[41] M. Skotiniotis, P. Sekatski, and W. Dür, Quantum metrology for the Ising Hamiltonian with transverse magnetic field, New J. Phys. **17**, 073032 (2015).
[42] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, Compatibility in multiparameter quantum metrology, Phys. Rev. A **94**, 052108 (2016).
[43] M. Gessner, L. Pezzè, and A. Smerzi, Sensitivity Bounds for Multiparameter Quantum Metrology, Phys. Rev. Lett. **121**, 130503 (2018).
[44] I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and G. Tóth, Precision bounds for gradient magnetometry with atomic ensembles, Phys. Rev. A **97**, 053603 (2018).
[45] See [46–51] for more detailed discussion on SPAM-robustness.
[46] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland, Randomized benchmarking of quantum gates, Phys. Rev. A **77**, 012307 (2008).
[47] E. Magesan, J. M. Gambetta, and J. Emerson, Scalable and Robust Randomized Benchmarking of Quantum Processes, Phys. Rev. Lett. **106**, 180504 (2011).
[48] A. Erhard, J. J. Wallman, L. Postler, M. Meth, R. Stricker, E. A. Martinez, P. Schindler, T. Monz, J. Emerson, and R. Blatt, Characterizing large-scale quantum computers via cycle benchmarking, Nat. Commun. **10**, 1 (2019).
[49] R. Harper, S. T. Flammia, and J. J. Wallman, Efficient learning of quantum noise, Nat. Phys. **16**, 1184 (2020).

[50] A. Elben, S. T. Flammia, H.-Y. Huang, R. Kueng, J. Preskill, B. Vermersch, and P. Zoller, The randomized measurement toolbox, Nat. Rev. Phys. **5**, 9 (2023).

[51] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.200403 for details of our algorithm and the proof of the main results.

[52] D. Aharonov, J. Cotler, and X.-L. Qi, Quantum algorithmic measurement, Nat. Commun. **13**, 1 (2022).

[53] S. Chen, J. Cotler, H.-Y. Huang, and J. Li, Exponential separations between learning with and without quantum memory, in *Proceedings of 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, New York, 2022), pp. 574–585.

[54] H.-Y. Huang, S. T. Flammia, and J. Preskill, Foundations for learning from noisy quantum experiments, arXiv:2204.13691.

[55] H.-Y. Huang, M. Broughton, J. Cotler, S. Chen, J. Li, M. Mohseni, H. Neven, R. Babbush, R. Kueng, J. Preskill *et al.*, Quantum advantage in learning from experiments, Science **376**, 1182 (2022).

[56] E. Campbell, Random Compiler for Fast Hamiltonian Simulation, Phys. Rev. Lett. **123**, 070503 (2019).

[57] D. W. Berry, A. M. Childs, Y. Su, X. Wang, and N. Wiebe, Time-dependent Hamiltonian simulation with L^1-norm scaling, Quantum **4**, 254 (2020).

[58] C.-F. Chen, H.-Y. Huang, R. Kueng, and J. A. Tropp, Concentration for random product formulas, PRX Quantum **2**, 040305 (2021).

[59] M. Suzuki, General theory of fractal path integrals with applications to manybody theories and statistical physics, J. Math. Phys. (N.Y.) **32**, 400 (1991).

[60] S. Lloyd, Universal quantum simulators, Science **273**, 1073 (1996).

[61] D. W. Berry, G. Ahokas, R. Cleve, and B. C. Sanders, Efficient quantum algorithms for simulating sparse Hamiltonians, Commun. Math. Phys. **270**, 359 (2007).

[62] L. Viola, E. Knill, and S. Lloyd, Dynamical Decoupling of Open Quantum Systems, Phys. Rev. Lett. **82**, 2417 (1999).

[63] K. Khodjasteh and D. A. Lidar, Performance of deterministic dynamical decoupling schemes: Concatenated and periodic pulse sequences, Phys. Rev. A **75**, 062310 (2007).

[64] A. Ajoy and P. Cappellaro, Quantum Simulation via Filtered Hamiltonian Engineering: Application to Perfect Quantum Transport in Spin Networks, Phys. Rev. Lett. **110**, 220503 (2013).

[65] G. T. Genov and N. V. Vitanov, Dynamical Suppression of Unwanted Transitions in Multistate Quantum Systems, Phys. Rev. Lett. **110**, 133002 (2013).

[66] A. Stark, N. Aharon, T. Unden, D. Louzon, A. Huck, A. Retzker, U. L. Andersen, and F. Jelezko, Narrow-bandwidth sensing of high-frequency fields with continuous dynamical decoupling, Nat. Commun. **8**, 1 (2017).

[67] S. Schmitt, T. Gefen, F. M. Stürner, T. Unden, G. Wolff, C. Müller, J. Scheuer, B. Naydenov, M. Markham, S. Pezzagna, J. Meijer, I. Schwarz, M. Plenio, A. Retzker, L. P. McGuinness, and F. Jelezko, Submillihertz magnetic spectroscopy performed with a nanoscale quantum sensor, Science **356**, 832 (2017).

[68] S. Schmitt, T. Gefen, D. Louzon, C. Osterkamp, N. Staudenmaier, J. Lang, M. Markham, A. Retzker, L. P. McGuinness, and F. Jelezko, Optimal frequency measurements with quantum probes, npj Quantum Inf. **7**, 1 (2021).

[69] M. C. Tran, Y. Su, D. Carney, and J. M. Taylor, Faster digital quantum simulation by symmetry protection, PRX Quantum **2**, 010323 (2021).

[70] D. Burgarth, P. Facchi, G. Gramegna, and K. Yuasa, One bound to rule them all: From Adiabatic to Zeno, Quantum **6**, 737 (2022).

[71] M. Mohseni, A. T. Rezakhani, and D. A. Lidar, Quantum-process tomography: Resource analysis of different strategies, Phys. Rev. A **77**, 032322 (2008).

[72] A. J. Scott, Optimizing quantum process tomography with unitary 2-designs, J. Phys. A **41**, 055308 (2008).

[73] J. L. O'Brien, G. J. Pryde, A. Gilchrist, D. F. V. James, N. K. Langford, T. C. Ralph, and A. G. White, Quantum Process Tomography of a Controlled-Not Gate, Phys. Rev. Lett. **93**, 080502 (2004).

[74] R. Levy, D. Luo, and B. K. Clark, Classical shadows for quantum process tomography on near-term quantum computers, arXiv:2110.02965.

[75] S. T. Merkel, J. M. Gambetta, J. A. Smolin, S. Poletto, A. D. Córcoles, B. R. Johnson, C. A. Ryan, and M. Steffen, Self-consistent quantum process tomography, Phys. Rev. A **87**, 062119 (2013).

[76] E. Nielsen, J. K. Gamble, K. Rudinger, T. Scholten, K. Young, and R. Blume-Kohout, Gate set tomography, Quantum **5**, 557 (2021).

[77] R. Blume-Kohout, J. K. Gamble, E. Nielsen, K. Rudinger, J. Mizrahi, K. Fortier, and P. Maunz, Demonstration of qubit operations below a rigorous fault tolerance threshold with gate set tomography, Nat. Commun. **8**, 1 (2017).

[78] A. M. Childs, Y. Su, M. C. Tran, N. Wiebe, and S. Zhu, Theory of Trotter Error with Commutator Scaling, Phys. Rev. X **11**, 011020 (2021).

[79] B. Şahinoğlu and R. D. Somma, Hamiltonian simulation in the low-energy subspace, npj Quantum Inf. **7**, 1 (2021).

[80] Y. Su, H.-Y. Huang, and E. T. Campbell, Nearly tight trotterization of interacting electrons, Quantum **5**, 495 (2021).

[81] D. An, D. Fang, and L. Lin, Time-dependent unbounded Hamiltonian simulation with vector norm scaling, Quantum **5**, 459 (2021).

[82] Y. Borns-Weil and D. Fang, Uniform observable error bounds of trotter formulae for the semiclassical Schrödinger equation, arXiv:2208.07957.

[83] Y. Tong, V. V. Albert, J. R. McClean, J. Preskill, and Y. Su, Provably accurate simulation of gauge theories and bosonic systems, Quantum **6**, 816 (2022).

[84] We use methods in [85] to obtain long-time evolution error bounds from short-time evolution error bounds.

[85] R. J. LeVeque, *Finite difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems* (SIAM, Philadelphia, 2007).

[86] The proof uses results in [87,88] and other works listed in the Supplemental Material thesupplement.

[87] I. Nechita, Z. Puchała, Ł. Pawela, and K. Życzkowski, Almost all quantum channels are equidistant, J. Math. Phys. (N.Y.) **59**, 052201 (2018).

[88] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).