Dynamical Fermionization in One-Dimensional Spinor Gases at Finite Temperature

Ovidiu I. Pâţu
Institute for Space Sciences, Bucharest-Măgurele, R 077125, Romania

(Received 7 November 2022; accepted 31 March 2023; published 21 April 2023)

Following the removal of axial confinement the momentum distribution of a Tonks-Girardeau gas approaches that of a system of noninteracting spinless fermions in the initial harmonic trap. This phenomenon, called dynamical fermionization, has been experimentally confirmed in the case of the Lieb-Liniger model and theoretically predicted in the case of multicomponent systems at zero temperature. We prove analytically that for all spinor gases with strong repulsive contact interactions at finite temperature the momentum distribution after release from the trap asymptotically approaches that of a system of spinless fermions at the same temperature but with a renormalized chemical potential which depends on the number of components of the spinor system. In the case of the Gaudin-Yang model we check numerically our analytical predictions using the results obtained from a nonequilibrium generalization of Lenard's formula describing the time evolution of the field-field correlators.

DOI: 10.1103/PhysRevLett.130.163201

Introduction.—In the last decade considerable effort has been devoted to understanding the nonequilibrium dynamics of one-dimensional (1D) integrable and near-integrable many-body systems after the realization that such systems do not thermalize [1–4]. This flurry of activity resulted in the introduction of powerful techniques like the quench action [5,6] and generalized hydrodynamics [7,8], and in the investigation of various nonequilibrium scenarios in both single component [9–33] and multicomponent systems [34–44].

At zero temperature the momentum distribution of 1D strongly interacting bosons released from a harmonic trap will asymptotically approach the momentum distribution of a similar number of spinless fermions in the initial trap. This phenomenon, dubbed dynamical fermionization (DF) was theoretically predicted in Refs. [45,46] (see also Refs. [47–52]) and experimentally confirmed recently using ultracold atomic gases [53]. DF was also theoretically predicted to occur in multicomponent systems, bosonic, fermionic [54], or mixtures [55] using the factorization of the wave functions in charge and spin components in the strongly interacting regime [56–64]. At finite temperature results in the literature regarding DF are almost nonexistent with the only example that we are aware of being the numerical confirmation in the case of single component bosons reported in Ref. [51]. Generalizing the method of Ref. [46] for finite temperature it can be shown in the Supplemental Material [65] that for a system of trapped impenetrable bosons described by the grandcanonical ensemble at temperature T and chemical potential μ that DF is present and the asymptotical momentum distribution is the same as the one for a system of spinless fermions at the same temperature and chemical potential. The situation in the case of multicomponent systems is, obviously, more complicated. Naively, one would expect that if DF occurs in a multicomponent system at finite temperature then the asymptotic momentum distribution would be expressed as a sum of momentum distributions of free fermions with different chemical potentials. Contrary to this expectation in this Letter we show that for a spinor system at finite temperature the asymptotic momentum distribution after release from the trap approaches that of a system of spinless fermions at the same temperature but with a *renormalized chemical potential*, denoted by μ', which depends on the number of components of the system (or magnetic field in the case of unbalanced systems) but not on the statistics of the particles. More precisely, for any harmonically trapped multicomponent gas, bosonic or fermionic, with strong repulsive contact interactions we will show that after release from the trap (0) the initial density profile of the spinor gas is the same as the density profile of spinless noninteracting fermions described by T and μ' (this is in general called fermionization); (1) the asymptotic momentum distribution has the same shape as the initial density profile; and (2) the asymptotic momentum distribution is the same as the one for spinless noninteracting fermions characterized by T and μ' which represents the dynamical fermionization of the gas. In the case of the Gaudin-Yang model we present results also for each component (spin-up and spin-down) and numerically check our analytical predictions by deriving an extremely efficient determinant representation for the correlators which can be understood as the nonequilibrium multicomponent generalization of Lenard's formula [66].

The Gaudin-Yang model.—It is instructive to look first at the two-component case which provides the general template for the proof of DF in spinor gases but also has the advantage of allowing one to investigate the contribution of each component (and not only the sum like in the general case) both analytically and numerically. The Gaudin-Yang model [67,68] describes one-dimensional fermions or bosons with contact interactions and is the natural two-component generalization of the Lieb-Liniger model [69]. In the presence of a time-dependent harmonic potential $V(x,t) = m\omega^2(t)x^2/2$ the Hamiltonian reads as

$$\mathcal{H} = \int dx \frac{\hbar^2}{2m}(\partial_x \Psi^\dagger \partial_x \Psi) + g{:}(\Psi^\dagger\Psi)^2{:} + [V(x,t)-\mu](\Psi^\dagger\Psi) + B(\Psi^\dagger\sigma_z\Psi), \tag{1}$$

where $\Psi = \binom{\Psi_\uparrow(x)}{\Psi_\downarrow(x)}$, $\Psi^\dagger = (\Psi^\dagger_\uparrow(x), \Psi^\dagger_\downarrow(x))$, σ_z is the third Pauli matrix, μ is the chemical potential, B the magnetic field and : : denotes normal ordering. $\Psi_{\uparrow,\downarrow}(x)$ are fermionic or bosonic fields which satisfy the commutation relations $\Psi_\alpha(x)\Psi^\dagger_\beta(y) - \varepsilon\Psi^\dagger_\beta(y)\Psi_\alpha(x) = \delta_{\alpha\beta}\delta(x-y)$ with $\varepsilon = 1$ in the bosonic case and $\varepsilon = -1$ in the fermionic case. In this Letter we will investigate the nonequilibrium dynamics in the Tonks-Girardeau (TG) regime characterized by $g = \infty$. In the TG regime, also known as the impenetrable regime, the system is integrable even in the presence of the external potential, and at $t = 0$ the eigenstates of a system of N particles of which M have spin-down are [$\boldsymbol{x} = (x_1, \ldots, x_N)$, $d\boldsymbol{x} = \prod_{i=1}^N dx_i$]:

$$|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle = \int d\boldsymbol{x} \sum_{\alpha_1,\ldots,\alpha_N=\{\downarrow,\uparrow\}} \chi^{\alpha_1\cdots\alpha_N}_{N,M}(\boldsymbol{x}|\boldsymbol{j},\boldsymbol{\lambda}) \Psi^\dagger_{\alpha_N}(x_N)\cdots\Psi^\dagger_{\alpha_1}(x_1)|0\rangle. \tag{2}$$

Here the summation is over the C^N_M sets of α's of which M are spin-down and $N-M$ are spin-up, and $|0\rangle$ is the Fock vacuum satisfying $\Psi_\alpha(x)|0\rangle = \langle 0|\Psi^\dagger_\alpha(x) = 0$ for all x and α. The eigenstates [Eq. (2)] are identified by two sets of unequal numbers $\boldsymbol{j} = (j_1, \ldots, j_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ which correspond to the charge and spin degrees of freedom. The normalized wave functions are

$$\chi^{\alpha_1\cdots\alpha_N}_{N,M}(\boldsymbol{x}|\boldsymbol{j},\boldsymbol{\lambda}) = \frac{1}{N!N^{M/2}}\left[\sum_{P\in S_N}(-\varepsilon)^P \eta^{\alpha_{P_1}\cdots\alpha_{P_N}}_{N,M}(\boldsymbol{\lambda}) \times\theta(P\boldsymbol{x})\right]\det_N[\phi_{j_a}(x_b)], \tag{3}$$

with the determinant expressed in terms of Hermite functions of frequency $\omega_0 = \omega(t \le 0)$, i.e., $\phi_j(x) = (2^j j!)^{-1/2}(m\omega_0/\pi\hbar)^{1/4}e^{-(m\omega_0 x^2/2\hbar)}H_j(\sqrt{(m\omega_0/\hbar)}x)$ with $H_j(x)$ the Hermite polynomials. In Eq. (3) the sum is over the permutations of N elements, and $\theta(P\boldsymbol{x}) = \theta(x_{P_1} < \cdots < x_{P_N}) = \prod_{j=2}^N \theta(x_{P_j} - x_{P_{j-1}})$ with $\theta(x)$ the Heaviside function. The $\eta_{N,M}$ functions describing the spin sector are the wave functions of the XX spin chain with periodic boundary conditions $\eta^{\alpha_1\cdots\alpha_N}_{N,M}(\boldsymbol{\lambda}) = \prod_{j>k}\mathrm{sign}(n_j - n_k)\det_M(e^{in_a\lambda_b})$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ with $e^{i\lambda_a N} = (-1)^{M+1}$ and $\boldsymbol{n} = (n_1, \ldots, n_M)$ is a set of integers, $n_a \in \{1, \ldots, N\}$, describing the positions of the spin-down particles in the ordered set $\{x_1, \ldots, x_N\}$. The wave functions [Eq. (3)] represent the natural generalization of the Bethe ansatz wave functions for the Gaudin-Yang model [57] in the presence of an external confining potential. They solve the many-body Schrödinger equation, have the appropriate symmetries when exchanging two particles of the same type, satisfy the hard-core condition (the wave functions vanish when two coordinates are equal), and form a complete system. We stress again that the wave functions [Eq. (3)] and all the results derived below are valid only in the TG regime ($g = \infty$). The eigenstates [Eq. (2)] are normalized $\langle\Phi_{N',M'}(\boldsymbol{j}',\boldsymbol{\lambda}')|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle = \delta_{N'N}\delta_{M'M}\delta_{\boldsymbol{j}'\boldsymbol{j}}\delta_{\boldsymbol{\lambda}'\boldsymbol{\lambda}}$, highly degenerate (their energies do not depend on $\boldsymbol{\lambda}$), and satisfy $\mathcal{H}|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle = E_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle$ with $E_{N,M}(\boldsymbol{j},\boldsymbol{\lambda}) = \sum_{i=1}^N[\hbar\omega_0(j_i + 1/2) - \mu + B] - 2BM$. It should be noted that the energy spectrum is independent of statistics.

Quench protocol.—We are interested in investigating the dynamics of the real space and momentum densities at finite temperature after release from the trap. Our quench protocol is the following. Initially the system is prepared in a grandcanonical thermal state with the density matrix

$$\boldsymbol{\rho}^{\mu,B,T} = \sum_{N=0}^\infty\sum_{M=0}^N\sum_{\{\boldsymbol{j}\}}\sum_{\{\boldsymbol{\lambda}\}} p^{\boldsymbol{j},\boldsymbol{\lambda}}_{N,M}(\mu,B,T) \times|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle\langle\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})|, \tag{4}$$

where $p^{\boldsymbol{j},\boldsymbol{\lambda}}_{N,M}(\mu,B,T) = e^{-E_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})/k_BT}/\mathcal{Z}(\mu,B,T)$, $\mathcal{Z}(\mu,B,T) = \mathrm{Tr}[e^{-\mathcal{H}^I/k_BT}]$ is the partition function of the Gaudin-Yang model and $\mathcal{H}^I$ is the Hamiltonian (1) at $t = 0$ [$\omega(t \le 0) = \omega_0$]. At $t > 0$ we remove the axial confinement, and the system evolves with $\mathcal{H}^F$ which is the Hamiltonian (1) with $\omega(t > 0) = 0$. Our main objects of study are the field-field correlators defined as ($\sigma = \{\uparrow,\downarrow\}$):

$$g^{\mu,B,T}_\sigma(\xi_1,\xi_2;t) = \mathrm{Tr}[\boldsymbol{\rho}^{\mu,B,T}\Psi^\dagger_\sigma(\xi_1,t)\Psi_\sigma(\xi_2,t)], \tag{5}$$

with $\Psi^\dagger_\sigma(\xi,t) = e^{i\mathcal{H}^F t}\Psi^\dagger_\sigma(\xi)e^{-i\mathcal{H}^F t}$. From the correlators one can obtain the real space densities $\rho^{\mu,B,T}_\sigma(\xi,t) = g^{\mu,B,T}_\sigma(\xi,\xi;t)$ and the momentum distributions $n^{\mu,B,T}_\sigma(p,t) = \int e^{ip(\xi_1-\xi_2)/\hbar}g^{\mu,B,T}_\sigma(\xi_1,\xi_2;t)d\xi_1 d\xi_2/2\pi$. Because $g^{\mu,B,T}_\uparrow(\xi_1,\xi_2;t) = g^{\mu,-B,T}_\downarrow(\xi_1,\xi_2;t)$ it is sufficient to consider only one of the correlators.

Time-evolution of the correlators.—The important observation which allows for the analytical investigation of the dynamics is that the spin component of the wave functions remains frozen during the time evolution due to the strong interactions between the particles [54,55]. The charge component of the wave functions [Eq. (3)] is expressed in terms of harmonic oscillator functions whose dynamics in the case of time-dependent frequency is known [70,71] and is implemented by the scaling transformation $\phi_j(x,t) = (1/\sqrt{b})\phi_j[(x/b),0]\exp[i(mx^2/2\hbar)(\dot{b}/b) - iE_j\tau(t)]$ with $E_j = \hbar\omega_0(j+1/2)$ and $\tau(t) = \int_0^t dt'/b^2(t')$. In the previous relations $b(t)$ is a solution of the Ermakov-Pinney equation $\ddot{b} = -\omega(t)^2 b + \omega_0^2/b^3$ with boundary conditions $b(0)=1$, $\dot{b}(0)=0$. Therefore, we can investigate the dynamics computing the correlators at $t=0$ and then applying the scaling transformation. At $t=0$ the correlators in the initial thermal state described by the density matrix [Eq. (4)] can be written as

$$g_\sigma^{\mu,B,T}(\xi_1,\xi_2) = \sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{\{j\}}\sum_{\{\lambda\}} p_{N,M}^{\boldsymbol{j},\boldsymbol{\lambda}} G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2), \quad (6)$$

with $G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2) = \langle\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})|\Psi_\sigma^\dagger(\xi_1)\Psi_\sigma(\xi_2)|\Phi_{N,M}(\boldsymbol{j},\boldsymbol{\lambda})\rangle$. The G functions are the normalized mean values of bilocal operators in arbitrary states described by $\boldsymbol{j}$ and $\boldsymbol{\lambda}$. Introducing a new parametrization [55,65,72] which makes the decoupling of the degree of freedom explicit then, for $\xi_1 \le \xi_2$, the G functions can be expressed as sums of products of spin and charge functions (for their explicit expressions see the Supplemental Material [65])

$$G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2) = \frac{1}{c_\sigma N^M}\sum_{d_1=1}^{N}\sum_{d_2=d_1}^{N} S_\sigma(d_1,d_2) \times I(d_1,d_2;\xi_1,\xi_2), \quad (7)$$

with $c_\downarrow = (N-M)!(M-1)!$ and $c_\uparrow = (N-M-1)!M!$. The time evolution of the correlators is obtained by plugging the scaling transformation of the Hermite functions in the expression for $G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2)$ in terms of wave functions (see the Supplemental Material [65]). We find ($l_0 = \sqrt{\hbar/(m\omega_0)}$)

$$G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2;t) = \frac{1}{b} G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}\left(\frac{\xi_1}{b},\frac{\xi_2}{b};0\right) e^{-\frac{i}{b\omega_0}\frac{\dot{b}}{}\frac{\xi_1^2-\xi_2^2}{2l_0^2}}, \quad (8)$$

and introducing the notation $\tilde{G}_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(p,t) = \int e^{ip(\xi_1-\xi_2)/\hbar} G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2;t) d\xi_1 d\xi_2/2\pi$ we have

$$\tilde{G}_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(p,t) = \frac{b}{2\pi}\int G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi_1,\xi_2;0) \times e^{-ib\left[\frac{\dot{b}}{\omega_0}\frac{\xi_1^2-\xi_2^2}{2l_0^2} - \frac{p(\xi_1-\xi_2)}{\hbar}\right]} d\xi_1 d\xi_2. \quad (9)$$

The dynamics of the real space density and momentum distribution is derived using (8) and (9) in Eq. (6).

Analytical derivation of dynamical fermionization.—As a preliminary step we will compute the partition function of the Gaudin-Yang (GY) model which appears in the definition of the state probabilities $p_{N,M}^{\boldsymbol{j},\boldsymbol{\lambda}}(\mu,B,T)$ describing the density matrix [Eq. (4)]. We should point out that the thermodynamics of trapped impenetrable particles with contact interactions is independent of statistics (the energy spectrum is identical and double occupancies are excluded). In the case of homogeneous systems a proof can be found in Ref. [57]. Using the identity $\sum_{M=0}^{N}\sum_{\{\lambda\}} e^{(2BM/k_BT)} = (1+e^{(2B/k_BT)})^N$ we obtain $\mathcal{Z}(\mu,B,T) = \sum_{N=0}^{\infty}\sum_{\{j\}}[2\cosh(B/k_BT)]^N e^{-E_N(\boldsymbol{j})/k_BT}$ with $E_N(\boldsymbol{j}) = \sum_{i=1}^{N}[\hbar\omega_0(j_i+1/2)-\mu]$ which shows that the partition function of the harmonically trapped GY model in the TG regime is the same as the one of trapped spinless free fermions $\mathcal{Z}_{\mathsf{FF}}(\mu',T)$ at the same temperature but with renormalized chemical potential (this is the generalization of the homogeneous result first obtained by Takahashi in Ref. [73]),

$$\mu' = \mu + k_BT\ln[2\cosh(B/k_BT)]. \quad (10)$$

Let us investigate the densities at $t=0$. From the definition [Eq. (6)] we have $\rho_\sigma^{\mu,B,T}(\xi) = \sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{\{j\}}\sum_{\{\lambda\}} \times p_{N,M}^{\boldsymbol{j},\boldsymbol{\lambda}} G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi,\xi)$ with $G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(\xi,\xi) = \sum_{d=1}^{N} S_\sigma(d,d) I(d,d;\xi,\xi)/(c_\sigma N^M)$. It can be shown in the Supplemental Material [65] that $S_\downarrow(d,d) = (N-M)!M!N^{M-1}$, $S_\uparrow(d,d) = (N-M-1)!M!N^{M-1}(N-M)$ and that $\sum_{d=1}^{N} I(d,d;\xi,\xi) = G_{N,FF}^{\mathbf{j}}(\xi,\xi)$ where $G_{N,FF}^{\mathbf{j}}(\xi,\xi)$ is the density of free fermions (in the state $\boldsymbol{j}$) at position ξ. Using these results we obtain

$$\rho_\downarrow^{\mu,B,T}(\xi) = \frac{e^{B/k_BT}}{2\cosh(B/k_BT)}\rho_{\mathsf{FF}}^{\mu',T}(\xi), \quad (11)$$

$\rho_\uparrow^{\mu,B,T}(\xi) = \rho_\downarrow^{\mu,-B,T}(\xi)$, and $\rho_\downarrow^{\mu,B,T}(\xi) + \rho_\uparrow^{\mu,B,T}(\xi) = \rho_{\mathsf{FF}}^{\mu',T}(\xi)$ proving that the initial densities are proportional to the densities of trapped spinless free fermions at the same temperature and chemical potential given by Eq. (10) (property 0 from the introduction).

Now we can investigate the dynamics. In the case of free expansion the solution of the Ermakov-Pinney equation is $b(t) = (1+\omega_0^2t^2)^{1/2}$ and in the large time limit we have $\lim_{t\to\infty} b(t) = \omega_0 t$ and $\lim_{t\to\infty}\dot{b}(t) = \omega_0$. The momentum distribution is

$$n_\sigma(p,t) = \sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{\{j\}}\sum_{\{\lambda\}} p_{N,M}^{\boldsymbol{j},\boldsymbol{\lambda}} \tilde{G}_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(p,t), \quad (12)$$

and we need $\lim_{t\to\infty}\tilde{G}_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(p,t)$. Using the method of stationary phase (Chap. 6 of Ref. [74] or Chap. 2.9 of Ref. [75]) in Eq. (9) with the points of stationary phase being $\xi_0 = p\omega_0 l_0^2/(\dot{b}\hbar)$ for both integrals we find $\tilde{G}_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}(p,t) \underset{t\to\infty}{\sim} |(\omega_0 l_0^2/\dot{b})| G_{N,M,\sigma}^{\boldsymbol{j},\boldsymbol{\lambda}}[(p\omega_0 l_0^2/\dot{b}\hbar),(p\omega_0 l_0^2/\dot{b}\hbar);0]$. We have

$G^{\boldsymbol{j},\boldsymbol{\lambda}}_{N,M,\downarrow}(\xi,\xi) = MG^{\boldsymbol{j}}_{N,\mathsf{FF}}(\xi,\xi)/N$ and $G^{\boldsymbol{j},\boldsymbol{\lambda}}_{N,M,\uparrow}(\xi,\xi) = (N-M)G^{\boldsymbol{j}}_{N,\mathsf{FF}}(\xi,\xi)/N$. Performing similar computations like in the case of the initial densities we obtain

$$n^{\mu,B,T}_{\downarrow}(p,t) \underset{t\to\infty}{\sim} l_0^2 \frac{e^{B/k_BT}}{2\cosh(B/k_BT)} \rho^{\mu',T}_{\mathsf{FF}}\left(\frac{pl_0^2}{\hbar}\right), \quad (13)$$

and $n^{\mu,B,T}_{\uparrow}(p,t) = n^{\mu,-B,T}_{\downarrow}(p,t)$ which shows that the asymptotic momentum distributions have the same shape as the initial densities (property 1). Finally, using the identity $n^{\mu,T}_{\mathsf{FF}}(p) = l_0^2\rho^{\mu,T}_{\mathsf{FF}}(pl_0^2/\hbar)$ (see Appendix E of Ref. [55]) we obtain

$$n^{\mu,B,T}_{\downarrow}(p,t) \underset{t\to\infty}{\sim} \frac{e^{B/k_BT}}{2\cosh(B/k_BT)} n^{\mu',T}_{\mathsf{FF}}(p), \quad (14)$$

and $n^{\mu,B,T}_{\downarrow}(p,t) + n^{\mu,B,T}_{\uparrow}(p,t) \underset{t\to\infty}{\sim} n^{\mu',T}_{\mathsf{FF}}(p)$ which proves the dynamical fermionization at finite temperature (property 2).

We can numerically check the analytical predictions given by Eq. (14) using a determinant representation for the field correlators which represents the other main result of this Letter. This representation obtained via summation of the form factors is the nonequilibrium multicomponent generalization of Lenards's formula [66] originally introduced for impenetrable bosons, and it reads as $[g^{\mu,B,T}_{\downarrow}(\xi_1,\xi_2;t) = g^{\mu,-B,T}_{\uparrow}(\xi_1,\xi_2;t)]$

$$g^{\mu,B,T}_{\downarrow}(\xi_1,\xi_2;t) = \det(1+\gamma\mathsf{V}+\mathsf{R}) - \det(1+\gamma\mathsf{V}), \quad (15)$$

with $\gamma = -(1+e^{2B/T}+\varepsilon)\mathrm{sign}(\xi_2-\xi_1)$, and the elements of the (infinite) matrices V,R are given by $\mathsf{V}_{a,b} = \sqrt{f(a)f(b)}\int_{\xi_1}^{\xi_2}\bar{\phi}_a(v,t)\phi_b(v,t)dv$ and $\mathsf{R}_{a,b} = \sqrt{f(a)f(b)}\bar{\phi}_a(\xi_1,t)\phi_b(\xi_2,t)$ where $f(a) = e^{-B/T}/[2\cosh(B/T) + e^{\{[\hbar\omega_0(a+1/2)-\mu]/T\}}]$ is the Fermi function and $\phi_a(v,t)$ are the time-evolved harmonic orbitals. In addition to representing the starting point for the rigorous derivation of various analytical properties of the correlators (for example one can show that $g^{\mu,B,T}_{\downarrow,\uparrow}(\xi_1,\xi_2;t)$ can be expressed in terms of Painlevé transcendents) Eq. (15) is also extremely efficient numerically due to the fact that the main computational effort is reduced to the calculation of partial overlaps of the single particle evolved wave functions and, therefore, can be used to investigate different experimentally relevant quench scenarios like breathing oscillations [25,26], quantum Newton's cradle [1,19], periodic modulation of the frequency [27], etc., which were not previously accessible in the case of multicomponent systems. Figure 1 presents the dynamics of $n_{\downarrow}(p,t)$ derived from Eq. (15) for an unbalanced system with $N=30$ particles and $N_{\downarrow}=20$ after release from the trap which shows the excellent agreement with the analytical result [Eq. (14)].

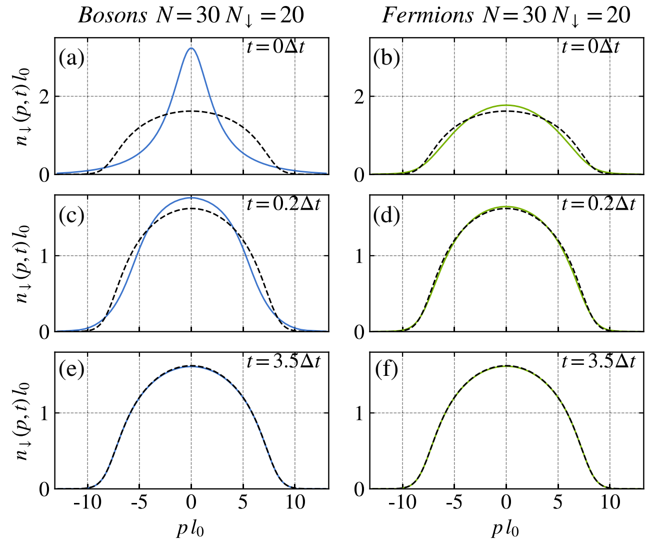

FIG. 1. Dynamics of the momentum distribution of spin-down particles after release from the trap in the GY model with $N=30$ and $N_{\downarrow}=20$. The temperature and initial trap frequency are $T=5$ and $\omega_0=1$ ($\mu=26.22$, $|B|=1.73$, $\Delta t=\pi/\omega_0$). The continuous line in (a), (c), and (e) [(b), (d), and (f)] represents the momentum distribution $n_{\downarrow}(p,t)$ for a bosonic (fermionic) system while the dashed line is the analytical prediction [Eq. (14)].

General case.—In the general case of a system with κ components the second line of the Hamiltonian [Eq. (1)] becomes $V(x,t)\Psi^\dagger\Psi - \Psi^\dagger\boldsymbol{\mu}\Psi$ where now $\Psi^\dagger = (\Psi_1^\dagger(x),\ldots,\Psi_\kappa^\dagger(x))$ with $\Psi_\sigma(x)$ ($\sigma=\{1,\ldots,\kappa\}$) fermionic or bosonic fields satisfying the commutation relations $\Psi_\sigma(x)\Psi^\dagger_{\sigma'}(y) - \varepsilon\Psi^\dagger_{\sigma'}(y)\Psi_\sigma(x) = \delta_{\sigma\sigma'}\delta(x-y)$, and $\boldsymbol{\mu}$ is a diagonal matrix with $(\mu_1,\ldots,\mu_\kappa)$ on the diagonal which are the chemical potentials of each component. The eigenstates of the system are described by κ sets of parameters [76,77] $\boldsymbol{j}=\{j_i\}_{i=1}^N$ and $[\boldsymbol{\lambda}] = (\{\lambda_i^{(1)}\}_{i=1}^{N_1},\ldots,\{\lambda_i^{(\kappa-1)}\}_{i=1}^{N_{\kappa-1}})$ with $N\geq N_1\geq\cdots\geq N_{\kappa-1}\geq 0$ and will be denoted by $|\Phi^\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])\rangle$. The number of particles in the state σ is $m_\sigma = N_{\sigma-1}-N_\sigma$ where we consider $N_0=N$ and $N_\kappa=0$ and $\mathcal{H}|\Phi^\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])\rangle = E_\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])|\Phi^\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])\rangle$ with $|E_\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])\rangle = \sum_{i=1}^N\hbar\omega_0(j_i+1/2) - \sum_{\sigma=1}^\kappa\mu_\sigma(N_{\sigma-1}-N_\sigma)$. The energies of the eigenstates do not depend on the spin configuration $[\boldsymbol{\lambda}]$ resulting in large degeneracies. From now on we will consider the case of pure Zeeman splitting which is described by $\mu_1 = \mu - B(\kappa-1)$ and $\mu_{i+1}-\mu_i = 2B$. The initial grandcanonical thermal state [analog of Eq. (4)] is

$$\rho^{\mu,B,T}_\kappa = \sum_{N=0}^{\infty}\sum_{N_1=0}^{N}\cdots\sum_{N_{\kappa-1}=0}^{N_{\kappa-2}}\sum_{\{\boldsymbol{j}\}}\sum_{\{\boldsymbol{\lambda}^{(1)}\}}\cdots\sum_{\{\boldsymbol{\lambda}^{(\kappa-1)}\}} \times p^{\boldsymbol{j},[\boldsymbol{\lambda}]}_\kappa(\mu,B,T)|\Phi^\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])\rangle\langle\Phi^\kappa(\boldsymbol{j},[\boldsymbol{\lambda}])|,$$

where now $p_\kappa^{\boldsymbol{j},[\lambda]}(\mu,B,T)=e^{-E_\kappa(\boldsymbol{j},[\lambda])/k_BT}/\mathcal{Z}_\kappa(\mu,B,T)$, with $\mathcal{Z}_\kappa(\mu,B,T)=\mathrm{Tr}[e^{-\mathcal{H}^I_\kappa/k_BT}]$ the partition function of the system with κ components at $t=0$. Like in the two-component case (see the Supplemental Material [65]) it can be shown that $\mathcal{Z}_\kappa(\mu,B,T)=Z_{\mathsf{FF}}(\mu'_\kappa,T)$ but now the renormalized chemical potential is (in the homogeneous case this result was first obtained by Schlottmann in Ref. [77])

$$\mu'_\kappa=\mu+k_BT\,\ln[\sinh(\kappa B/k_BT)/\sinh(B/k_BT)]. \quad (16)$$

The calculations in the general case are very similar with the ones for the GY model. We now have κ field correlators $g_\sigma^{\mu,B,T}(\xi_1,\xi_2;t)=\mathrm{Tr}[\boldsymbol{\rho}_\kappa^{\mu,B,T}\Psi_\sigma^\dagger(\xi_1,t)\Psi_\sigma(\xi_2,t)]$, $(\sigma=\{1,\ldots,\kappa\})$ and the same number of densities $\rho_\sigma^{\mu,B,T}(\xi,t)=g_\sigma^{\mu,B,T}(\xi,\xi;t)$ and momentum distributions $n_\sigma^{\mu,B,T}(p,t)$. Similar to the GY case the wave function has a product form with the charge component given by a Slater determinant of Hermite functions and the spin component given by an arbitrary function of an appropriate spin chain [58,59]. This means that the mean values of bilocal operators $G_\sigma^{\boldsymbol{j},[\lambda]}(\xi_1,\xi_2)=\langle\Phi^\kappa(\boldsymbol{j},[\lambda])|\Psi_\sigma^\dagger(\xi_1)\Psi_\sigma(\xi_2)|\Phi^\kappa(\boldsymbol{j},[\lambda])\rangle$ appearing in the generalization of Eq. (6) also have a product representation generalizing Eq. (7) and given by (explicit expressions for the components can be found in Refs. [54,65,78,79]) $G_\sigma^{\boldsymbol{j},[\lambda]}(\xi_1,\xi_2)=\sum_{d_1,d_2=1}^N S_\sigma(d_1,d_2)I(d_1,d_2;\xi_1,\xi_2)$. Unfortunately we do not know the value of $S_\sigma(d,d)$ [a reasonable conjecture would be $S_\sigma(d,d)=m_\sigma/N$] only that $\sum_{\sigma=1}^\kappa S_\sigma(d,d)=1$ [54]. Using this relation we obtain for the real space densities at $t=0$ $\sum_{\sigma=1}^\kappa\rho_\sigma^{\mu,B,T}(\xi)=\rho_{\mathsf{FF}}^{\mu'_\kappa,T}(\xi)$ with μ'_κ defined in Eq. (16). In the large t limit performing the stationary phase analysis like in the GY case we obtain that the total asymptotic momentum distribution has the same shape as the real space density profile $\sum_{\sigma=1}^\kappa n_\sigma^{\mu,B,T}(p,t)\underset{t\to\infty}{\sim}l_0^2\rho_{\mathsf{FF}}^{\mu'_\kappa,T}(pl_0^2/\hbar)$ and using $n_{\mathsf{FF}}^{\mu,T}(p)=l_0^2\rho_{\mathsf{FF}}^{\mu,T}(pl_0^2/\hbar)$ we find

$$\sum_{\sigma=1}^{\kappa}n_\sigma^{\mu,B,T}(p,t)\underset{t\to\infty}{\sim}n_{\mathsf{FF}}^{\mu'_\kappa,T}(p), \quad (17)$$

which is the dynamical fermionization of the strongly interacting κ component gas.

Finite interaction case.—In the case of large, but finite, repulsion, we expect that most of the features presented above to remain valid [54]. In this case, to first order in g, the wave functions still have a product form [78] with the charge degrees of freedom characterized by a Slater determinant and the spin part described by a spin chain [antiferromagnetic (ferromagnetic) in the fermionic (bosonic) case] with position dependent coefficients C_i. Fortunately, the time evolution of these coefficients during expansion is given by $C_i(t)=b^{-3}(t)C_i(0)$ [80] which means that spin dynamics of the system remains frozen like in the impenetrable case, and the same considerations apply. For arbitrary repulsion it is also sensible to assume that the system will dynamically fermionize after expansion and that the initial quasimomenta of the trapped gas will be mapped to real momenta of the expanded cloud similar to the case of single component bosons [10,12,50]. This is due to the fact that at long time after release the dimensionless parameter $\gamma(x)=c/n(x)$ $(c=mg/\hbar^2)$, which characterizes the strength of the interaction, will become very large [the density $n(x)$ decreases] and, therefore, the dynamics will be described by the TG Hamiltonian [(1) with $g=\infty$]. We expect that these considerations can be made rigorous using the Yudson representation for integrable systems [81] generalizing the proof for the Lieb-Liniger model derived in Ref. [12].

Conclusions.—We have proved that DF occurs in all bosonic and fermionic impenetrable 1D spinor gases at finite temperature. At long times after release from the trap the asymptotic momentum distribution approaches that of a system of spinless noninteracting fermions at the same temperature and a renormalized chemical potential which depends on the number of the components of the spinor system and magnetic field but not on the statistics. Using the same method one can prove the existence of DF in the case of an arbitrary Bose-Fermi mixture [55,72,76,82,83] using the fact that the wave functions in the TG regime also factorize with the spin component given by wave functions of an appropriate graded spin chain while the charge part is still described by a Slater determinant of Hermite functions. The proof runs along the same lines taking into account that the thermodynamics (partition function) of impenetrable particles is independent of the statistics of the constituent particles.

Financial support from the Grants No. 16N/2019 and No. 30N/2023 of the National Core Program of the Romanian Ministry of Research, Innovation and Digitization is gratefully acknowledged.

[1] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).
[2] M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
[3] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[4] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, and B. L. Lev, Science **371**, 296 (2021).
[5] J.-S. Caux and F. H. L. Essler, Phys. Rev. Lett. **110**, 257203 (2013).
[6] J.-S. Caux, J. Stat. Mech. (2016) 064006.
[7] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Phys. Rev. X **6**, 041065 (2016).

[8] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Phys. Rev. Lett. **117**, 207201 (2016).
[9] R. Pezer and H. Buljan, Phys. Rev. Lett. **98**, 240403 (2007).
[10] D. Jukić, R. Pezer, T. Gasenzer, and H. Buljan, Phys. Rev. A **78**, 053602 (2008).
[11] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. **106**, 227203 (2011).
[12] D. Iyer and N. Andrei, Phys. Rev. Lett. **109**, 115304 (2012).
[13] J. De Nardis, B. Wouters, M. Brockmann, and J.-S. Caux, Phys. Rev. A **89**, 033601 (2014).
[14] B. Pozsgay, J. Stat. Mech. (2013) P10028.
[15] B. Wouters, J. De Nardis, M. Brockmann, D. Fioretto, M. Rigol, and J.-S. Caux, Phys. Rev. Lett. **113**, 117202 (2014).
[16] B. Pozsgay, M. Mestyán, M. A. Werner, M. Kormos, G. Zaránd, and G. Takács, Phys. Rev. Lett. **113**, 117203 (2014).
[17] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Phys. Rev. Lett. **115**, 157201 (2015).
[18] L. Piroli, B. Pozsgay, and Eric Vernier, J. Stat. Mech. (2017) 023106.
[19] R. van den Berg, B. Wouters, S. Eliëns, J. De Nardis, R. M. Konik, and J.-S. Caux, Phys. Rev. Lett. **116**, 225302 (2016).
[20] M. Collura, S. Sotiriadis, and P. Calabrese, Phys. Rev. Lett. **110**, 245301 (2013).
[21] M. A. Rajabpour and S. Sotiriadis, Phys. Rev. A **89**, 033620 (2014).
[22] M. Kormos, M. Collura, G. Takács, and P. Calabrese, Nat. Phys. **13**, 246 (2017).
[23] M. Collura, M. Kormos, and P. Calabrese, Phys. Rev. A **97**, 033609 (2018).
[24] L. Vidmar, D. Iyer, and M. Rigol, Phys. Rev. X **7**, 021012 (2017).
[25] Y. Y. Atas, D. M. Gangardt, I. Bouchoule, and K. V. Kheruntsyan, Phys. Rev. A **95**, 043622 (2017).
[26] Y. Y. Atas, I. Bouchoule, D. M. Gangardt, and K. V. Kheruntsyan, Phys. Rev. A **96**, 041605(R) (2017).
[27] Y. Y. Atas, S. A. Simmons, and K. V. Kheruntsyan, Phys. Rev. A **100**, 043602 (2019).
[28] O. I. Pâţu, Phys. Rev. A **102**, 043303 (2020).
[29] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, Phys. Rev. Lett. **122**, 090601 (2019).
[30] P. Ruggiero, Y. Brun, and J. Dubail, SciPost Phys. **6**, 051 (2019).
[31] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, Phys. Rev. Lett. **124**, 140603 (2020).
[32] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, Science **373**, 1129 (2021).
[33] S. Scopa, A. Krajenbrink, P. Calabrese, and J. Dubail, J. Phys. A **54**, 404002 (2021).
[34] L. Yang and H. Pu, Phys. Rev. A **94**, 033614 (2016).
[35] E. Ilievski and J. De Nardis, Phys. Rev. B **96**, 081118(R) (2017).
[36] M. Mestyán, B. Bertini, L. Piroli, and P. Calabrese, J. Stat. Mech. (2017) 083103.
[37] P. Siegl, S. I. Mistakidis, and P. Schmelcher, Phys. Rev. A **97**, 053626 (2018).
[38] Y. Zhang, L. Vidmar, and M. Rigol, Phys. Rev. A **99**, 063605 (2019).
[39] S. Wang, X. Yin, Y.-Y. Chen, Y. Zhang, and X.-W. Guan, J. Phys. A **53**, 464002 (2020).
[40] Y. Nozawa and H. Tsunetsugu, Phys. Rev. B **101**, 035121 (2020).
[41] Y. Nozawa and H. Tsunetsugu, Phys. Rev. B **103**, 035130 (2021).
[42] S. Scopa, P. Calabrese, and L. Piroli, Phys. Rev. B **104**, 115423 (2021).
[43] C. Rylands, B. Bertini, and P. Calabrese, J. Stat. Mech. (2022) 103103.
[44] S. Scopa, P. Calabrese, and L. Piroli, Phys. Rev. B **106**, 134314 (2022).
[45] M. Rigol and A. Muramatsu, Phys. Rev. Lett. **94**, 240403 (2005).
[46] A. Minguzzi and D. M. Gangardt, Phys. Rev. Lett. **94**, 240404 (2005).
[47] A. del Campo, Phys. Rev. A **78**, 045602 (2008).
[48] D. M. Gangardt and M. Pustilnik, Phys. Rev. A **77**, 041604 (R) (2008).
[49] C. J. Bolech, F. Heidrich-Meisner, S. Langer, I. P. McCulloch, G. Orso, and M. Rigol, Phys. Rev. Lett. **109**, 110602 (2012).
[50] A. S. Campbell, D. M. Gangardt, and K. V. Kheruntsyan, Phys. Rev. Lett. **114**, 125302 (2015).
[51] W. Xu and M. Rigol, Phys. Rev. A **95**, 033617 (2017).
[52] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, SciPost Phys. **6**, 070 (2019).
[53] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol, and D. S. Weiss, Science **367**, 1461 (2020).
[54] S. S. Alam, T. Skaras, L. Yang, and H. Pu, Phys. Rev. Lett. **127**, 023002 (2021).
[55] O. I. Pâţu, Phys. Rev. A **105**, 063309 (2022).
[56] M. Ogata and H. Shiba, Phys. Rev. B **41**, 2326 (1990).
[57] A. G. Izergin and A. G. Pronko, Nucl. Phys. **B520**, 594 (1998).
[58] F. Deuretzbacher, K. Fredenhagen, D. Becker, K. Bongs, K. Sengstock, and D. Pfannkuche, Phys. Rev. Lett. **100**, 160405 (2008).
[59] L. Guan, S. Chen, Y. Wang, and Z. Q. Ma, Phys. Rev. Lett. **102**, 160402 (2009).
[60] A. G. Volosniev, D. V. Fedorov, A. S. Jensen, M. Valiente, and N. T. Zinner, Nat. Commun. **5**, 5300 (2014).
[61] J. Levinsen, P. Massignan, G. M. Bruun, and M. M. Parish, Sci. Adv. **1**, e1500197 (2015).
[62] L. Yang and X. Cui, Phys. Rev. A **93**, 013617 (2016).
[63] F. Deuretzbacher, D. Becker, J. Bjerlin, S. M. Reimann, and L. Santos, Phys. Rev. A **95**, 043630 (2017).
[64] L. Yang, S. S. Alam, and H. Pu, J. Phys. A **55**, 464005 (2022).
[65] See Suplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.163201 for a proof of finite temperature DF in the Lieb-Liniger model and other technical details.
[66] A. Lenard, J. Math. Phys. (N.Y.) **7**, 1268 (1966).
[67] M. Gaudin, Phys. Lett. **24A**, 55 (1967).
[68] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[69] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[70] V. S. Popov and A. M. Perelomov, Zh. Eksp. Teor. Fiz. **57**, 1684 (1970) [J. Exp. Theor. Phys. **30**, 910 (1970)], http://jetp.ras.ru/cgi-bin/e/index/e/30/5/p910?a=list.
[71] A. M. Perelomov and Y. B. Zeldovich, *Quantum Mechanics* (World Scientific, Singapore, 1998).

[72] A. Imambekov and E. Demler, Ann. Phys. (NY) **321**, 2390 (2006).
[73] M. Takahashi, Progr. Theor. Phys. **46**, 1388 (1971).
[74] N. Bleistein and R. A. Handelsman, *Asymptotic Epansions of Integrals* (Dover Publications, New York, USA, 1986).
[75] A. Erdélyi, *Asymptotic Expansions* (Dover Publications, New York, USA, 1956).
[76] B. Sutherland, Phys. Rev. Lett. **20**, 98 (1968).
[77] P. Schlottmann, J. Phys. C **5**, 5869 (1993).
[78] L. Yang, L. Guan, and H. Pu, Phys. Rev. A **91**, 043634 (2015).
[79] F. Deuretzbacher, D. Becker, and L. Santos, Phys. Rev. A **94**, 023606 (2016).
[80] A. G. Volosniev, H.-W. Hammer, and N. T. Zinner, Phys. Rev. B **93**, 094414 (2016).
[81] V. I. Yudson, Sov. Phys. JETP **61**, 1043 (1985); Phys. Lett. A **129**, 17 (1988).
[82] B. Fang, P. Vignolo, M. Gattobigio, C. Miniatura, and A. Minguzzi, Phys. Rev. A **84**, 023626 (2011).
[83] J. Decamp, J. Jünemann, M. Albert, M. Rizzi, A. Minguzzi, and P. Vignolo, New J. Phys. **19**, 125001 (2017).