Kähler Geometry of Black Holes and Gravitational Instantons

Steffen Aksteiner* and Bernardo Araneda†

Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, D-14476 Potsdam, Germany

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We obtain a closed formula for the Kähler potential of a broad class of four-dimensional Lorentzian or Euclidean conformal "Kähler" geometries, including the Plebański-Demiański class and various gravitational instantons such as Fubini-Study and Chen-Teo. We show that the Kähler potentials of Schwarzschild and Kerr are related by a Newman-Janis shift. Our method also shows that a class of supergravity black holes, including the Kerr-Sen spacetime, is Hermitian. We finally show that the integrability conditions of complex structures lead naturally to the Weyl double copy.

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Introduction.—Complex methods as a tool to investigate spacetime structure in general relativity (GR) have a long and fruitful history of remarkable developments. Profound constructions, pioneered by Penrose, Newman, Plebański, Robinson, and Trautman [1–6] among others, include twistor theory and heavenly structures, but there are also simple yet intriguing results such as the "Newman-Janis shift" relating special solutions via complex coordinate transformations.

An important insight regarding complex structures in GR is provided by Flaherty [7], who showed that type D vacuum and Einstein-Maxwell spacetimes possess an analog of the Hermitian structures of Riemannian geometry. In Lorentz signature, a Hermitian structure must necessarily be complex-valued, so its integrability properties are more subtle than in the Euclidean case. Flaherty gave a comprehensive analysis of such properties [8,9], and he found that the above classes of type D spacetimes are not only Hermitian but also satisfy the Lorentzian analog of the conformal Kähler condition.

In Riemannian geometry, Kähler metrics are encoded in "generating functions" or scalar Kähler potentials. An analogous feature in GR occurs in perturbation theory, where perturbative fields are generated by scalar Debye potentials. These potentials are instrumental for modern studies of black hole stability and gravitational wave physics (e.g., [10–14]). The increasing interest in nonperturbative structures for gravitational wave science [15], together with the importance of scalar potentials for perturbation theory, motivate the question of whether there are "Debye potentials" for exact, astrophysically relevant solutions of GR. Moreover, the recently discovered applications of the Newman-Janis shift [16,17] suggest that complex structures in GR may play an important role in the understanding of such nonperturbative structures.

Motivated by the above considerations, in this Letter we develop a method to find the Kähler potentials of a broad class of geometries, including black holes and gravitational instantons, and we show intimate connections of this approach with other theoretical structures of modern interest for gravitational wave physics such as the Newman-Janis shift and the double copy relation between gauge and gravity theories.

As an example, consider the Kerr metric g with parameters M, a for mass and angular momentum per mass, respectively. In Boyer-Lindquist coordinates (t, r, θ, ϕ), the metric is block diagonal in $(dt, d\phi)$ and $(dr, d\theta)$, with components in the first block given by

$$g_{tt} = (\Delta - a^2\sin^2\theta)\Sigma^{-1},$$
$$g_{t\phi} = -a[\Delta - (r^2 + a^2)]\Sigma^{-1}\sin^2\theta,$$
$$g_{\phi\phi} = [a^2\Delta\sin^2\theta - (r^2 + a^2)^2]\Sigma^{-1}\sin^2\theta,$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$. Following Flaherty [8], one can find four complex scalar fields, $(z^0, z^1, \tilde{z}^0, \tilde{z}^1)$, defined by

$$dz^0 = dt - (a^2 + r^2)\Delta^{-1}dr - ia\sin\theta d\theta,$$
$$dz^1 = d\phi - a\Delta^{-1}dr - i\csc\theta d\theta,$$
$$d\tilde{z}^0 = dt + (a^2 + r^2)\Delta^{-1}dr + ia\sin\theta d\theta,$$
$$d\tilde{z}^1 = d\phi + a\Delta^{-1}dr + i\csc\theta d\theta$$

such that the Kerr metric is

$$g = g_{tt}dz^0d\tilde{z}^0 + g_{t\phi}(dz^0d\tilde{z}^1 + dz^1d\tilde{z}^0) + g_{\phi\phi}dz^1d\tilde{z}^1.$$

In addition, letting $\Omega^{-2} = (r - ia\cos\theta)^2$, there must exist a scalar K such that $g_{tt} = \Omega^{-2}\partial_{z^0}\partial_{\tilde{z}^0}K$, $g_{t\phi} = \Omega^{-2}\partial_{z^0}\partial_{\tilde{z}^1}K$, $g_{\phi\phi} = \Omega^{-2}\partial_{z^1}\partial_{\tilde{z}^1}K$. However, expressions for K do not seem to have been obtained in the literature.

The method developed in this Letter computes the generating function and Kähler potential of Kerr to be

$$K = 4\int r\Delta^{-1}dr + 4\log\sin\theta.$$

As a function $K = K(z, \tilde{z})$, the potential fully generates the spacetime geometry. Moreover, using Kähler transformations, we shall show that the Kähler potentials for Kerr and Schwarzschild are simply related by a Newman-Janis shift. More generally, the geometries studied in this Letter include the general Plebański-Demiański class [18] as well as the Chen-Teo family [19] of gravitational instantons. Our method also allows us to prove that a general class of supergravity black holes [20], including the Kerr-Sen spacetime [21], has a Hermitian (not conformal Kähler) structure.

In addition, we shall show that the integrability of complex structures leads to the existence of special scalar and massless free fields associated to the geometry, that can be combined to give a unified geometric description of the "Weyl double copy" [22–24]. Our results contain not only the type D and N double copies, but also provide new examples of this relation for both vacuum and nonvacuum geometries, including, e.g., the general Einstein-Maxwell Plebański-Demiański class and the Fubini-Study and Chen-Teo instantons.

Importantly, we shall not impose any field equations: the conformal Kähler property of the geometries we study does not depend on a particular field theory. This means that the conformal factor does not, in principle, play a role in our construction, but we shall nevertheless include it since it arises naturally in GR, where the Einstein and Kähler metrics are conformally related.

Complexified Kähler geometry.—Given a four-dimensional complex geometry (M, g), we define an almost-Hermitian structure [25] as a (1,1) tensor field J such that $J^2 = -\mathbb{I}$ and $g(J\cdot, J\cdot) = g(\cdot,\cdot)$. The tangent bundle decomposes as $TM = T^+ \oplus T^-$, where $T^\pm$ corresponds to vectors with eigenvalue $\pm$i under J. We say that the almost-Hermitian structure is integrable, and is thus a Hermitian structure, if $[T^\pm, T^\pm] \subset T^\pm$ (for *both* signs), where $[\cdot,\cdot]$ is the Lie bracket of vector fields. One can show that a Hermitian structure implies that there are four complex scalars $(z^i, \tilde{z}^i)$ such that

$$g = g_{i\tilde{j}}dz^i d\tilde{z}^j, \tag{1}$$

where $g_{i\tilde{j}} = g(\partial_{z^i}, \partial_{\tilde{z}^j})$, with $i = 0,\, 1$, $\tilde{j} = \tilde{0}, \tilde{1}$.

The fundamental 2-form is defined by $\kappa(\cdot,\cdot) \coloneqq g(J\cdot,\cdot)$. We say that a Hermitian geometry is Kähler if $d\kappa = 0$, and conformal Kähler if there is a scalar field Ω^2 such that $d\hat{\kappa} = 0$, where $\hat{\kappa} = \Omega^2\kappa$. By the complex version of the Poincaré Lemma, if $d\hat{\kappa} = 0$ then there exists, locally, a complex scalar K such that

$$\hat{g}_{i\tilde{j}} = \partial_{z^i}\partial_{\tilde{z}^j}K, \tag{2}$$

where $\hat{g}_{i\tilde{j}} = \Omega^2 g_{i\tilde{j}}$, cf. [[8], Theorem IX.8]. We say that K is a Kähler potential. It is not unique: one has the freedom to perform "Kähler transformations":

$$K \to K + F(z^i) + \tilde{F}(\tilde{z}^i). \tag{3}$$

The Kähler potential can be found by integrating Eq. (2). Define $p_j \coloneqq \partial K/\partial\tilde{z}^j$, then $p_j = \int \hat{g}_{i\tilde{j}}dz^i$. Integrating once again, the potential is $K = \int p_i d\tilde{z}^i$.

In this Letter, we shall study geometries whose metric has the block-diagonal form

$$g = a_{ij}d\sigma^i d\sigma^j + b_{IJ}dx^I dx^J \tag{4}$$

for some coordinates $\sigma^i = (\tau, \phi)$ and $x^I = (x, y)$, and known functions a_{ij}, b_{IJ}. Introduce an orthonormal coframe $e^1 = c_i^1 d\sigma^i$, $e^2 = c_I^2 dx^I$, $e^3 = c_I^3 dx^I$, $e^4 = c_i^4 d\sigma^i$, for some functions $c_i^1, c_I^2, c_I^3, c_i^4$; such that $g = e^1 \otimes e^1 + \cdots + e^4 \otimes e^4$. Define now a null coframe by

$$\ell = \frac{1}{\sqrt{2}}(e^1 + ie^2), \qquad n = \frac{1}{\sqrt{2}}(e^1 - ie^2),$$
$$m = \frac{1}{\sqrt{2}}(e^3 + ie^4), \qquad \tilde{m} = \frac{1}{\sqrt{2}}(-e^3 + ie^4). \tag{5}$$

The metric is $g = 2(\ell \odot n - m \odot \tilde{m})$. We shall consider almost-Hermitian structures whose fundamental 2-forms are $\kappa_\pm = \mathrm{i}(\ell \wedge n \pm m \wedge \tilde{m})$. For concreteness, let us focus on $\kappa_- \equiv \kappa$.

Let $V^i = (-\partial_\phi, \partial_\tau)$. We define the 1-forms

$$\omega^i \coloneqq \mu V^i \lrcorner (\ell \wedge m), \qquad \tilde{\omega}^i \coloneqq \tilde{\mu} V^i \lrcorner (n \wedge \tilde{m}), \tag{6}$$

where $\mu^{-1} = (\ell \wedge m)(\partial_\tau, \partial_\phi)$ and $\tilde{\mu}^{-1} = (n \wedge \tilde{m})(\partial_\tau, \partial_\phi)$. A calculation shows that

$$\omega^i = d\sigma^i + E_I^i dx^I, \qquad \tilde{\omega}^i = d\sigma^i - E_I^i dx^I, \tag{7}$$

for some functions $E_I^i = \omega^i(\partial_I)$, where $\partial_I = \partial/\partial x^I$. In addition, the metric and fundamental 2-form are

$$g = g_{i\tilde{j}}\omega^i \odot \tilde{\omega}^j, \qquad \kappa = \frac{i}{2}g_{i\tilde{j}}\omega^i \wedge \tilde{\omega}^j \tag{8}$$

where $g_{i\tilde{j}} = g(\partial_{\sigma^i}, \partial_{\sigma^j})$. Note that this implies that $g_{0\tilde{0}} = g_{\tau\tau}$, $g_{0\tilde{1}} = g_{1\tilde{0}} = g_{\tau\phi}$, $g_{1\tilde{1}} = g_{\phi\phi}$.

The almost-Hermitian structure is integrable if and only if $d\omega^i = 0 = d\tilde{\omega}^i$: if this is satisfied, then there will be (locally) $z^i, \tilde{z}^i$ such that $\omega^i = dz^i$ and $\tilde{\omega}^i = d\tilde{z}^i$, and from the first equation in Eq. (8) we see that the metric, Eq. (4), will have the Hermitian expression, Eq. (1). Using Eq. (7), this integrability condition has a simple form: $d\omega^i = 0 = d\tilde{\omega}^i$ if and only if

$$E^i_I = E^i_I(x^J) \quad \text{and} \quad \partial_{[I}E^i_{J]} = 0. \tag{9}$$

The second equation implies that, locally, there are functions $\psi^0(x^I), \psi^1(x^I)$ such that $E^i_I = \partial_I \psi^i$. The $(z^i, \tilde{z}^i)$ coordinates will then be given by

$$z^i = \sigma^i + \psi^i, \qquad \tilde{z}^i = \sigma^i - \psi^i. \tag{10}$$

The associated vector fields are $\partial_{z^i} = \frac{1}{2}(\partial_{\sigma^i} + \partial_{\psi^i})$, $\partial_{\tilde{z}^i} = \frac{1}{2}(\partial_{\sigma^i} - \partial_{\psi^i})$. In terms of x^I, we have $\partial_{\psi^i} = E^I_i \partial_I$, where E^I_i is the inverse of E^i_I (thought of as a 2×2 matrix).

We shall now assume that $\partial_{\sigma^1} = \partial_\tau$ and $\partial_{\sigma^2} = \partial_\phi$ are Killing vectors. This includes all of the examples studied in this Letter. Using $\kappa = -i g_{i\tilde{j}} d\sigma^i \wedge d\psi^j$ and $\hat{g}_{i\tilde{j}} \coloneqq \Omega^2 g_{i\tilde{j}}$, a short calculation shows that the conformal Kähler condition $d(\Omega^2\kappa) = 0$ is

$$\partial_{\psi^0}\hat{g}_{\tau\phi} - \partial_{\psi^1}\hat{g}_{\tau\tau} = 0 = \partial_{\psi^0}\hat{g}_{\phi\phi} - \partial_{\psi^1}\hat{g}_{\tau\phi}. \tag{11}$$

Assuming the above conditions, the formula $K = \int p_i d\tilde{z}^i$ for the Kähler potential can be rewritten as follows. From Cartan's formula $\pounds_v \hat{\kappa} = d(v \lrcorner \hat{\kappa}) + v \lrcorner d\hat{\kappa}$, we deduce that the Killing fields have Hamiltonians, i.e., functions H_0, H_1 such that $dH_i = \partial_{\sigma^i} \lrcorner \hat{\kappa} = -i\hat{g}_{i\tilde{j}} d\psi^j$, where the second equality follows from the expression of $\hat{\kappa}$ in terms of σ^i, ψ^i. Choosing K to be independent of σ^i, we get

$$K = -4i \int H_i d\psi^i. \tag{12}$$

The integration in ψ^i can be replaced by an integration in x^I by using $d\psi^i = E^i_I dx^I$.

To recover real metrics with different signatures, we impose reality conditions on the null coframe $(\ell, n, m, \tilde{m})$. Euclidean signature $(++++)$ corresponds to requiring $n = \overline{\ell}$ and $\tilde{m} = -\bar{m}$. The functions E^i_I in Eq. (7) are then purely imaginary, so $\tilde{\omega}^i = \bar{\omega}^i$ and $\tilde{z}^i = \bar{z}^i$. Lorentzian signature $(+---)$ corresponds to ℓ, n real and $\tilde{m} = \bar{m}$. The functions E^i_I in Eq. (7) are generally complex, so z^i and $\tilde{z}^i$ in Eq. (10) are not complex conjugates.

Black holes and instantons.—Diagonal metrics: Consider the special case of Eq. (4) where $g = g_{\tau\tau}d\tau^2 + g_{\phi\phi}d\phi^2 + g_{xx}dx^2 + g_{yy}dy^2$. We choose the frame such that the functions E^i_I in Eq. (7) are $E^\tau_x = i\sqrt{g_{xx}/g_{\tau\tau}}$, $E^\tau_y = 0$, $E^\phi_x = 0$, $E^\phi_y = -i\sqrt{g_{yy}/g_{\phi\phi}}$. The Hermitian condition is equivalent to $\partial_y(g_{xx}/g_{\tau\tau}) = 0$, $\partial_x(g_{yy}/g_{\phi\phi}) = 0$, and the conformal Kähler condition is $\partial_x(\Omega^2 g_{\tau\tau}) = 0$, $\partial_y(\Omega^2 g_{\phi\phi}) = 0$.

A simple example is an arbitrary static, spherically symmetric spacetime $g = f(r)d\tau^2 - h(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Using $\Omega^2 = 1/r^2$, and *regardless* of the form of $f(r)$, $h(r)$, the geometry is conformal Kähler. This includes not only the well-known spherical black hole spacetimes but also solutions from the Einstein-scalar field system such as the Janis-Newman-Winicour wormhole [26]. In the special case $h = f^{-1}$, the Kähler potential is given by $K = 4\{\int [rf(r)]^{-1} dr + \log\sin\theta\}$.

The Plebański-Demiański class: Consider the metric Eq. (4) with

$$\begin{aligned} &g_{\tau\tau} = [\Delta_r - a^2\Delta_x]/(\Pi\Sigma), \\ &g_{\tau\phi} = a[(r^2 + a^2)\Delta_x - (1 - x^2)\Delta_r]/(\Pi\Sigma), \\ &g_{\phi\phi} = [a^2(1 - x^2)^2\Delta_r - (r^2 + a^2)^2\Delta_x]/(\Pi\Sigma), \\ &g_{xx} = -\Sigma/(\Pi\Delta_x), \quad g_{xy} = 0, \quad g_{yy} = -\Sigma/(\Pi\Delta_r), \end{aligned} \tag{13}$$

where $y \equiv r$, $\Sigma = r^2 + a^2x^2$, $a = \text{const}$, and $\Pi = \Pi(r, x)$, $\Delta_x = \Delta_x(x)$, $\Delta_r = \Delta_r(r)$ are arbitrary functions of their arguments. We find that, *regardless* of the specific form of Π, Δ_r, Δ_x, the geometry is conformal Kähler, with complex coordinates

$$\begin{aligned} &z^0 = \tau - (r^* - iax^*), \qquad z^1 = \phi - (ar^\sharp - ix^\sharp), \\ &\tilde{z}^0 = \tau + (r^* - iax^*), \qquad \tilde{z}^1 = \phi + (ar^\sharp - ix^\sharp), \end{aligned} \tag{14}$$

where $r^*, x^*, r^\sharp, x^\sharp$ are defined by

$$\begin{aligned} &dr^* = (r^2 + a^2)\Delta_r^{-1}dr, \qquad dx^* = (1 - x^2)\Delta_x^{-1}dx, \\ &dr^\sharp = \Delta_r^{-1}dr, \qquad dx^\sharp = \Delta_x^{-1}dx, \end{aligned} \tag{15}$$

and the conformal factor is

$$\Omega^2 = \Pi/(r - iax)^2. \tag{16}$$

The Kähler form $\hat{\kappa} = \Omega^2\kappa$ is given by

$$\begin{aligned} \hat{\kappa} = \frac{i}{(r - iax)^2}\{&-d\phi \wedge [a(1 - x^2)dr - i(r^2 + a^2)dx] \\ &+ d\tau \wedge (dr - iadx)\}. \end{aligned} \tag{17}$$

Notice that this is independent of Δ_r, Δ_x. The Hamiltonians are $H_0 = -i/(r - iax)$ and $H_1 = i(a + irx)/(r - iax)$, hence, using Eq. (12), we find that the Kähler potential is

$$K = 4\left[\int \frac{r}{\Delta_r}dr - \int \frac{x}{\Delta_x}dx\right]. \tag{18}$$

We stress that the existence of this potential is independent of the explicit form of the functions Δ_r, Δ_x.

The Plebański-Demiański family [18,27,28] is Eq. (13) with $\Pi = (1-\alpha r x)^2$ and

$$\Delta_x = 1 + \frac{2N}{a}x - x^2 + 2\alpha M x^3 - \left[\frac{\lambda}{3}a^2 + \alpha^2(Q^2+a^2)\right]x^4,$$
$$\Delta_r = Q^2 + a^2 - 2Mr + r^2 - \frac{2\alpha N}{a}r^3 - \left(\alpha^2 + \frac{1}{3}\lambda\right)r^4,$$

where $\alpha = \text{const}$, $Q^2 = q_e^2 + q_m^2$, and λ, q_e, q_m correspond, respectively, to cosmological constant and electric and magnetic charges. The rest of the parameters can be related to mass, angular momentum, acceleration, and Newman-Unti-Tamburino charge, cf. [27] for details. This is the general type D solution (assuming non-null orbits of the isometry group) of the Einstein equations with an aligned electromagnetic field.

We note that, for the case $Q = 0$, the transformation $(r, M) \leftrightarrow \pm(iax, iN)$ leaves the Kähler potential and the metric invariant, and the coordinates, Eq. (14), change according to $z^i \leftrightarrow \tilde{z}^i$ for $+$ and are invariant for $-$. A detailed analysis of this and other dualities will be given in a separate work [29].

Newman-Janis shifts: For the Schwarzschild and Kerr spacetimes (putting $x = \cos\theta$), we find the Kähler potentials to be

$$K_{\text{schw}} = 4[\log|r-2M| + \log\sin\theta], \tag{19a}$$

$$K_{\text{kerr}} = 4\left[\frac{1}{2}\log|r^2 - 2Mr + a^2| + \log\sin\theta - \frac{M}{\sqrt{M^2-a^2}} f\left(\frac{r-M}{\sqrt{M^2-a^2}}\right)\right], \tag{19b}$$

where $f = \tanh^{-1}$ if r is in between the two roots of Δ_r and $f = \coth^{-1}$ outside the roots and we assume $M^2 \neq a^2$.

Using Eq. (14) and Kähler transformations, Eq. (3), a calculation shows that Eqs. (19a) and (19b) are equivalent to

$$K_{\text{schw}} = 4\left[-\frac{r}{2M} + \log\sin\theta\right], \tag{20a}$$

$$K_{\text{kerr}} = 4\left[-\frac{(r - ia\cos\theta)}{2M} + \log\sin\theta\right], \tag{20b}$$

where we assume $M \neq 0$. Thus, the Kähler potentials are related by a Newman-Janis shift $r \to r - ia\cos\theta$ [5], although it is not at all obvious from Eq. (19).

For $M = 0$, which corresponds (locally) to flat spacetime [30], we can see the Newman-Janis shift as follows. Consider complexified Minkowski space, in complexified spherical coordinates (r_c, θ_c, ϕ_c). In terms of complexified inertial coordinates (t_c, x_c, y_c, z_c), we have the usual relations $x_c^2 + y_c^2 = r_c^2\sin^2\theta_c$, $z_c = r_c\cos\theta_c$. The Kähler potential can be shown to be $K = 4\log(r_c\sin\theta_c)$. Consider first the real slice $\mathbb{M}$ given by $\{t_c = t, x_c = x, y_c = y, z_c = z\}$, where t, x, y, z are real. Then (r_c, θ_c, ϕ_c) become ordinary real spherical coordinates, and the Kähler potential is

$$K|_{\mathbb{M}} = 4\log(r\sin\theta). \tag{21}$$

Now consider a different real slice $\mathbb{M}'$ given by a Newman-Janis shift [31]: $\{t_c = t, x_c = x, y_c = y, z_c = z - ia\}$, where a is a real constant. Choosing the complex radius to be $r_c = r - ia\cos\theta$, a calculation gives $x^2 + y^2 = (r^2 + a^2)\sin^2\theta$, so

$$K|_{\mathbb{M}'} = 4\left[\frac{1}{2}\log(r^2 + a^2) + \log\sin\theta\right]. \tag{22}$$

Equations (21) and (22) correspond, respectively, to the $M \to 0$ limits in Eqs. (19a) and (19b).

Supergravity black holes: Consider the metric, Eq. (4), with

$$g_{\tau\tau} = (R-U)/W, \quad g_{\tau\phi} = (RW_u + UW_r)/W,$$
$$g_{\phi\phi} = (RW_u^2 - UW_r^2)/W,$$
$$g_{xx} = -W/R, \quad g_{xy} = 0, \quad g_{yy} = -W/U, \tag{23}$$

where $x \equiv r$, $y \equiv u$, (R, W_r) and (U, W_u) are arbitrary functions of r and u, respectively, and $W = a(W_r + W_u)$, with real constant a. The metric, Eq. (23), includes a general class of black hole solutions of supergravity [20], in particular the Kerr-Sen black hole [21].

Using the almost-Hermitian structure associated to the frame given in [[20], Eq. (4.79)], our method shows that the geometry, Eq. (23), is Hermitian, with complex coordinates, Eq. (10), where $\psi^0 = r^* + iu^*$, $\psi^1 = r^\sharp - iu^\sharp$, and $dr^* = a(W_r/R)dr$, $du^* = a(W_u/U)du$, $dr^\sharp = (a/R)dr$, $du^\sharp = (a/U)du$. However, the conformal Kähler condition, Eq. (11), does not hold for this Hermitian structure.

Gravitational instantons: We now specialize to Euclidean signature. Consider first the metric, Eq. (4), with

$$g_{\tau\tau} = \frac{a^2x^2}{4(1+x^2)^2}, \quad g_{\phi\phi} = g_{\tau\tau}(1 + x^2\sin^2 y), \quad g_{xy} = 0,$$
$$g_{\tau\phi} = g_{\tau\tau}\cos y, \quad g_{xx} = \frac{4}{a^2x^2}g_{\tau\tau}, \quad g_{yy} = (1+x^2)g_{\tau\tau},$$

where a is an arbitrary constant. Using "$(\mp)$" to denote quantities associated to $\kappa_\mp$, one can choose frames such that the functions in Eq. (7) are $E^\tau_{(\mp)x} = 2i/(ax)$, $E^\tau_{(\mp)y} = \pm i\cot y$, $E^\phi_{(\mp)x} = 0$, $E^\phi_{(\mp)y} = \mp i\csc y$. Then a calculation shows that the geometry is conformal Kähler with regard to both sides, with $\Omega^2_\mp = [(1+x^2)/x^2]^{1\mp 1/a}$.

For $a = \pm 1$, one side becomes Kähler and the metric is Einstein: this is the Fubini-Study metric in $\mathbb{CP}^2$.

A new family of gravitational instantons was discovered by Chen and Teo [19]. This is a toric, Ricci-flat geometry of the form, Eq. (4), that depends on seven parameters $k, \nu, a_0 \ldots a_4$. The nontrivial metric components are $g_{\tau\tau}, g_{\tau\phi}, g_{\phi\phi}, g_{xx}, g_{yy}$, and depend on functions F, G, H, X, Y given explicitly in [[19], Eq. (2.1)]. The family contains other known instantons such as Eguchi-Hanson and Euclidean Plebański-Demiański. It was recently shown [32] that the Chen-Teo geometry is one-sided type D, and thus (from Ricci-flatness) conformal Kähler, with $\Omega^2 = (x-y)^2/(\nu x + y)^2$. See also [33]. Our method computes the complex coordinates to be $dz^0 = d\tau + d\psi^0$, $dz^1 = d\phi + d\psi^1$ [cf. Eq. (10)], where $d\psi^0 = E^\tau_x dx + E^\tau_y dy$, $d\psi^1 = E^\phi_x dx + E^\phi_y dy$, and

$$E^\tau_x = i\frac{\sqrt{k}}{F}\left[\frac{Gx}{X} + \frac{Hy}{(x-y)}\right], \qquad E^\phi_x = \frac{-i\sqrt{k}x}{X},$$
$$E^\tau_y = -i\frac{\sqrt{k}}{F}\left[\frac{Gy}{Y} + \frac{Hx}{(x-y)}\right], \qquad E^\phi_y = \frac{i\sqrt{k}y}{Y}. \tag{24}$$

The Hermitian condition, Eq. (9), reduces to $\partial_y E^\tau_x - \partial_x E^\tau_y = 0$, which provides an interpretation for Eq. (3.49) in [32]. The Hamiltonians are

$$H_0 = \frac{\sqrt{k}(x-y)}{(1+\nu)(\nu x + y)}, \qquad H_1 = \frac{\sqrt{k} f(x,y)}{(\nu x + y)(x-y)},$$

where $f(x,y) = (\nu - 1)(a_0 + a_4 x^2 y^2) - a_2 x[\nu(x-2y) + y] + (a_1 + a_3 xy)(\nu x - y)$. The Kähler potential can now be computed using Eq. (12):

$$K = \frac{4k}{1+\nu}\left[4(1-\nu)\log(x-y) - \int \frac{h_1}{X} dx + \int \frac{h_2}{Y} dy\right],$$

where $h_1(x) = a_1 + a_2(1-2\nu)x + a_3(2-\nu)x^2 + 2a_4(1-\nu)x^3$ and $h_2(y) = a_1\nu - a_2 y - a_3(1-2\nu)y^2 - 2a_4(1-\nu)y^3$.

Double copy structures.—In string theory, the Kawai-Lewellen-Tye relations [34] imply that gravitational amplitudes are closely related to the square of Yang-Mills amplitudes. The extension of these relations to field theory is known as the "double copy." At the classical level, a recent formulation is the "curved Weyl double copy" [23], which asserts that for some vacuum gravity solutions, the Weyl curvature spinor is $\Psi_{ABCD} = (1/S)\Phi_{(AB}\Phi_{CD)}$ for some scalar field S ("zeroth copy") and symmetric spinor field Φ_{AB} ("single copy"), where S satisfies a wave equation and Φ_{AB} satisfies Maxwell's equations. (We refer to [35,36] for background on the 2-spinor formalism.) The relation has been proven for vacuum type D and type N spacetimes [23,24]. We shall now show that the integrability conditions of complex structures give automatically this sort of relations among scalar, Maxwell, and gravitational fields.

Consider a conformal Kähler geometry, with conformal factor Ω, Kähler form $\hat{\kappa}_{ab} = \varphi_{AB}\epsilon_{A'B'}$, Weyl spinor Ψ_{ABCD}, and Ricci spinor $\Phi_{ABA'B'}$. Note that, as $\hat{\kappa}_{ab}$ is a Kähler form, it must necessarily be (anti-)self-dual (cf. [[37], Theorem 3.1]); we choose anti-self-dual for concreteness. Then one can show the following identities:

$$(\Box + 2\Psi_2 + R/6)\Omega = 0, \tag{25a}$$

$$d\hat{\kappa} = 0 = d^*\hat{\kappa}, \tag{25b}$$

$$\Psi_{ABCD} = \Psi_2 \Omega^{-4} \varphi_{(AB}\varphi_{CD)}, \tag{25c}$$

$$\Phi_{ABA'B'} = \Phi_{11}|\Omega|^{-4}\varphi_{AB}\bar{\varphi}_{A'B'}. \tag{25d}$$

Equation (25a) follows by first noticing that the Lee form f_a (defined by $d\kappa = -2f \wedge \kappa$) is $f_a = \partial_a \log\Omega$, then taking a divergence and using the identity $\nabla^a f_a + f^a f_a = -(2\Psi_2 + R/6)$ (which can be proved using the Newman-Penrose formalism). Equation (25b) follows from the conformal Kähler condition. Finally, Eqs. (25c) and (25d) follow from the integrability conditions of a conformal Kähler structure, where for Eq. (25d) we assume Lorentz signature [and that the Ricci tensor satisfies $R(J\cdot, J\cdot) = R(\cdot,\cdot)$].

From Eq. (25) we see that any conformal Kähler geometry combines scalar Ω, Maxwell $\hat{\kappa}$, and gravitational fields in a double copylike structure, without assuming any field equations. For Einstein manifolds ($\Phi_{ABA'B'} = 0$), Bianchi identities imply $\Omega = \Psi_2^{1/3}$, so we recover the type D double copy [23] (extended to nontrivial cosmological constant). More generally, all of the conformal Kähler examples of the previous sections have the structure, Eq. (25), so they represent double copy relations. New examples include the Fubini-Study and Chen-Teo instantons, but also the whole (nonvacuum) Plebański-Demiański class.

Furthermore, in the Plebański-Demiański case, the fields Ω and $\hat{\kappa}$ solve flat spacetime equations. More precisely, we see from Eq. (16) that Ω is independent of $\{M, N, q_e, q_m, \lambda\}$, and since the case in which these parameters vanish corresponds to Minkowski, we immediately get $\eta^{ab}\partial_a\partial_b\Omega = 0$. In addition, from Eq. (17) we see that the Kähler form $\hat{\kappa}$ depends only on a, so it must solve Maxwell's equations in Minkowski.

The type N double copy [24] is not included in the above construction, but it also arises from the integrability of complex structures. First, consider a Petrov type II spacetime whose repeated principal spinor o_A satisfies

$$o^A o^B \nabla_{AA'} o_B = 0. \tag{26}$$

Equation (26) is the condition for "half-integrability" of a complex structure (see [[38], Section 2.4]). One can show [[38], Prop. 2.6] that there is a scalar Ω such that the Lee form satisfies $o^A f_{AA'} = o^A \nabla_{AA'} \log \Omega$. Applying $\iota^A \nabla_A^{A'}$ to this equation (where $o_A \iota^A = 1$), after some computations we again find that Ω satisfies Eq. (25a). Notice that Ω is not unique: we have the freedom to add $\Omega \to \Omega + \nu$, where ν is any function such that $o^A \nabla_{AA'} \nu = 0$.

In addition, from [[36], Lemma (7.3.15)], Eq. (26) implies that there are two complex scalars $z^i = (z^0, z^1)$ such that $dz^i = o_A Z^i_{A'} dx^{AA'}$, for some spinors $Z^i_{A'}$. It follows that the 2-form $dz^0 \wedge dz^1$ is anti-self-dual and closed, so it is a Maxwell field. Note that $F(z^0, z^1) dz^0 \wedge dz^1$ is also a Maxwell field for any function $F(z^0, z^1)$.

Finally, the conditions on o_A imply that there is a scalar λ such that $o^A \nabla_{AA'} (\lambda o_B) = 0$. This leads to $\nabla^{AA'} (\lambda \Omega o_A) = 0$, which in turn implies that $\varphi_{A_1 \ldots A_n} = \Omega \lambda^n o_{A_1} \ldots . o_{A_n}$ is a massless free field: $\nabla^{A_1 A'_1} \varphi_{A_1 \ldots A_n} = 0$. For $n = 2$ and $n = 4$, we get the spin 1 and 2 fields $\varphi_{AB} = \varphi_2 o_A o_B$ and $\psi_{ABCD} = \psi_4 o_A o_B o_C o_D$, with $\varphi_2 = \Omega \lambda^2$ and $\psi_4 = \Omega \lambda^4$. These are related by

$$\psi_4 = \frac{1}{\Omega} (\varphi_2)^2. \tag{27}$$

In the special case in which the spacetime is type N, ψ_{ABCD} can be chosen to be the Weyl curvature spinor, and Eq. (27) is the type N double copy relation [[24], Eq. (6)]. The nonuniqueness noticed in [24] is due to the freedom to include the functions $\nu(z^0, z^1)$, $F(z^0, z^1)$ mentioned before.

Discussion.—A general expression for the Kähler potential K for the class of conformal Kähler (Lorentzian or Euclidean) geometries of the form, Eq. (4), with two Killing fields is given by Eq. (12). This includes the Plebański-Demiański and Chen-Teo families. The potential K generates not only the metric, but also the Maxwell field $\hat{\kappa}$. Notice that this electromagnetic field is exactly the Coulomb field of the Schwarzschild solution, or the $\sqrt{\text{Kerr}}$ or magic field of the Kerr solution [16,39].

In $(z, \tilde{z})$ coordinates, K is not necessarily expressible in terms of elementary functions. For example, while a potential for Minkowski is

$$K(z, \tilde{z}) = 4 \log \left[\frac{\tilde{z}^0 - z^0}{1 + e^{\mathrm{i}(z^1 - \tilde{z}^1)}} \right],$$

in the Schwarzschild case, Eq. (20a), r and $(z, \tilde{z})$ are related by $r + 2M \log(r - 2M) = (\tilde{z}^0 - z^0)/2$, which can be solved in terms of the Lambert W function.

Nevertheless, since the Kähler potential contains (locally and up to Kähler transformations) all the information of the geometry, it represents a fully nonlinear version of the Debye potentials of perturbation theory in GR. As such, it is of intrinsic interest for the investigation of nonperturbative results for gravitational wave physics, further supported by the intriguing manifestation of the Newman-Janis shift in K found in this Letter, and by the fact that, as we showed, Kähler and complex geometry in GR contain naturally the known instances of the Weyl double copy.

The general framework and results obtained in this Letter motivate applications to a variety of exciting problems in different areas of interest. In mathematical GR, potential applications include the analysis of waves on black hole spacetimes, analytic compactifications, and possible generalizations of the Chen-Teo instanton. In gravitational wave science, it would be interesting to make explicit connections to modern techniques used in scattering amplitudes and quantum field theory [15,40,41]. The relation between Kähler potentials and the Newman-Janis shift motivates further investigation into the geometric origin of this trick, together with connections with its interpretation as a generation of intrinsic spin [[8], Chapter X], see also [16,17]. A detailed description of dualities in the Plebánski-Demiański family will appear elsewhere [29].

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[*]Corresponding author.
steffen.aksteiner@aei.mpg.de
[†]Corresponding author.
bernardo.araneda@aei.mpg.de

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