

Global Topological Synchronization on Simplicial and Cell Complexes

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Topological signals, i.e., dynamical variables defined on nodes, links, triangles, etc. of higher-order networks, are attracting increasing attention. However, the investigation of their collective phenomena is only at its infancy. Here we combine topology and nonlinear dynamics to determine the conditions for global synchronization of topological signals defined on simplicial or cell complexes. On simplicial complexes we show that topological obstruction impedes odd dimensional signals to globally synchronize. On the other hand, we show that cell complexes can overcome topological obstruction and in some structures signals of any dimension can achieve global synchronization.

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Synchronization is a widespread phenomenon at the root of several biological rhythms or human-made technological systems [1,2]. Synchronization refers to the spontaneous ability of coupled oscillators to operate at unison and thus exhibit a coherent collective behavior. Global synchronization is the resulting phenomenon where all oscillators behave in the same way.

Traditionally synchronization has been studied when identical [3,4] or nonidentical oscillators [5,6] are defined on the nodes of a network and are coupled by the network links. However, to capture the function of many complex systems, e.g., brain networks [7,8], social networks [9] and protein interaction networks [10], it is important to go beyond pairwise interactions and consider higher-order interactions [11] between two or more nodes instead. For instance, collaborations typically require the cooperation of a team of more than two individuals, and to perform a function in the cell, proteins form complexes formed by several different types of proteins. Many-body interactions are captured by higher-order networks [12,13] (such as hypergraphs, simplicial, and cell complexes) and are dramatically changing our understanding of the interplay between structure and dynamics of complex systems [11,13,14]. Note that higher-order networks by definition are constructed by higher-order building blocks, but like networks they can have different structure [13]. In particular, a higher-order network can display a very regular (latticelike) structure, a fractal [15], or a very random structure [16]. Moreover higher-order networks can also be built from network motifs according to clique-complex construction, by identifying each clique with a simplex, and its generalizations [13].

Lately, synchronization of identical and nonidentical oscillators defined on the nodes of higher-order networks

has been a field of intense research activity. Global synchronization of identical oscillators was first formulated for special topologies (a p -regular hypergraph) [17] and for a peculiar Laplace operator obtained from the hyperadjacency matrix [18] while recently a general and comprehensive theoretical framework was proposed in Ref. [19] to study dynamical systems defined on hypergraph with heterogeneous hyperedges size distribution, the latter also influencing the Laplace matrix. Partial synchronization of nonidentical nodes oscillators has been investigated using a variation of the Kuramoto model leading to explosive transitions [20].

Simplicial and cell complexes also sustain *topological signals* [13,21–23], i.e., dynamical variables that can be defined not only on nodes but also on links, triangles, and higher-dimensional simplices or cells. Examples of real topological signals are edge signals such as biological transportation fluxes or traffic signals [23], synaptic and brain edge signals [24], or climate data such as currents in the ocean or speed of wind at different locations [25]. In the framework of quantum systems, it has been shown that synchronization and quantum entanglement are strongly intertwined [26,27]; this observation can be relevant in quantum computations involving anyons [28] or bosons [29] where signals defined on links of some array should be protected from noise. Finally, topological signals are attracting increasing attention in signal processing and machine learning [23,25,30–32]. However, the study of their collective phenomena is only at its infancy [21,22,33–36].

Recently the formulation of a higher-order topological Kuramoto model [21,22,36] has demonstrated that topological signals of any dimension can synchronize leading to either continuous or explosive synchronization transitions.

These results concern partial synchronization while an important question is whether global synchronization of topological signals can ever be achieved.

The aim of this work is to determine the topological and dynamical conditions under which global topological synchronization of identical topological oscillators can be observed. Relying on the use of higher-order Laplacian matrices [37–40] and the extension of the master stability function (MSF) to simplicial and cell complexes dynamics, we are able to tackle this problem emphasizing the difference existing among the two frameworks.

Anticipating on our results we can state that on simplicial complex we observe topological obstruction: given a simplicial complex of dimension K , if the topological signal is defined on an odd-dimensional simplex of dimension $k < K$, then global synchronization is not possible. On the other hand, if the simplex has an even dimension, then we can have global synchronization provided the simplex is *balanced* (see below) and the model parameters allow for it. Interestingly we show that cell complexes can overcome topological obstruction and some topologies can sustain global synchronization of signals of any dimension.

Definition of simplicial and cell complexes.—Simplicial and cell complexes are generalized network structures that besides nodes and links do also contain triangles, polygons, tetrahedra, hypercubes, orthoplexes, and so on. Given a non-negative integer k , a k -simplex σ^k is a set of $k + 1$ different nodes, $\sigma^k = [v_0, v_1, \dots, v_k]$. A $(k - 1)$ -face σ^{k-1} of a k -simplex is a $(k - 1)$ -simplex obtained by removing one vertex from σ^k . Therefore a node is a 0-simplex, a link is a 1-simplex, a 2-simplex is a triangle, and so on. A simplicial complex \mathcal{X} of dimension K is a finite collection of simplices of dimensions $k \leq K$, closed under the inclusion of faces; namely, if $\sigma \in \mathcal{X}$, then also all the faces of σ should belong to \mathcal{X} . Simplicial complexes are thus a natural generalization of networks, which are recovered for $k = 1$. In algebraic topology a simplex σ^k is assigned an *orientation*, typically induced by the node labels. A coherent orientation of the face with the orientation of the simplex will be denoted by $\sigma^{k-1} \sim \sigma^k$, otherwise we will write $\sigma^{k-1} \not\sim \sigma^k$.

A more general structure retaining the algebraic richness of simplicial complexes is given by the *cell complexes* [38,41–44]. The latter differs from simplicial complexes because they are not just built by simplices but instead they are obtained by gluing cells (i.e., regular polytopes) along their faces. In particular, 0-cells are nodes, 1-cells links, while 2-cells are generic polygons, and 3-cells are the Platonic polytopes.

In algebraic topology the boundary operator is defined on chains [13,38], linear combinations of oriented simplices of the simplicial complex. The k boundary operator retains the information about the faces of a k -simplex σ^k and their relative orientation. The boundary operator is encoded into the *incidence matrices* \mathbf{B}_k whose elements

are given for all $k = 1, \dots, K$ by $B_k(i, j) = 1$ if $\sigma_i^{k-1} \sim \sigma_j^k$, $B_k(i, j) = -1$ if $\sigma_i^{k-1} \not\sim \sigma_j^k$, and $B_k(i, j) = 0$ otherwise [45], where K is the dimension of the simplicial complex, namely the size of the largest simplex. One key property of the boundary operator is that the boundary of the boundary is null; hence, $\mathbf{B}_k \mathbf{B}_{k+1} = 0$. To be coherent an oriented simplicial complex needs to satisfy this condition for every k with $0 \leq k \leq K$. Let us conclude this section by noting that the boundary operators for cell complexes have the same definition [23].

The higher-order Laplacians [13,33,38–40] allow us to define diffusion among k -simplices and capture the topology of the higher-order network; they are defined in terms of the incidence matrix as

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top, \quad k = 1, \dots, K - 1. \quad (1)$$

For $k = 0$ and $k = K$, we have instead $\mathbf{L}_0 = \mathbf{B}_1 \mathbf{B}_1^\top$ and $\mathbf{L}_K = \mathbf{B}_K^\top \mathbf{B}_K$. If we denote by N_k , $k = 0, \dots, K$ the number of k -simplices, then it follows that \mathbf{B}_k is a $N_{k-1} \times N_k$ matrix, while the size of \mathbf{L}_k is $N_k \times N_k$. The matrix \mathbf{L}_0 coincides with the graph Laplacian. The higher-order Laplacian \mathbf{L}_k is a semidefinite operator and one of its most celebrated properties is that the dimension of its kernel is given by the k -Betti number, i.e., $\dim(\ker \mathbf{L}_k) = \beta_k$, where β_k indicates the number of k -dimensional cavities in the simplicial complex. Moreover, there is a basis of the kernel \mathbf{L}_k which is formed by eigenvectors localized on each of the k -dimensional cavities of the simplicial complex.

Simplicial and cell-complex dynamics.—Let us now consider a topological k -dimensional signal encoded in a k -dimensional cochain $\mathbf{x}: C_k \rightarrow \mathbb{R}^d$ which assigns to every chain C_k (linear combination of k -simplices) values on \mathbb{R}^d . The k -topological signal has elements $\mathbf{x}_i = \mathbf{x}(\sigma_i^k) = (x_i^1, \dots, x_i^d)$ defined on the i th oriented k -simplex σ_i^k [see Supplemental Material (SM) [46]]. According to the properties of the k -cochains [38,39] we have $\mathbf{x}(-\sigma_i^k) = -\mathbf{x}(\sigma_i^k)$, being the discrete analogous of differential forms on manifolds. For instance, for $k = 1$ and $d = 1$, $x_i = x(\sigma_i^k)$ indicates a flux defined on the link i that is positive if going in the same direction of the positive orientation of the link and negative otherwise; i.e., $x(-\sigma_i^k) = -x_i$. Let us assume that the value of the topological signal on every simplex i follows the same dynamics and evolves according to $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$, for some odd nonlinear function $\mathbf{f}: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Assume now the k -simplex to belong to a K -simplicial complex, $K \geq k$, and assume the existence of a diffusivelike nonlinear interaction among adjacent simplices of the same dimension:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sum_{j=1}^{N_k} L_k(i, j) \mathbf{h}(\mathbf{x}_j), \quad \forall i = 1, \dots, N_k, \quad (2)$$

where $\mathbf{h}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is some odd nonlinear coupling function. This equation generalizes the dynamics of identical oscillators anchored to each node [3] to the

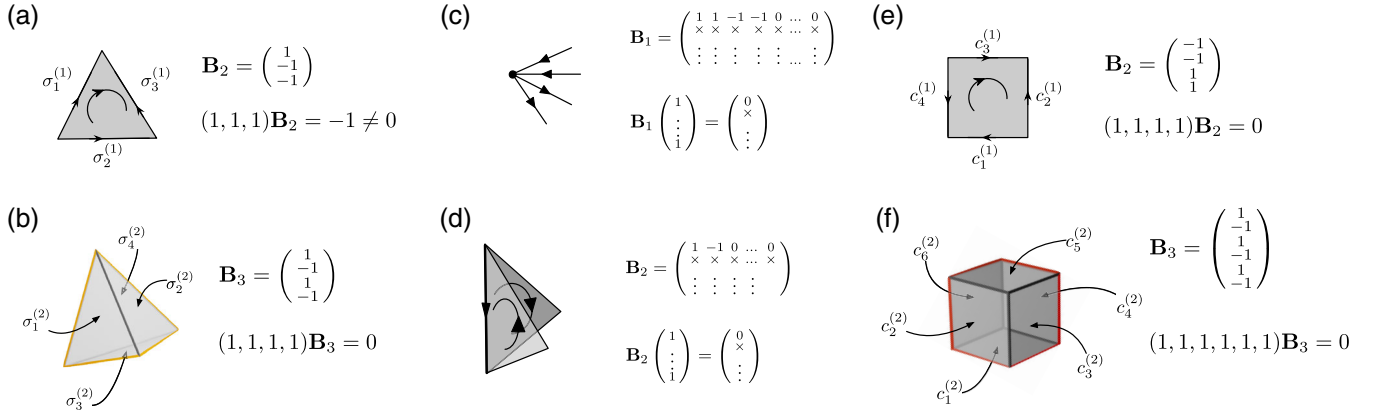


FIG. 1. Schematic description of conditions (i) [(a),(b),(e),(f)] and (ii) [(c),(d)] for topological signals defined on one-dimensional cells (links, panels in top row) and two-dimensional cells (triangles or squares, panels in bottom row) in which we assume that there is an orientation such that $\mathbf{w} = \mathbf{u} = (1, 1, \dots)^\top$. In the case of simplicial complexes (a),(d) condition (i) cannot be satisfied for signals defined on one-dimensional simplices, while in the case of cell complexes (e),(f) condition (i) can be satisfied. Condition (ii) can be satisfied on simplicial and cell complexes as long as the simplices are balanced [see (c) and (d) for the simplicial complex case].

scenario in which identical oscillators are defined on higher-dimensional simplices or cells. Please note that requiring odd functions $\mathbf{f}(\mathbf{x}_i)$ and $\mathbf{h}(\mathbf{x}_i)$ is necessary for higher-order topological signals with $k > 0$ in order to ensure invariance under change of orientation of each simplex i (see SM [46]). For node dynamics ($k = 0$) the existence of a global synchronized state is automatically determined by the properties of the graph Laplacian whose kernel is spanned by $\mathbf{u} = (1, \dots, 1)^\top$; indeed we have $\beta_0 = 1$, one connected component. Its stability is instead determined by the celebrated MSF [3,4].

Given the growing interest in topological signals, a key question is how these classic results of nonlinear dynamics on networks extend to nonlinear dynamics of topological signals on simplicial complexes. Anticipating our results, we will show that topology and combinatorics of the higher-order Laplacian will not always ensure existence of a globally synchronized state, and moreover, since the dimension of the kernel of \mathbf{L}_k can be bigger than one, also the MSF will differ from the network case.

Let us then fix a reference stable solution $\mathbf{s}(t)$ of the uncoupled system, $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$. We are interested in determining the conditions under which the state having each simplex i either in the state $\mathbf{x}_i = \mathbf{s}(t)$ or in $\mathbf{x}_i = -\mathbf{s}(t)$ is also a stable solution of the coupled system Eq. (2). Namely, the latter exhibits a *global synchronous* behavior in which all simplices display the same dynamics of the isolated simplices when we account for differences of sign, determined by their orientation (see SM [46]).

Let us now introduce the vector $\mathbf{v} = (v_1, \dots, v_{N_k})^\top \in \{-1, 1\}^{N_k}$, such that the globally synchronized state is given by $\mathbf{x}_i = v_i \mathbf{s}(t)$. A *necessary condition* to observe global synchronization is that $\sum_j L_k(i, j) \mathbf{v}_j = 0$ (see SM [46]). Let us recall that $\ker \mathbf{L}_k = \ker \mathbf{B}_k \cap \ker \mathbf{B}_{k+1}^\top$; thus the latter condition ultimately returns to require $\mathbf{B}_k \mathbf{v} = 0$ [condition (i)] and $\mathbf{v}^\top \mathbf{B}_{k+1} = 0$ [condition (ii)] (see SM).

The first condition has a striking consequence. If k is an odd number, because any $(k + 1)$ -simplex contains an odd number of k -faces, then condition (i) cannot ever be satisfied. On the contrary, if k is even, then any $(k + 1)$ -simplex contains an even number of k -faces thus condition (i) can be realized [see Figs. 1(a) and 1(b)]. On the other hand, condition (ii) can be satisfied by imposing a suitable condition of the $(k - 1)$ -faces of the k -simplex, which we call *balanced condition*. In particular, if $\mathbf{v} = \mathbf{u} = (1, \dots, 1)^\top$, this condition can be satisfied by requiring every $(k - 1)$ -face to be adjacent to an even number of k -simplices and to be coherently oriented with half of them [see Figs. 1(c) and 1(d)].

Therefore, for even values of k , global synchronization can be achieved, while if k is odd, we observe, as long as $\mathbf{v}^\top \mathbf{B}_{k+1} \neq \mathbf{0}$, a *topological obstruction* to the onset of global synchronization. Interestingly, for K -dimensional signals having $\mathbf{B}_{K+1} = \mathbf{0}$, only the balanced condition remains [i.e., condition (ii)], which is automatically satisfied for the vector $\mathbf{v} = \mathbf{u}$ if the simplicial complex is a closed manifold without boundary. Hence, K -dimensional topological signals defined on closed K -dimensional manifolds can always achieve global synchronization for arbitrary value of K .

A similar derivation can be generalized and extended to topological signals defined on the k -dimensional cells of cell complexes. In particular, the conditions to achieve global synchronization on a cell complex are unchanged and given again by conditions (i) and (ii). However, the combinatorics of cell complexes is different from the one of simplicial complexes. Take for instance a cell complex whose network skeleton is formed by a d -dimensional square lattice with periodic boundary conditions, i.e., a regular tessellation of d -dimensional torus. Then every cell of dimension $k + 1 > 0$ has an even number of

k -dimensional faces; therefore condition (i) can be satisfied also if k is odd [see Figs. 1(e) and 1(f)]. This implies that on cell complexes we can overcome topological obstruction. Until now we have focused on the combinatorial implication of conditions (i) and (ii). However, these conditions also have topological consequences. In fact, since on manifolds the eigenvectors of the kernel of the Hodge Laplacian \mathbf{L}_k are localized on the k -dimensional holes, manifolds that will display an eigenvector with the properties of the above defined \mathbf{v} are characterized by holes spanning the whole structure as, for instance, $(k + 1)$ -dimensional hyperspheres or d -dimensional tori with $d > k$.

Master stability equation for topological signals.—Let us now assume the reference solution $\mathbf{s}(t)$ to also be a solution of the coupled system Eq. (2), then by introducing the distance from the reference orbit, $\delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{s}(t)$, we can derive its time evolution by linearizing Eq. (2):

$$\frac{d\delta\mathbf{x}_i}{dt} = \mathbf{J}_f(\mathbf{s})\delta\mathbf{x}_i - \sum_{j=1}^{N_k} L_k(i, j)\mathbf{J}_h(\mathbf{s})\delta\mathbf{x}_j, \quad \forall i = 1, \dots, N_k,$$

with $\mathbf{J}_f(\mathbf{s})$ [$\mathbf{J}_h(\mathbf{s})$] the Jacobian of the function \mathbf{f} [\mathbf{h}] evaluated on the reference solution.

The matrix \mathbf{L}_k being symmetric, it admits an orthonormal basis $\boldsymbol{\phi}_k^{(\alpha)}$ associated to eigenvalues $\Lambda_k^{(\alpha)}$, $\alpha = 1, \dots, N_k$; namely, $\mathbf{L}_k\boldsymbol{\phi}_k^{(\alpha)} = \Lambda_k^{(\alpha)}\boldsymbol{\phi}_k^{(\alpha)}$. In particular, since we work under the assumption that the simplicial complex is balanced, $\boldsymbol{\phi}_k^{(1)} \sim (1, \dots, 1)^\top \in \mathbb{R}^{N_k}$, $\Lambda_k^{(\alpha)} = 0$ for $1 \leq \alpha \leq \beta_k$ and $\Lambda_k^{(\alpha)} > 0$ for all $\alpha > \beta_k$.

Let us decompose the deviation vectors $\delta\mathbf{x}_i$ onto this eigenbasis: $\delta\mathbf{x}_i = \sum_{\alpha} \delta\mathbf{x}^{(\alpha)}\boldsymbol{\phi}_k^{(\alpha)}(i)$. Then linearizing the dynamical equation, we obtain

$$\frac{d\delta\mathbf{x}^{(\alpha)}}{dt} = [\mathbf{J}_f(\mathbf{s}) - \Lambda_k^{(\alpha)}\mathbf{J}_h(\mathbf{s})]\delta\mathbf{x}^{(\alpha)} \quad \forall \alpha = 1, \dots, N_k.$$

Perturbations aligned with the kernel do not change the stability of the uncoupled system; therefore only the perturbations orthogonal to the kernel can modify the stability of the reference solution. This is the MSF in the framework of simplicial synchronization of topological signals. It is a linear, in general nonautonomous, ordinary differential equation parametrized by the eigenvalues $\Lambda_k^{(\alpha)}$, allowing us to infer the stability character of the reference solution by looking at its spectrum.

Simplicial Stuart-Landau (SL) model.—As an application of the general theory introduced above, let us consider the Stuart-Landau model [47–49] defined for topological signals of dimension k and $d = 2$. For $k = 0$ the model describes a nonlinear oscillator anchored at each node, while for $k = 1$ it can describe an oscillating flux associated to an edge linking two nodes. More precisely, let us define $w_j = x_j + ix_j^2$ and let us consider the “local reaction” function $f(\mathbf{x}) = \tilde{f}(w) = \sigma w - \beta w|w|^2$, where $\sigma = \sigma_{\Re} + i\sigma_{\Im}$

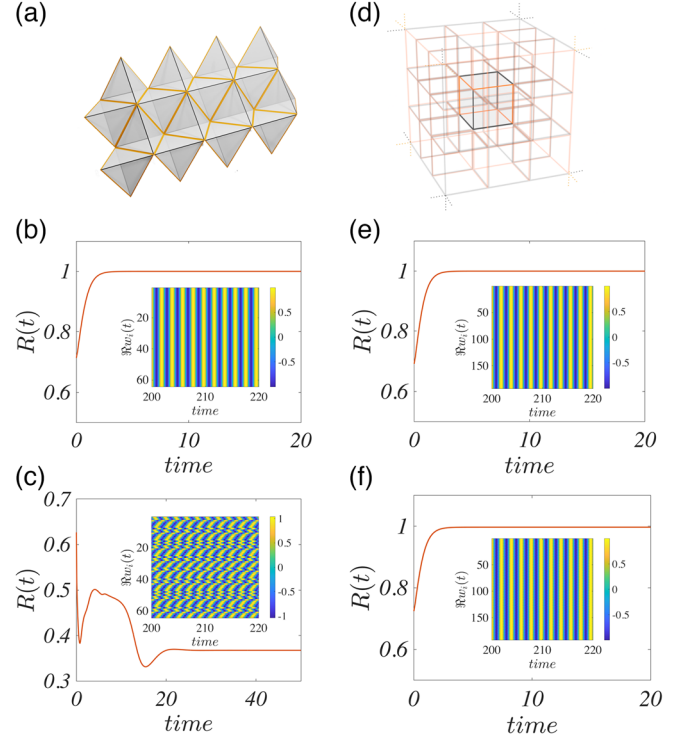


FIG. 2. The Kuramoto order parameter R is plotted versus time t for the Stuart-Landau model of topological oscillators of the balanced simplicial and cell complexes represented in (a) and (d), respectively. Panels (b) and (c) refer, respectively, to the order parameter of triangles and links of the simplicial complex in (a). Panels (e) and (f) refer, respectively, to the order parameter of the squares and links of the cell complex in (d). The insets display the dynamical time series of the topological signals. It is clear that while on the links of the simplicial complex the oscillators do not globally synchronize, the links of the cell complex do support synchronization. The model parameters are $\sigma = 1.0 + 4.3i$, $\beta = 1.0 + 1.1i$, $\mu = 1.0 - 0.5i$, and $m = 3$, ensuring the negativity of the dispersion relation (see SM [46]).

and $\beta = \beta_{\Re} + i\beta_{\Im}$ are complex control parameters. We can prove that the uncoupled dynamics $\dot{w}_j = f(w_j)$ in each simplex j admits a limit cycle solution $\hat{z}(t) = \sqrt{\sigma_{\Re}/\beta_{\Re}}e^{i\omega t}$, where $\omega = \sigma_{\Im} - \beta_{\Im}\sigma_{\Re}/\beta_{\Re}$, that is stable provided $\sigma_{\Re} > 0$ and $\beta_{\Re} > 0$, conditions that we hereby assume. We now consider the coupled dynamics Eq. (2) with nonlinear coupling function $h(\mathbf{x}) = \tilde{h}(w) = \mu w|w|^{m-1}$, where m is a positive integer and $\mu = \mu_{\Re} + i\mu_{\Im}$ a complex parameter that sets the coupling strength [50]. We study the stability of the reference limit cycle solution $\hat{z}(t)$ (see SM [46]) and we prove that the system can globally synchronize, i.e., the dispersion relation is negative, only if the model parameters do satisfy $\mu_{\Re} + \mu_{\Im}\beta_{\Im}/\beta_{\Re} > 0$, and the simplicial complex is such that $\mathbf{u} \in \ker \mathbf{L}_k$. To measure global synchronization we compute the (generalized) Kuramoto order parameter $R(t) = |\sum_j \rho_j(t)e^{i\theta_j(t)}|/N_k$, where $w_j(t) = \rho_j(t)e^{i\theta_j(t)}$ is the polar form of the complex signal. Then $R(t) \rightarrow 1$

testifies the existence of phase and amplitude synchronization. Results shown in Fig. 2 provide numerical evidence of our theoretical predictions. In Figs. 2(a)–2(c) we show the results obtained by studying the SL model defined on top of a designed balanced 3-simplicial complex (see SM). The model parameters have been set to values allowing for a negative dispersion relation (see SM) and indeed once the complex amplitudes are defined on 2-faces, i.e., triangles, the system globally synchronizes [see Fig. 2(b)]. On the other hand, once the SL oscillators are defined on links the system cannot globally synchronize [see Fig. 2(c)]. In Figs. 2(d)–2(f) we provide an example of a cell complex which overcomes topological obstruction: a 3D square lattice with periodic boundary conditions. Such cell complex is made of nodes, links, squares, and cubes (see SM). In this case, global synchronization can be achieved for signals of every dimension [see Figs. 2(e) and 2(f) for global synchronization of links and squares].

In conclusion, in this work we have studied global synchronization of identical topological oscillators on simplicial or cell complexes. We have found that global synchronization of topological signals cannot be observed on arbitrary simplicial or cell complexes but that only some special higher-order network topologies can sustain such a dynamical state. This is in stark contrast with the corresponding scenario in networks where global synchronization can be observed in every network topology given proper dynamical conditions. By combining topology, and in particular the spectral properties of higher-order Laplacians, to nonlinear dynamics techniques such as the MSF, we have identified the topological and dynamical conditions under which identical topological oscillators can achieve global synchronization on simplicial or cell complexes. We have proved that global synchronization of odd-dimensional topological signals is obstructed in simplicial complexes. This topological obstruction implies that on a K -dimensional simplicial complex we can never observe global synchronization of odd-dimensional topological signals of dimension $d < K$. However, such obstruction is not present in cell complexes. In particular, we show evidence that in specific topologies such as the d -dimensional square lattice with periodic boundary conditions, global synchronization of topological signal of any dimension can be observed.

These results significantly enrich our understanding of the relation between higher-order network topology and dynamics revealing collective phenomena of topological signals. Our study is relevant, for its inherent simplicity, to a wide spectrum of applications (neuroscience, biology, and social sciences) where many-body interactions involve higher-order interacting units. Therefore, we hope that this Letter will stimulate further research work in physics and beyond.

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