Sufficient Condition for Gapless Spin-Boson Lindbladians, and Its Connection to Dissipative Time Crystals

Leonardo da Silva Souza, Luis Fernando dos Prazeres, and Fernando Iemini
Instituto de Física, Universidade Federal Fluminense,
Avenida General Milton Tavares de Souza s/n, Gragoatá, 24210-346 Niterói, Rio de Janeiro, Brazil

(Received 6 October 2022; revised 6 February 2023; accepted 12 April 2023; published 3 May 2023)

We discuss a sufficient condition for gapless excitations in the Lindbladian master equation for collective spin-boson systems and permutationally invariant systems. The condition relates a nonzero macroscopic cumulant correlation in the steady state to the presence of gapless modes in the Lindbladian. In phases arising from competing coherent and dissipative Lindbladian terms, we argue that such gapless modes, concomitant with angular momentum conservation, can lead to persistent dynamics in the spin observables with the possible formation of dissipative time crystals. We study different models within this perspective, from Lindbladians with Hermitian jump operators, to non-Hermitian ones composed by collective spins and Floquet spin-boson systems. We also provide a simple analytical proof for the exactness of the mean-field semiclassical approach in such systems based on a cumulant expansion.

DOI: 10.1103/PhysRevLett.130.180401

Nonequilibrium quantum many-body dynamics constitutes a fundamental and open research field [1–3]. The dissipative dynamics of a quantum system embedded in an environment can, in general, be quite cumbersome due to its high complexity. A common useful approach relies on Born-Markovian approximation, for which the effective dynamics for the quantum system is described by a Lindbladian master equation [4]. Among its emergent phases, a new form of spontaneous symmetry breaking (SSB) so-called dissipative time crystal (TC) [5], has gained much attention recently. These nonequilibrium phases break spontaneously the time-translation symmetry of the system, leading to persistent oscillations of macroscopic observables in the thermodynamic limit. Despite intense theoretical and experimental activity (see Refs. [6,7] for interesting reviews) many aspects of these new phases are still being unraveled, with particular attention to the precise role of its many-body correlations [8,9], symmetries [10–13] and basic mechanisms for the stabilization of such peculiar nonequilibrium phases.

The spectral properties of a Lindbladian master equation host valuable information on the system dynamics and phases [14–19]. The Lindbladian gap in particular can characterize the critical behavior in dissipative phase transitions, the emergence of symmetry breaking phases as well as the asymptotic relaxation dynamics towards the steady states of the system. In dissipative TCs the Lindbladian spectrum features gapless excitations generating long-lived asymptotic dynamics towards the steady state, with a divergent lifetime in the thermodynamic limit [18,19]. These gapless excitations appear along with coherent dynamics within their subspace inducing the persistent oscillations of the system observables. The determination of the Lindbladian gap, however, is not, in general, a trivial task. Apart from models sharing specific structures [20–23] (as quadratic fermion or boson Lindbladians, symmetries, integrability) for which one can determine it spectral properties and steady states analytically (or quasi-analytically), for general interacting systems its computation relies either in the diagonalization of the Lindbladian superoperator in an extended Hilbert space or inferring from the dynamics of the observables of the system in the asymptotic limit; in both cases an often nontrivial and challenging task.

In this Letter we discuss a simple sufficient condition to ensure the gapless nature of a Lindbladian, based only on its steady state correlations. Although (long-range) correlations are expected to be connected to gapless excitations and ground state SSB phases in closed Hamiltonian settings, in this Letter we obtain an analytical proof of this correspondence for a class of open-systems driven by Lindbladian master equation. The condition is based on the exactness of mean-field semiclassical approach for such systems, which we proof using a cumulant expansion, and the fact that the macroscopic spin magnetizations of the nonequilibrium steady state (NESS) cannot be dynamically reached in the thermodynamic limit due to spin total angular momentum conservation [as illustrated in Fig. 1]. Studying different collective spin-boson models we see that such gapless excitations, or the inability of the dynamics to reproduce the spin NESS magnetizations, are usually associated to the appearance of persistent dynamics and possible dissipative TC phase, indicating an intimate connection among them.

The model.—We consider M spin ensembles, each composed of N spin-$1/2$ subsystems, interacting with a

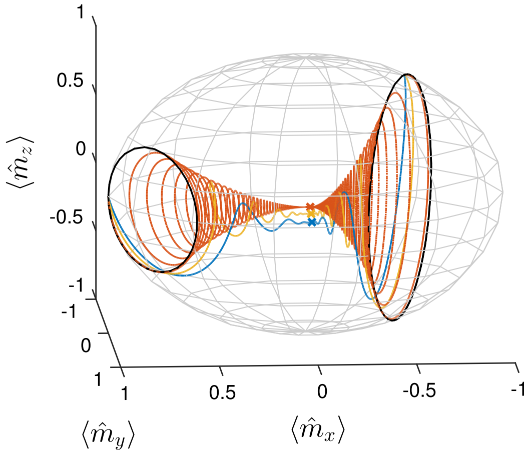

FIG. 1. Dynamics of macroscopic spin magnetizations $\langle \hat{m}^\alpha \rangle$ for a collective spin-1/2 model with competing coherent Hamiltonian (ω_x) and dissipation κ [Eq. (12)], for $\omega_x/\kappa = 2$. We show the dynamics for different initial conditions and system sizes—$N = 2^2$ (blue), 2^3 (yellow), 2^6 (red), and semiclassical limit $N \to \infty$ (black). The crosses are the corresponding NESS, and the spherical shell represents the semiclassical set of states with null macroscopic cumulants, $\langle \hat{m}^x \rangle^2 + \langle \hat{m}^y \rangle^2 + \langle \hat{m}^z \rangle^2 = 1$. The NESS lie inside the spherical shell (i.e., with nonzero macroscopic cumulant correlations). In the thermodynamic limit the dynamics is constrained to the shell and therefore cannot reach such NESS leading to the appearance of gapless Lindbladian excitations, featuring in this case a persistent dynamics.

bosonic mode and coupled to a Markovian environment. The time evolution for the system is described by the Lindbladian master equation [4],

$$\frac{d}{dt}\hat{\rho} = \mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_{i=1}^{M} \mathcal{D}_i[\rho] + \mathcal{D}_a[\rho], \tag{1}$$

with $\mathcal{L}$ the Lindbladian superoperator. The coherent driving term is given by $\hat{H} = \hat{H}_{\text{spin}} + \hat{H}_{\text{boson}} + \hat{H}_{\text{spin-boson}}$, where $\hat{H}_{\text{spin (boson)}}$ corresponds to the spin (boson) term and $\hat{H}_{\text{spin boson}}$ to the spin-boson interaction. Specifically,

$$\begin{aligned}
\hat{H}_{\text{spin}} &= \sum_{i=1}^{M}\sum_{\alpha} \omega_\alpha^{(i)} \hat{S}_i^\alpha + \frac{1}{S}\sum_{i,j=1}^{M}\sum_{\alpha,\beta} \omega_{\alpha,\beta}^{(i,j)} \hat{S}_i^\alpha \hat{S}_j^\beta, \\
\hat{H}_{\text{boson}} &= \omega_b \hat{a}^\dagger \hat{a}, \\
\hat{H}_{\text{spin boson}} &= \sum_{i=1}^{M}\sum_{\alpha} \frac{g_\alpha}{\sqrt{N}} (\hat{a} + \hat{a}^\dagger) \hat{S}_i^\alpha,
\end{aligned} \tag{2}$$

where $\hat{S}_i^\alpha = \sum_{k=1}^{N} \hat{\sigma}_{i,k}^\alpha/2$ [24], with $\alpha = x,\ y,\ z$ are the collective spin operators for the ith ensemble, $\hat{\sigma}_{i,k}^\alpha$ are the Pauli spin operators for the kth spin in the ith ensemble, $S = N/2$ is the total angular momentum of the system and $\hat{a}$ the annihilation operator of the bosonic mode satisfying $[\hat{a}, \hat{a}^\dagger] = 1$. The parameter $w_\alpha^{(i)}$ describe local fields on the collective spins, $w_{\alpha,\beta}^{(i,j)}$ the collective spin-spin interactions, ω_b the bosonic mode excitation energy and g_α the spin-boson couplings. The collective spin operators inherit the SU(2) algebra of their subsystems satisfying the commutation relations $[\hat{S}_i^\alpha, \hat{S}_j^\beta] = i\epsilon^{\alpha\beta\gamma}\hat{S}_i^\gamma \delta_{i,j}$. Because of the collective nature of the interactions, the model conserves the total spin $S_i^2 = (\hat{S}_i^x)^2 + (\hat{S}_i^y)^2 + (\hat{S}_i^z)^2$.

The dissipative boson and spin terms of the Lindbladians are defined by

$$\mathcal{D}_a[\rho] = \kappa_b\left(\hat{a}\,\hat{\rho}\,\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \hat{\rho}\}\right), \tag{3}$$

$$\mathcal{D}_i[\rho] = \frac{1}{S}\sum_{\alpha,\beta}\Gamma_{\alpha,\beta}^{(i)}\left(\hat{S}_i^\alpha\hat{\rho}\hat{S}_i^\beta - \frac{1}{2}\{\hat{S}_i^\beta\hat{S}_i^\alpha, \hat{\rho}\}\right), \tag{4}$$

with κ_b representing the boson loss rate and $\Gamma_{\alpha,\beta}^{(i)} \in \mathbb{C}$ the elements of the dissipative spin matrix $\mathbf{\Gamma^{(i)}} \in \mathbb{C}^{3\text{x}3}$ with $\alpha, \beta = x,\ y,\ z$. Although in the examples discussed in the Letter we consider cases with $\mathbf{\Gamma^{(i)}}$ positive semidefinite, representing in fact a Lindbladian master equation, our proofs are independent of such constraint, thus valid for more general dynamics not fully in Lindblad form.

Exactness of the mean-field (MF) approach.—Because of the collective character of the spin operators, the spins within each ensemble are permutationally invariant. This symmetry simplifies considerably the description of the macroscopic observables

$$\hat{x} = \frac{\hat{a}^\dagger + \hat{a}}{\sqrt{2N\omega_b}}, \qquad \hat{p} = i\frac{\hat{a}^\dagger - \hat{a}}{\sqrt{2N/\omega_b}}, \qquad \hat{m}_i^\alpha = \hat{S}_i^\alpha/S \tag{5}$$

in the thermodynamic limit. Specifically, in this limit MF is proven exact as we show using a cumulant expansion approach [25–28] (different proofs are also known [29,30] using different methods). Such an approach shall be useful both to (i) define variables of interest for our gapless condition, (ii) set clear limits of validation for the MF; and (iii) extend the proof for more general systems (discussed later in the manuscript). The reasoning behind our proof is as follows. The first, second, and third order cumulants of general observables are defined, respectively, by

$$\begin{aligned}
K(\hat{O}_j) &= \langle \hat{O}_j \rangle, \\
K(\hat{O}_j, \hat{O}_\ell) &= \langle \hat{O}_j\hat{O}_\ell \rangle - \langle \hat{O}_j \rangle\langle \hat{O}_\ell \rangle, \\
K(\hat{O}_j, \hat{O}_\ell, \hat{O}_m) &= \langle \hat{O}_j\hat{O}_\ell\hat{O}_m \rangle + 2\langle \hat{O}_j \rangle\langle \hat{O}_\ell \rangle\langle \hat{O}_m \rangle \\
&\quad - \langle \hat{O}_j\hat{O}_\ell \rangle\langle \hat{O}_m \rangle - \langle \hat{O}_j\hat{O}_m \rangle\langle \hat{O}_\ell \rangle \\
&\quad - \langle \hat{O}_\ell\hat{O}_m \rangle\langle \hat{O}_j \rangle.
\end{aligned} \tag{6}$$

Deriving the Heisenberg equations of motion for the second cumulant of the macroscopic observables, which we denote with a lower case notation $\hat{o}_j = \hat{m}_i^\alpha$, $\hat{x}$ or $\hat{p}$, we observe that $\dot{K}(\hat{o}_j,\hat{o}_\ell) = f[hK(\hat{o}_p)K(\hat{o}_q,\hat{o}_r), hK(\hat{o}_p,\hat{o}_q,\hat{o}_r), hh'/N, h/(h'N)]$ with f a linear function of its arguments, p, q ranging from the possible observables of the system and h, h' the possible coupling parameters ($\omega^{[\ldots]}_{[\ldots]}$, g_α, κ or Γ) (see Supplemental Material [25] for detailed calculations). The function has no independent first-order cumulant terms. Therefore, assuming h as finite coupling constants, given an initial uncorrelated state (e.g., a product state) with

$$\lim_{N\to\infty} K(\hat{o}_q, \hat{o}_r) = 0, \qquad \lim_{N\to\infty} K(\hat{o}_p, \hat{o}_q, \hat{o}_r) = 0, \quad (7)$$

one has $\dot{K}(\hat{o}_j, \hat{o}_\ell) = 0$ implying that the state remains uncorrelated, and therefore the dynamics shall be constrained to the MF first order cumulants, proving its exactness. We remark that despite MF is usually assumed exact for collective spin-boson systems, this may not always be the case. Recall that the derivative function f also depends on the system couplings and its scaling with system size. Recently it was proposed collective spin systems acting as quantum heat engines for which the coupling strength scales nontrivially with system size [31]; despite still having a well-defined thermodynamic limit, it leads to the failure of MF due to the unusual scaling and, consequently, to a nontrivial (enhanced) performance of the heat engine. Our proof thus provides a simple understanding for these limitations, and can shed light for engineering more general systems failing MF with possibly unusual emergent features. In summary, the MF approach is exact as long as the weight product between couplings and the initial state is negligible according to the derivative function f and Eq. (7). Moreover, since the resulting first order cumulants describe macroscopic observables, the corresponding dynamical rates in the Heisenberg equations of motion must be extensive with the system size (a subtler condition often not discussed [25]).

Sufficient condition.—For simplicity, we discuss here the case of a continuous time-independent Lindbladian. The case of Floquet Lindbladian follows similar reasoning, as we discuss later. The evolution of a quantum state $\hat{\rho}(t)$ with the Lindblad master equation is given by

$$\hat{\rho}(t) = \hat{\rho}_{\text{NESS}} + \sum_i e^{\lambda_i t}\mathcal{P}_i[\hat{\rho}(0)], \quad (8)$$

where $\hat{\rho}_{\text{NESS}}$ is the nonequilibrium steady state of the dynamics (i.e., $\mathcal{L}[\hat{\rho}_{\text{NESS}}] = 0$), λ_i are the generalized eigenvalues of the Lindbladian and $\mathcal{P}_i$ their corresponding superoperators. The gap of the Lindbladian is defined as,

$$\Delta_N = \max_i \text{Re}(\lambda_i), \quad (9)$$

which are always nonpositive. Dissipative TCs breaking a continuous time-symmetry feature gapless excitations (i.e., $\lim_{N\to\infty}\Delta_N = 0$) along with a nonzero imaginary part for such eigenvalues $[\text{Im}(\lambda_i) \neq 0]$ inducing nontrivial coherent oscillation in the system dynamics.

In the case of a nondegenerate Lindbladian ($\Delta_N \neq 0, \ \forall \ N$) we see directly that both limits commute $\lim_{N,t\to\infty}\hat{\rho}(t) = \lim_{t,N\to\infty}\hat{\rho}(t) = \hat{\rho}_{\text{NESS}}$, where we use the notation $\lim_{A,B\to\infty} \equiv \lim_{A\to\infty}\lim_{B\to\infty}$. On the other hand, if the Lindbladian has gapless excitations in the thermodynamic limit—and only in this limit, thus excluding possible decoherence-free subspaces with $\Delta_N = 0$ for finite N—one may have a noncommutativity between these two limits,

$$\lim_{N,t\to\infty} \hat{\rho}(t) \neq \lim_{t,N\to\infty} \hat{\rho}(t). \quad (10)$$

The noncommutativity of the NESS properties works as a sufficient condition for gapless modes in general nondegenerate Lindbladian, a main property we will explore in the Letter.

In the thermodynamic limit the dynamics becomes exact within a mean-field approach, therefore a natural "order parameter" to seek for the existence of gapless excitations follows from the study of cumulant correlations. Specifically, given an initial uncorrelated state $\lim_{t,N\to\infty} K(\hat{o}_j, \hat{o}_\ell)(t) = 0$ [Eq. (7)], a sufficient condition for noncommutativity follows from the inverse limit,

$$\lim_{N,t\to\infty} K(\hat{o}_j, \hat{o}_\ell)(t) = \lim_{N\to\infty} [K(\hat{o}_j, \hat{o}_\ell)]_{\text{NESS}} \neq 0, \quad (11)$$

where $[K(\hat{o}_j, \hat{o}_\ell)]_{\text{NESS}} = (\langle \hat{o}_j \hat{o}_\ell \rangle - \langle \hat{o}_i \rangle \langle \hat{o}_\ell \rangle)_{\text{NESS}}$, showing that nonzero macroscopic cumulant correlations in the NESS come along with gapless excitations.

We focus here in the discussion of cumulants, but it is worth remarking that any other noncommuting feature could also be employed as a sufficient criterium for gapless modes. In particular, a class of states with null macroscopic cumulant correlations are those of coherent pure states—the mean-field pure state ansatz. Assuming that this relation is bijective, i.e., any state with null macroscopic cumulants corresponds to a coherent pure state, our gapless condition could be rephased in terms of any property not shared by coherent pure states, which may be simpler to determine depending on the system. An equivalent condition could rely in this way on the purity of the state, as studied for spin systems with p-order interactions [10] or with a modified parity-time symmetry [12], for which the mixedness of the steady state was indeed observed in association to gapless modes and furthermore to the presence of boundary time crystals.

Considering Eq. (11) is satisfied the mean-field dynamics can behave in different forms due to the existence of conserved quantities in the system: (i) if the correlations concern to boson degrees of freedom, the mean-field may still reproduce the NESS one-body macroscopic observables correctly, i.e., $[K(\hat{o}_j)]_{\text{NESS}} = [K(\hat{o}_j)]_{\text{MF}}$ since there are no constraints to their expectation values; (ii) however, dealing with collective spins this can never be reached due to the angular momentum conservation $\sum_\alpha \langle (\hat{m}_i^\alpha)^2 \rangle = 1$, and this is a crucial observation. Specifically, this conservation can be rewritten as $\sum_\alpha [K(\hat{m}_i^\alpha)]^2_{\mathrm{NESS}} + [K(\hat{m}_i^\alpha, \hat{m}_i^\alpha)]_{\mathrm{NESS}} = 1 = \sum_\alpha [K(\hat{m}_i^\alpha)]^2_{\mathrm{MF}}$. Given Eq. (11) and from the fact that $K(\hat{m}_i^\alpha, m_i^\alpha) \geq 0 \ \forall \ j$, there must be at least an α' such that $[K(\hat{m}_i^{\alpha'})]_{\mathrm{NESS}} \neq [K(\hat{m}_i^{\alpha'})]_{\mathrm{MF}}$ in the equality. Therefore, in this case the MF fails completely in the attempt to reproduce the one-body NESS macroscopic observables, and the dynamics can become "lost" due to its inability to match these expectation values, as illustrated in Fig. 1. We study below different models for which the cumulant gapless condition is satisfied, and find an interesting connection to persistent dynamics with the possible formation of dissipative time-crystal behavior [32]. The inability of MF to reach the NESS values seems to lie at the core of these behaviors, showing the importance of conservation laws (as also considered in different models Refs. [33–36]) and correlations for such phases.

Hermitian Lindblad operators.—Perhaps the simplest case correspond to Lindbadians with Hermitian jump operators, for which the dissipation leads to a collective dephasing on the spins or boson degrees of freedom suppressing the off-diagonal terms in the density matrix with respect to their eigenstate basis. If the Lindbladian has no degeneracy for finite system sizes, given any initial state the dynamics is driven towards the maximally mixed state $\hat{\rho}_{\mathbb{I}} = \mathbb{I}/d$, with d the normalization constant, which by definition has nonzero macroscopic cumulant correlations. The steady state in this case is trivial for any strength of dissipation, with no specific ordering among the spins. This class of Lindbladians satisfy Eq. (11) and therefore always support gapless modes. A simple example follows a single spin ensemble ($M = 1$) driven by a coherent field Hamiltonian along the x direction (ω_x) and a dissipation along the orthogonal z direction ($\Gamma_{z,z}$), with all other parameters null in the Lindbladian [Eq. (1)]. The gap eigenvalue λ_1 (the one with largest nonzero real part) of the Lindbladian shows in the limit of large system sizes a gapless scaling $-\mathrm{Re}(\lambda_1) \sim 1/N$ with an imaginary term $|\mathrm{Im}(\lambda_1)| \sim \omega_x$. While the decay rate (real part) arises from the dephasing, the coherent oscillations (imaginary term) follow directly from the field applied to the spins, with both features acting roughly independently of each other (see Ref. [25]). Although one may still observe persistent dynamics on its observables, it arises simply from the applied field on the spins (notice that the frequency is independent of the dissipation), and not due to a correlated dynamics. A different situation arises, however, for systems with non-Hermitian Lindblad operators. In this case the competing coherent and dissipative dynamics can generate nontrivial steady states and possibly ordered dissipative time-crystal phases. We discuss examples below.

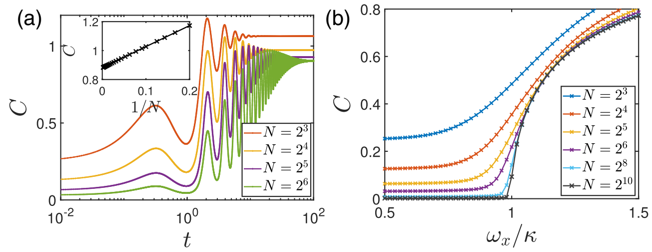

FIG. 2. (a) Dynamics of cumulants $C = \sum_\alpha K(\hat{m}_i^\alpha, \hat{m}_i^\alpha)$ for different system sizes, with $\omega_x/\kappa = 2$. The inset shows the finite size scaling analysis, converging to a non-null value in the thermodynamic limit. (b) NESS cumulants for varying ratio ω_x/κ and system sizes.

Collective spin-1/2 model.—A model with non-Hermitian Lindblad operators supporting nontrivial steady states corresponds to a single spin ensemble ($M = 1$) driven by competing coherent transverse field ω_x and a dissipative decay with

$$\Gamma_{x,x} = \Gamma_{y,y} = \sqrt{\kappa}, \qquad \Gamma_{y,x} = i\sqrt{\kappa} = \Gamma^*_{x,y}, \tag{12}$$

[equivalently, the dissipation corresponds to a decay Lindblad jump operator $\sqrt{\kappa}\hat{S}_-$, with $\hat{S}_\pm = (\hat{S}_x \pm i\hat{S}_y)$], while all other parameters are null. As discussed in Ref. [18] in the thermodynamic limit (and only in this limit) the model features persistent oscillations of its macroscopic magnetization (dissipative TC) for the weak dissipative regime $\omega_x/\kappa > 1$ (see Fig. 1), while in the strong dissipative case $\omega_x/\kappa < 1$ it shows a relaxation to its steady state. We show in Fig. 2(a) the dynamics for the cumulants in the dissipative TC phase with its oscillations and nonvanishing thermodynamic limit (inset). The gapless excitations tend to spread correlations among the spin constituents of the ensemble. In Fig. 2(b) we compute the cumulant phase diagram for the steady state of the model. While for stronger dissipation the spins dominantly decay, roughly pointing all down along the z direction and therefore with null correlations among them, for weaker dissipation there are indeed nonzero macroscopic correlations in the NESS. The region with dissipative TC phase is therefore precisely the one with non-null macroscopic cumulants, corroborating our arguments. We also analyzed extended models composed by a pair of interacting spin-1/2 systems ($M = 2$). Our results show again a connection between cumulant correlations to dissipative TCs (see Ref. [25] for details).

Floquet spin-boson system.—We also consider time-dependent Floquet Lindbladian dynamics. Specifically, a modulated open Dicke model with a single spin ensemble ($M = 1$) interacting with a single-mode cavity. This model of interacting systems supports dissipative time-crystals phase robust to perturbations, as examined in detail by Gong *et al.* in Ref. [19]. The Lindbladian is defined as

Eq. (1) where the spin-boson coupling $g_x(t)$ is modulated periodically,

$$g_x(t+T) = g_x(t) = \begin{cases} g, & 0 \le t \le \frac{T}{2}, \\ 0, & \frac{T}{2} \le t \le T, \end{cases} \tag{13}$$

with T the Floquet period, the cavity (spin) field is given by ω_b ($\omega_z^{(1)}$) and cavity loss by the rate κ_b. All other parameters are null. The model can break its discrete time-translation symmetry showing subharmonic oscilations with period nT, for $n > 1$.

In the case of Floquet dynamics we must revisit the gap definition. It is now appropriate dealing with the Floquet-Lindblad superoperator $\mathcal{U}_F = \mathcal{T} e^{\int_0^T dt \mathcal{L}(t)}$. The Floquet steady state is described by the eigenvalue of the operator $\mathcal{U}_F$ with $\lambda_i = 1$. The gap describing the slowest relaxation mode of the system is the closest to the unit ratio, i.e.,

$$\Delta_N^{[\text{Floquet}]} = 1 - \max_{\{\lambda_i\}} |\lambda_i|. \tag{14}$$

With this definition all our gapless condition arguments remain unchanged, based on the cumulant correlations of the Floquet steady state.

The analysis of the dynamics for the system is very demanding in the limit of large number of spins due to the high dimensionality of the Hilbert space. Nevertheless, one can work with an effective description of the model, within an adiabatic elimination of the spin degrees of freedom and under specific conditions [19]. The boson-only description is given by the coherent Hamiltonian,

$$\hat{H} = \omega_b \hat{a}^\dagger \hat{a} - \frac{\Omega_2(t)}{4}(\hat{a}^\dagger + \hat{a})^2 + \frac{\Omega_4(t)}{32N}(\hat{a}^\dagger + \hat{a})^4, \tag{15}$$

and losses as Eq. (3). The couplings are given by

$$\Omega_2(t+T) = \Omega_2(t) = \begin{cases} 1.5\omega_b, & 0 \le t \le \frac{\pi}{\omega_b}, \\ 0, & \frac{\pi}{\omega_b} \le t \le (2-\epsilon)\frac{\pi}{\omega_b}, \end{cases} \tag{16}$$

with $\Omega_4(t) = \Omega_2(t)$. The dynamics of the system can be studied numerically, truncating the maximum boson occupancy sufficiently large. Given an initial coherent state, we show in Fig. 3(a) the Floquet dynamics for the system in the thermodynamic limit, with couplings $\kappa_b = 0.1\omega_b$ and $\epsilon = 0.1$ supporting a dissipative TC. The macroscopic position operator, after an initial transient time, is characterized by a stable period doubling dynamics. Performing a finite-size scaling for the macroscopic cumulant correlations we obtain their scaling with system size shown in Fig. 3(b). Once more, concomitant with macroscopic correlations, the system supports gapless Floquet excitations, as shown in the inset of Fig. 3(b).

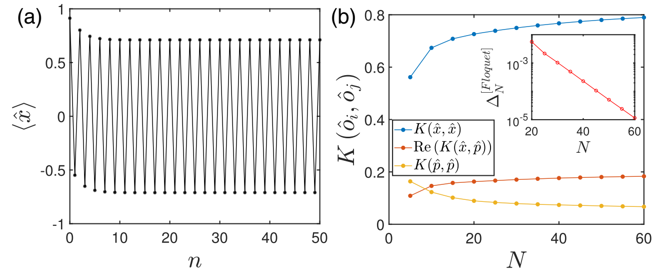

FIG. 3. (a) Stroboscopic dynamics ($t_n = nT$) for the macroscopic position operator with $N = 60$, considering a coherent state as initial condition. (b) Cumulant correlations $K(\hat{o}_i, \hat{o}_j)$ in the NESS for different macroscopic observables. The inset shows the floquet spectral gap $\Delta_N^{[\text{Floquet}]}$ [Eq. (14)] with the number of spins N, highlighting its exponential decay. In all plots we used a maximum boson occupation of $d_b = 65$.

Permutationally invariant systems.—Our MF proof and gapless condition can be extended to more general systems, as we illustrate below. Specifically, we generalize to Lindbladians satisfying a weaker restriction, that the expectation values of the spins be permutationally invariant in the dynamics:

$$\left\langle \prod_{j=1}^{n} \hat{\sigma}_{i,k_j}^{\alpha_j} \right\rangle = \left\langle \prod_{j=1}^{n} \hat{\sigma}_{i,k'_j}^{\alpha_j} \right\rangle, \quad \forall \ k_j, k'_j, i, n, \tag{17}$$

where $\langle \hat{O} \rangle \equiv Tr(\hat{\rho}(t)\hat{O})$, $k_p \neq k_q$, and $k'_p \neq k'_q$, for $p \neq q$, i.e., for each ith spin ensemble, the n-body correlated magnetization is independent on the spin microscopic labels (k_j and k'_j). We consider as an example a dissipative spin channel with possible spatial dependence,

$$\mathcal{D}_i[\hat{\rho}] = \sum_{j,k=1}^{N} \sum_{\alpha,\beta} \gamma_{\alpha,\beta}^{(i)j,k} \left(\hat{\sigma}_{i,j}^{\alpha} \hat{\rho} \hat{\sigma}_{i,k}^{\beta} - \frac{1}{2} \{ \hat{\sigma}_{i,k}^{\beta} \hat{\sigma}_{i,j}^{\alpha}, \hat{\rho} \} \right), \tag{18}$$

with well-defined thermodynamic couplings, i.e., $\lim_{N\to\infty} \sum_{j=1}^{N} \gamma_{\alpha,\beta}^{(i)j,j}/N$ and $\lim_{N\to\infty} \sum_{j\neq k}^{N} \gamma_{\alpha,\beta}^{(i)j,k}/N$ finite. The previously discussed collective spin dissipation [Eq. (4)], as well sufficiently long-range [13] and strictly local ones ($\gamma_{\alpha,\beta}^{i,j} = \delta_{i,j}\tilde{\gamma}_{\alpha,\beta}$) lie as specific cases of the above channel [37]. The proof follows similarly to the previous discussions (see Ref. [25] for details).

Conclusion.—In this Letter we discussed a condition for gapless excitations in a class of Lindbladians described by collective spin-boson models and general permutationally invariant systems. The condition based only on the macroscopic cumulant correlations is important not only for analytically establishing the fundamental relationship between NESS properties and system dynamics, but also from a practical point of view in the spectral determination without the need for a time evolution integration of the master equation or its exact diagonalization. These results can be insightful towards a proper understanding of the basic mechanisms for persistent dynamics in the thermodynamic limit. Interesting perspectives rely on the generalization for different models, i.e., nonpermutationally invariant systems [38]. Moreover, since our condition is a sufficient one, it would be important to extend to the necessary conditions for gapless excitations as well. Finally, we expect that our results will have an impact on the experimental implementation. Such phases have recently been observed in atom-cavity systems [39–42]. Envisioned with state-of-the-art quantum simulation platforms [43–47], one could engineer different forms of correlated dissipative steady states in spin-boson systems, and therefore different dissipative time-crystal implementations.

The codes for the numerical simulations have been constructed using the open source QuTiP library and free software Octave [48].

We acknowledge enlightening discussions with A. Russomanno, G. Passarelli, F. Carollo, and I. Lesanovsky. The authors acknowledge financial support from the Brazilian funding agencies CAPES, CNPQ, and FAPERJ (Grants No. 308205/2019-7, No. E-26/211.318/2019, No. 151064/2022-9, and No. E-26/201.365/2022).

[1] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).

[2] Dmitry A. Abanin, Ehud Altman, Immanuel Bloch, and Maksym Serbyn, Many-body localization, thermalization, and entanglement, Rev. Mod. Phys. **91**, 021001 (2019).

[3] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.

[4] Heinz-Peter Breuer and Francesco Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2017).

[5] F. Wilczek, Quantum Time Crystals, Phys. Rev. Lett. **109**, 160401 (2012).

[6] Krzysztof Sacha and Jakub Zakrzewski, Time crystals: A review, Rep. Prog. Phys. **81**, 016401 (2018).

[7] Dominic V. Else, Christopher Monroe, Chetan Nayak, and Norman Y. Yao, Discrete time crystals, Annu. Rev. Condens. Matter Phys. **11**, 467 (2020).

[8] Antônio C. Lourenço, Luis Fernando dos Prazeres, Thiago O. Maciel, Fernando Iemini, and Eduardo I. Duzzioni, Genuine multipartite correlations in a boundary time crystal, Phys. Rev. B **105**, 134422 (2022).

[9] F. Carollo and I. Lesanovsky, Exact solution of a boundary time-crystal phase transition: Time-translation symmetry breaking and non-Markovian dynamics of correlations, Phys. Rev. A **105**, L040202 (2022).

[10] Giulia Piccitto, Matteo Wauters, Franco Nori, and Nathan Shammah, Symmetries and conserved quantities of boundary time crystals in generalized spin models, Phys. Rev. B **104**, 014307 (2021).

[11] Luis Fernando dos Prazeres, Leonardo da Silva Souza, and Fernando Iemini, Boundary time crystals in collective d-level systems, Phys. Rev. B **103**, 184308 (2021).

[12] Yuma Nakanishi and Tomohiro Sasamoto, Dissipative time crystals originated from parity-time symmetry, Phys. Rev. A **107**, L010201 (2023).

[13] Gianluca Passarelli, Procolo Lucignano, Rosario Fazio, and Angelo Russomanno, Dissipative time crystals with long-range Lindbladians, Phys. Rev. B **106**, 224308 (2022).

[14] Fabrizio Minganti, Alberto Biella, Nicola Bartolo, and Cristiano Ciuti, Spectral theory of Liouvillians for dissipative phase transitions, Phys. Rev. A **98**, 042118 (2018).

[15] Victor V. Albert and Liang Jiang, Symmetries and conserved quantities in Lindblad master equations, Phys. Rev. A **89**, 022118 (2014).

[16] Tomaž Prosen, Third quantization: A general method to solve master equations for quadratic open Fermi systems, New J. Phys. **10**, 043026 (2008).

[17] Berislav Buča, Joseph Tindall, and Dieter Jaksch, Non-stationary coherent quantum many-body dynamics through dissipation, Nat. Commun. **10**, 1730 (2019).

[18] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, Boundary Time Crystals, Phys. Rev. Lett. **121**, 035301 (2018).

[19] Zongping Gong, Ryusuke Hamazaki, and Masahito Ueda, Discrete Time-Crystalline Order in Cavity and Circuit QED Systems, Phys. Rev. Lett. **120**, 040404 (2018).

[20] Masaya Nakagawa, Norio Kawakami, and Masahito Ueda, Exact Liouvillian Spectrum of a One-Dimensional Dissipative Hubbard Model, Phys. Rev. Lett. **126**, 110404 (2021).

[21] Pedro Ribeiro and Tomaž Prosen, Integrable Quantum Dynamics of Open Collective Spin Models, Phys. Rev. Lett. **122**, 010401 (2019).

[22] A. McDonald and A. A. Clerk, Exact Solutions of Interacting Dissipative Systems via Weak Symmetries, Phys. Rev. Lett. **128**, 033602 (2022).

[23] Aleksandra A. Ziolkowska and Fabian H. L. Essler, Yang-Baxter integrable Lindblad equations, SciPost Phys. **8**, 044 (2020).

[24] In its most general form, one could alternatively consider the $\{S^\alpha\}$ operators as an abstract SU(2) group, with N representing the dimension of the irredutible representation for the algebra. Our discussion for the gapless sufficient condition follows similarly in this case.

[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.180401 for further details.

[26] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Oxford University Press, New York, 1985).

[27] Ryogo Kubo, Generalized cumulant expansion method, J. Phys. Soc. Jpn. **17**, 1100 (1962).

[28] A. Wünsche, Quantum-mechanical cumulant expansions and their application to phase-space and to phase distributions, Phys. Scr. **90**, 074063 (2015).

[29] F. Benatti, F. Carollo, R. Floreanini, and H. Narnhofer, Quantum spin chain dissipative mean-field dynamics, J. Phys. A **51**, 325001 (2018).

[30] Federico Carollo and Igor Lesanovsky, Exactness of Mean-Field Equations for Open Dicke Models with an Application to Pattern Retrieval Dynamics, Phys. Rev. Lett. **126**, 230601 (2021).

[31] Leonardo da Silva Souza, Gonzalo Manzano, Rosario Fazio, and Fernando Iemini, Collective effects on the performance and stability of quantum heat engines, Phys. Rev. E **106**, 014143 (2022).

[32] Despite our analytical proof is performed for an initial uncorrelated initial state, we believe it has a wider range for general initial states, as discussed in [25].

[33] Angelo Russomanno, Fernando Iemini, Marcello Dalmonte, and Rosario Fazio, Floquet time crystal in the Lipkin-Meshkov-Glick model, Phys. Rev. B **95**, 214307 (2017).

[34] Federica Maria Surace, Angelo Russomanno, Marcello Dalmonte, Alessandro Silva, Rosario Fazio, and Fernando Iemini, Floquet time crystals in clock models, Phys. Rev. B **99**, 104303 (2019).

[35] H. P. Ojeda Collado, Gonzalo Usaj, C. A. Balseiro, Damián H. Zanette, and José Lorenzana, Emergent parametric resonances and time-crystal phases in driven Bardeen-Cooper-Schrieffer systems, Phys. Rev. Res. **3**, L042023 (2021).

[36] H. P. Ojeda Collado, Gonzalo Usaj, C. A. Balseiro, Damián H. Zanettea, and José Lorenzana, Dynamical phases transitions in periodically driven Bardeen-Cooper-Schrieffer systems, Phys. Rev. Res. **5**, 023014 (2023).

[37] Nathan Shammah, Shahnawaz Ahmed, Neill Lambert, Simone De Liberato, and Franco Nori, Open quantum systems with local and collective incoherent processes: Efficient numerical simulations using permutational invariance, Phys. Rev. A **98**, 063815 (2018).

[38] Joseph Tindall, Amy Searle, Abdulla Alhajri, and Dieter Jaksch, Quantum physics in connected worlds, Nat. Commun. **13**, 7445 (2022).

[39] Hans Keßler, Jayson G. Cosme, Michal Hemmerling, Ludwig Mathey, and Andreas Hemmerich, Emergent limit cycles and time crystal dynamics in an atom-cavity system, Phys. Rev. A **99**, 053605 (2019).

[40] Hans Keßler, Jayson G. Cosme, Christoph Georges, Ludwig Mathey, and Andreas Hemmerich, From a continuous to a discrete time crystal in a dissipative atom-cavity system, New J. Phys. **22**, 085002 (2020).

[41] Hans Keßler, Phatthamon Kongkhambut, Christoph Georges, Ludwig Mathey, Jayson G. Cosme, and Andreas Hemmerich, Observation of a Dissipative Time Crystal, Phys. Rev. Lett. **127**, 043602 (2021).

[42] Phatthamon Kongkhambut, Jim Skulte, Ludwig Mathey, Jayson G. Cosme, Andreas Hemmerich, and Hans Keßler, Observation of a continuous time crystal, Science **377**, 670 (2022).

[43] Lorenza Viola, Evan M. Fortunato, Marco A. Pravia, Emanuel Knill, Raymond Laflamme, and David G. Cory, Experimental realization of noiseless subsystems for quantum information processing, Science **293**, 2059 (2001).

[44] J. M. Fink, R. Bianchetti, M. Baur, M. Göppl, L. Steffen, S. Filipp, P. J. Leek, A. Blais, and A. Wallraff, Dressed Collective Qubit States and the Tavis-Cummings Model in Circuit QED, Phys. Rev. Lett. **103**, 083601 (2009).

[45] Minghui Xu, D. A. Tieri, E. C. Fine, James K. Thompson, and M. J. Holland, Synchronization of Two Ensembles of Atoms, Phys. Rev. Lett. **113**, 154101 (2014).

[46] Athreya Shankar, John Cooper, Justin G. Bohnet, John J. Bollinger, and Murray Holland, Steady-state spin synchronization through the collective motion of trapped ions, Phys. Rev. A **95**, 033423 (2017).

[47] D. A. Gangloff, G. Éthier Majcher, C. Lang, E. V. Denning, J. H. Bodey, D. M. Jackson, E. Clarke, M. Hugues, C. L. Gall, and M. Atatüre, Quantum interface of an electron and a nuclear ensemble, Science **364**, 62 (2019).

[48] The dynamics for the density matrix with finite N spins and mean-field equations were computed using Runge-Kutta numerical integration. We obtain the Lindbladian spectrum with numerical exact diagonalization of the superoperator matrix.