Measurement Noise Susceptibility in Quantum Estimation

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Fisher information is a key notion in the whole field of quantum metrology. It allows for a direct quantification of the maximal achievable precision of the estimation of the parameters encoded in quantum states using the most general quantum measurement. It fails, however, to quantify the robustness of quantum estimation schemes against measurement imperfections, which are always present in any practical implementations. Here, we introduce a new concept of Fisher information measurement noise susceptibility that quantifies the potential loss of Fisher information due to small measurement disturbance. We derive an explicit formula for the quantity, and demonstrate its usefulness in the analysis of paradigmatic quantum estimation schemes, including interferometry and superresolution optical imaging.

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Introduction.—Noise, decoherence, and implementation imperfections are the main factors hindering the transfer of quantum enhanced technologies (e.g., quantum computing and communication) from proof-of-principle experiments to real-life applications [1]. These issues also affect the development of quantum metrology, whose goal is to utilize sophisticated properties of light and matter to enhance sensing instruments [2–5]. Quantum estimation theory [6,7] laid theoretical grounds for present-day quantum metrology—one of its greatest achievements is identification of protocols that perform optimally in the presence of noise [8–12].

One of the key elements affecting the precision of metrological protocols is the imperfect realization of the final measurement step, where information is being extracted from quantum sensors. In order to assess the effect of imperfect measurement implementation, the standard route is to characterize the type of noise present, e.g., detector dark counts, measurement output crosstalks, etc., and then analyze its impact on the relevant figures-of-merit [13]. A more systematic and general study of the effect of readout noise on the measurement precision is provided in Ref. [14].

Still, from a fundamental point of view, it would be much more advantageous to be able to determine the noise robustness of a given measurement scheme without specifying the actual form of the noise. A similar motivation lays behind measurement robustness considerations that can be found in the context of other quantum information tasks, see, e.g., [15,16], but have never been applied to quantum estimation theory. In this Letter, we focus on the most common figure of merit in quantum estimation theory—the Fisher information—and propose a quantity, Fisher information measurement noise susceptibility (FI MENOS), which characterizes the maximal relative decrease of FI due to small measurement disturbance of the most general type. This quantity allows us to obtain a deep insight into fundamental noise-robustness properties of different measurement schemes without assuming any particular noise form. We illustrate the fruitfulness of this approach by analyzing paradigmatic quantum enhanced metrological schemes including interferometry and superresolution imaging.

Quantum estimation theory preliminaries.—In a paradigmatic quantum estimation scenario, a continuous parameter θ is encoded in a state ρ_θ of a probe system with associated Hilbert space $\mathcal{H}_S$. In order to describe the process of extraction of information on the parameter θ from the state in full generality, one considers an external measuring device whose Hilbert space is $\mathcal{H}_M$. The device is initialized in a pure state $|0\rangle_M$ and the generalized measurement of ρ_θ consists of two stages: (i) interaction between S and M described by a unitary operation U_{SM} and (ii) projective measurement of the postevolution state of M, which returns an outcome $i \in \{1, 2, \ldots, K\}$ with a probability

$$p_\theta(i) = \mathrm{Tr}(\rho_\theta M_i), \quad M_i = {}_M\langle 0|U^\dagger_{SM}|i\rangle_M \langle i|U_{SM}|0\rangle_M \tag{1}$$

where M_i are effective measurement operators acting on S—note that scalar products in the above formula are partial, they act on subsystem M only, leaving part S intact. A set $\boldsymbol{M} = \{M_i\}_i$ is called a positive operator-valued measure (POVM), where M_i satisfy (i) $\sum_i M_i = \mathbb{1}$ and (ii) $M_i \geq 0$. The set of all POVMs will be denoted as $\mathcal{M}$, so we will write $\boldsymbol{M} \in \mathcal{M}$. Different choices of U_{SM} lead to different POVMs, the number of possible outcomes is $K = \dim \mathcal{H}_M$. Each POVM can be physically implemented with the help of an appropriate choice of $\mathcal{H}_M$ and U_{SM}. The projective measurement in a basis $|i\rangle$ of $\mathcal{H}_S$ corresponds to $\boldsymbol{M}$ with $M_i = |i\rangle\langle i|$.

When N copies of ρ_θ are measured independently with the same POVM $\boldsymbol{M}$ this leads to N i.i.d. random variables sampled from $p_\theta(i)$. According to the Cramér-Rao bound (CRB) [6,7], the mean squared error (MSE) of any (locally) unbiased estimator $\tilde{\theta}$, that estimates θ based on this data, will be lower bounded as

$$\Delta^2\tilde{\theta} \geq \frac{1}{NF_C}, \qquad F_C = \sum_i p_i l_i^2 \tag{2}$$

where F_C is the classical Fisher information (CFI), $p_i = p_\theta(i)$, and $l_i = \partial_\theta \log p_\theta(i)$ is the logarithmic derivative of $p_\theta(i)$. Intuitively, the CFI quantifies how sensitive $p_\theta(i)$ is to the change of θ—the larger l_i^2, the greater the CFI. The CRB is tight—it is always possible to find a locally unbiased estimator whose MSE saturates (2), and for $N \to \infty$ one can construct a globally unbiased CRB-saturating estimator [17].

For a fixed quantum state ρ_θ, the CFI depends only on the measurement $\boldsymbol{M}$, and in order to highlight this we will denote it as $F_C[\boldsymbol{M}]$. Combining (2) with (1) the explicit form of the CFI reads

$$F_C[\boldsymbol{M}] = \sum_i \mathrm{Tr}(\rho_\theta M_i) l_i^2, \qquad l_i = \frac{\mathrm{Tr}(\dot{\rho}_\theta M_i)}{\mathrm{Tr}(\rho_\theta M_i)}, \tag{3}$$

where the dot denotes the derivative over θ. It is natural to ask what the greatest possible CFI is for a given ρ_θ—the answer is given by the quantum Fisher information (QFI) [6,7], which is the maximum of the CFI over all POVMs $\boldsymbol{M}$, and can be computed as

$$F_Q = \max_{\boldsymbol{M}\in\mathcal{M}} F_C[\boldsymbol{M}] = \mathrm{Tr}(\rho_\theta \Lambda_\theta^2), \tag{4}$$

where Λ_θ is the symmetric logarithmic derivative matrix defined by the equation $\partial_\theta \rho_\theta = \frac{1}{2}(\rho_\theta\Lambda_\theta + \Lambda_\theta\rho_\theta)$. For a given ρ_θ, and arbitrary $\boldsymbol{M}$, the MSE of any locally unbiased estimator of θ is lower-bounded by the quantum Cramér-Rao bound (QCRB), which is similar to (2), but F_C is replaced with F_Q. The projective measurement on eigenstates of Λ_θ is always QCRB saturating (its CFI is equal to QFI), but sometimes there are many different QCRB sat. measurements, see Refs. [18] and [19] Section D for a detailed discussion.

Fisher information measurement noise susceptibility.—Let us assume, that due to a small disturbance, $\boldsymbol{M}$ changes to $\tilde{\boldsymbol{M}} = (1-\epsilon)\boldsymbol{M} + \epsilon\boldsymbol{N}$ (summation of two POVMs is done elementwise), which can be viewed as the replacement of desired POVM $\boldsymbol{M}$ with an unwanted one $\boldsymbol{N}$ with a probability $\epsilon \ll 1$. This type of noise may be caused by inaccurate initialization of a measuring device M in a mixed state $(1-\epsilon)|0\rangle\langle 0| + \epsilon\rho'_M$ instead of $|0\rangle\langle 0|$, or it may be the result of other small imperfections, such as signal losses, dark counts, crosstalks etc. (see Ref. [19] Sections A and B).

The measurement noise affects the CFI, the effect of which we quantify using

$$\chi[\boldsymbol{M},\boldsymbol{N}] = \lim_{\epsilon\to 0} \frac{F_C[\boldsymbol{M}] - F_C[(1-\epsilon)\boldsymbol{M} + \epsilon\boldsymbol{N}]}{\epsilon F_C[\boldsymbol{M}]}, \tag{5}$$

which can be understood as the relative *decrease* of the CFI under infinitesimally added noise $\boldsymbol{N}$—the effect of ϵ noise results in F_C in the CRB, Eq. (2), being replaced with $F_C[\boldsymbol{M}](1-\epsilon\chi[\boldsymbol{M},\boldsymbol{N}])$. After inserting (3) into (5), we obtain, after straightforward calculations,

$$\chi[\boldsymbol{M},\boldsymbol{N}] = 1 + F_C[\boldsymbol{M}]^{-1} G[\boldsymbol{N}], \tag{6}$$

where

$$G[\boldsymbol{N}] = \sum_i \mathrm{Tr}(A_i N_i), \qquad A_i = l_i^2\rho_\theta - 2l_i\dot{\rho}_\theta. \tag{7}$$

To get an intuition regarding this quantity, consider a simple example where $\boldsymbol{M} = (0, M_2, \ldots, M_K)$ and $\boldsymbol{N} = (\mathbb{1}, 0, \ldots, 0)$, so noise only activates a noninformative measurement outcome 1. Then, $G[\boldsymbol{N}] = 0$ and hence $\chi[\boldsymbol{M},\boldsymbol{N}] = 1$, which means that the relative decrease of the CFI is equal to the probability of obtaining a useless, noisy result. Clearly, the decrease of CFI will be more substantial, when the disturbance affects the statistics of informative outcomes, and noise cannot be separated from the signal easily. Our goal is to figure out, what is the maximal shrinkage rate of the CFI caused by an infinitesimal measurement noise described by an arbitrary POVM $\boldsymbol{N}$. The answer is given by a quantity

$$\chi[\boldsymbol{M}] = \max_{\boldsymbol{N}\in\mathcal{M}} \chi[\boldsymbol{M},\boldsymbol{N}], \tag{8}$$

which we call FI MENOS because it tells us how *susceptible* CFI is to small disturbances of the measurement. Note that the larger χ implies potentially stronger decrease of CFI as a result of measurement disturbance, so strictly speaking this is a *negative* susceptibility (cf. "menos" in Spanish). Notably, it does not depend on $\boldsymbol{N}$—this allows us to compare the robustness against noise of different measurements without invoking any specific noise model.

Explicit formula for FI MENOS.—We now present an explicit solution to the maximization problem from (8), which, according to (6), boils down to finding the maximum of $G[\boldsymbol{N}]$. Without loss of generality, we can relabel the elements of POVM $\boldsymbol{M}$ such that logarithmic derivatives satisfy $l_1 \leq l_2 \leq \ldots \leq l_K$. Let $\boldsymbol{N} = (N_1, \ldots, N_i, \ldots, N_K)$ be an arbitrary POVM while $\Gamma_i^1(\boldsymbol{N}) = (N_1 + N_i, \ldots, 0, \ldots, N_K)$ and $\Gamma_i^K(\boldsymbol{N}) = (N_1, \ldots, 0, \ldots, N_K + N_i)$ be two POVMs constructed from $\boldsymbol{N}$ with zeros at ith positions, $i \in \{2, \ldots, K-1\}$. Using (7), we obtain

$$G[\Gamma_i^1(\boldsymbol{N})] - G[\boldsymbol{N}] = f_i(l_1) - f_i(l_i), \tag{9}$$

$$G[\Gamma_i^K(\boldsymbol{N})] - G[\boldsymbol{N}] = f_i(l_k) - f_i(l_i), \tag{10}$$

where $f_i(x) = x^2\mathrm{Tr}(\rho_\theta N_i) - 2x\mathrm{Tr}(\dot{\rho}_\theta N_i)$ is a convex quadratic function, and therefore $f_i(l_1) \geq f_i(l_i) \vee f_i(l_K) \geq f_i(l_i)$, so from (9) and (10), $G[\Gamma_i^1(\boldsymbol{N})] \geq G[\boldsymbol{N}] \vee G[\Gamma_i^K(\boldsymbol{N})] \geq G[\boldsymbol{N}]$. Therefore, for each $\boldsymbol{N}$ and i, there is $j_i \in \{1, K\}$ such that $G[\Gamma_i^{j_i}(\boldsymbol{N})] \geq G[\boldsymbol{N}]$, so we can choose $i_2, \ldots, i_{K-1} \in \{1, K\}$ such that $\tilde{\boldsymbol{N}} = \Gamma_2^{j_2} \circ \ldots \circ \Gamma_{K-1}^{j_{K-1}}(\boldsymbol{N})$ satisfies $G[\tilde{\boldsymbol{N}}] \geq G[\boldsymbol{N}]$, and from the construction of $\tilde{\boldsymbol{N}}$, $\tilde{N}_i = 0$ for $i \in \{2, \ldots, K-1\}$. It means, that for arbitrary $\boldsymbol{N}$, it is possible to construct $\tilde{\boldsymbol{N}} = (\tilde{N}_1, 0, \ldots, 0, \mathbb{1} - \tilde{N}_1)$ satisfying $\chi[\boldsymbol{M}, \tilde{\boldsymbol{N}}] \geq \chi[\boldsymbol{M}, \boldsymbol{N}]$. Hence, the worst-case scenario noise will affect only the outcomes with the smallest and the largest logarithmic derivatives. This observation allows us to perform the maximization from (8) analytically, (see Ref. [19] Section C), and obtain the general expression for the FI MENOS:

$$\chi[\boldsymbol{M}] = 1 + \frac{1}{2F_C[\boldsymbol{M}]}(l_1^2 + l_K^2 + \|A_1 - A_K\|_1), \tag{11}$$

where $\|A\|_1 = \mathrm{Tr}\sqrt{AA^\dagger}$ is the trace norm. Notice, that χ depends on F_C, ρ_θ, $\dot{\rho}_\theta$, and the extremal logarithmic derivatives only. When an outcome i has a vanishing probability, $p_i \to 0$ but its contribution to the CFI, $p_i l_i^2$, remains finite and nonzero, then $l_i^2 \to \infty$, which implies that either l_1^2 or l_k^2 diverges, and hence, χ diverges as well according to (11). This reflects the fact, that the contribution to the CFI resulting from an outcome with a very low probability may be completely washed out by a very small measurement noise, and the chosen measurement is not likely to be practical.

When there are many measurements which lead to the same CFI, the FI MENOS may help to judge which one is more robust and hence more suitable for practical purposes—the one with lower χ. It is especially interesting to find the minimum of $\chi[M]$ over all QCRB sat. measurements,

$$\chi_Q = \min_{\{\boldsymbol{M} \in \mathcal{M}, F_C[\boldsymbol{M}] = F_Q\}} \chi[\boldsymbol{M}], \tag{12}$$

since the corresponding measurement $\boldsymbol{M}$ should be regarded as the most robust among the most informative measurements. This task is tractable thanks to the exact formula (11)—we demonstrate exemplary solutions to this problem in the next two paragraphs.

Pure state models.—Let us start with a simple, yet important case when ρ_θ is pure, $\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|$. We focus on the local estimation paradigm and assume θ is close to some known parameter value θ_0. For any θ_0 it is possible to fix orthonormal vectors $|0\rangle, |1\rangle \in \mathrm{span}(|\psi_{\theta_0}\rangle, |\dot{\psi}_{\theta_0}\rangle)$ such that $\rho_{\theta_0} = |+\rangle\langle+|$, $\dot{\rho}_{\theta_0} = \frac{1}{2}\sqrt{F_Q}\sigma_y$, where $|+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$, σ_y is a Pauli matrix, F_Q is the QFI. Then, the measurement $\boldsymbol{M}$ is QCRB sat. if and only if all its elements are of the form $M_i = \lambda_i|\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle = (1/\sqrt{2})(|0\rangle + e^{i\varphi_i}|1\rangle)$ (see Ref. [19] Section E1). As we prove in [19] Section E1, $\chi[\boldsymbol{M}] \geq 4F_Q$ for all such measurements, and the inequality is only saturated for the projective measurement on the eigenstates of σ_y. Notice, that we used a qubit subspace to describe the evolution of any pure state locally even though $\mathcal{H}_S$ can be arbitrarily large.

This parametrization allows us to represent any pure state problem as a phase estimation in a Mach-Zender interferometer with a single photon input. When the phase θ between the upper ($|0\rangle$) and lower ($|1\rangle$) arm is acquired, then the photon state is $|\psi_\theta\rangle = (1/\sqrt{2})(|0\rangle + e^{i\theta}|1\rangle)$. After fixing $\theta_0 = 0$, our problem reduces to the one already defined with $F_Q = 1$. All QCRB sat. projective measurements can be implemented with the help of two single-photon detectors followed by a beam-splitter, and a well-controlled phase difference between two arms, φ (see Fig. 1). The upper and lower detectors click with probabilities p_+ and p_-, respectively, where $p_\pm = \frac{1}{2}[1 \pm \cos(\theta + \varphi)]$. Straightforward calculations confirm, that $F_C[p_\pm] = 1$ independently of φ [23]. However, the FI MENOS depends on φ, and for $\theta_0 = 0$ we have

$$\chi(\varphi) = 1 + \cos^{-2}(\varphi/2) + \tan^{-2}(\varphi/2), \tag{13}$$

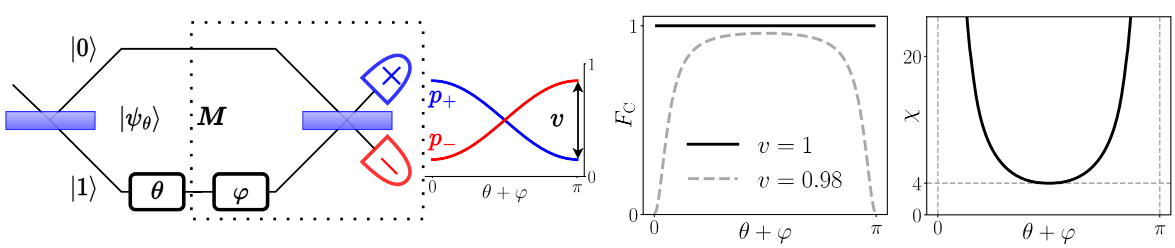

FIG. 1. Phase θ is measured using a Mach-Zender interferometer; φ is an extra, well-controlled phase. The resulting CFI (F_C) does not depend on φ when $v = 1$, which is never achieved in practice. For any smaller visibility (e.g., $v = 0.98$), the CFI is maximal for $\theta + \varphi = \pi/2$, and vanishes for $\theta + \varphi \in \{0, \pi\}$. We can deduce that $\theta + \varphi = \pi/2$ is the optimal working point without assuming nonunit visibility because FI MENOS (χ) is minimal there, so the precision of the estimation of θ is the least vulnerable to a general measurement noise.

which means that the optimal working point is at $\varphi = \pi/2$ (balanced interferometer), while our scheme is extremely sensitive to a measurement noise for $\varphi \to 0$ and $\varphi \to \pi$ (unbalanced interferometer) as in this case $\chi(\varphi) \to \infty$, see Fig. 1.

Similar conclusions follow from a standard analysis of the nonunit visibility ($v < 1$) interferometer model, where the detection probabilities are $p_\pm = \frac{1}{2}[1 \pm v\cos(\theta)]$. Then, the CFI is maximal for $\varphi = \pi/2$, and reaches 0 for $\varphi \in \{0, \pi\}$ even for v very close to 1, see Fig. 1. The advantage of the approach based on FI MENOS, is that one does not need to consider any particular noise model, and it is guaranteed that the worst case scenario has been taken into account.

Superresolution optical imaging.—Quantum estimation theory allows for a rigorous study of fundamental limits in optical microscopy, and serves as a tool for a systematic search for the most precise imaging schemes [24–30]. In the elementary scenario, two equally bright incoherent weak point sources are imaged using a translationally invariant system [25] (see discussion of limitations of this approximation in [31]). The state of a single photon in the image plane is

$$\rho_\theta = \frac{1}{2}(|u_{+,\theta}\rangle\langle u_{+,\theta}| + |u_{-,\theta}\rangle\langle u_{-,\theta}|), \qquad (14)$$

where $\langle x|u_{\pm,\theta}\rangle = u(x \pm \theta/2)$, $\{|x\rangle\}$ is the position basis, $|u(x)|^2 = (2\pi\sigma^2)^{-1/2}e^{-x^2/2\sigma^2}$ is the system point spread function. The only unknown parameter is the separation between two sources, θ, the centroid of two points is known *a priori*. Intuitively, it should be hard to estimate θ when $\theta \ll \sigma$ because then images of two points overlap significantly. This is true for a standard measurement in the position basis $|x\rangle$ because $F_C[\{|x\rangle\langle x|\}] \to 0$ when $\theta \to 0$. Surprisingly, the QFI does not depend on θ at all, $F_Q[\rho_\theta] = 1/4\sigma^2$ [25]. Therefore, it seems to be no fundamental difference between small and large separations θ, when all quantum measurements are allowed. Unfortunately, this is a highly idealized statement since the estimation precision for small θ is highly affected by detection noise, system misalignment, crosstalk noise, and other imperfections [28–30]—in fact, even for the most clever choice of the measurement, the CFI vanishes with $\theta \to 0$ for all practical scenarios.

At this point, we want to demonstrate the fundamental difficulty of resolving two sources whose images overlap, without referring to any specific noise model, but rather employing the newly introduced FI MENOS figure of merit. In the most commonly studied superresolution protocol, the state in the image plane is measured in the basis of orthogonal Hermite-Gaussian modes $|\phi_q\rangle$ whose center lies in the centroid of two observed sources [25]—see Ref. [27] for an overview of implementations of this measurement based on holography, interferometry, etc. In most of these implementations, we can extract and separate first $K-1$ modes, and the rest of the signal is collected in the Kth outcome, such that our POVM consists of elements $M_i = |\phi_i\rangle\langle\phi_i|$ for $i \in \{1, \ldots, K-1\}$, $M_K = \mathbb{1} - \sum_{i=1}^{K-1} M_i$. The CFI increases with K, but for $\theta = 0$, it approaches the QFI already for $K = 2$. The QCRB is saturated in the full range of θ only for $K \to \infty$, but the precision is close to optimal for a wide range of θ already for $K = 4$—see Fig. 2. In the figure we also plot $\chi(\theta)$ for different values of K. Unfortunately, $\chi \to \infty$ for $\theta \to 0$ in all cases. Consequently, for $\theta \ll \sigma$, it is impossible to achieve the high CFI in the presence of noise using the family of measurements considered so far.

However, there are many other ways to saturate QCRB locally for any fixed value of θ, and some of them may be less susceptible to noise. As for the pure states model, we systematically study all QCRB sat. measurements to find $\chi_Q(\theta)$. Following the technique from [25], we reduce the problem to four-dimensional Hilbert space $\mathcal{H}^{(\theta)} = \mathrm{span}\{|u_{\pm,\theta}\rangle, \partial_\theta|u_{\pm,\theta}\rangle\}$, which is a direct sum of two orthogonal two-dimensional subspaces $\mathcal{H}_s$ and $\mathcal{H}_a$, containing symmetric and antisymmetric modes, respectively. We construct the orthonormal basis of both subspaces, $|0\rangle_s$, $|1\rangle_s$ and $|0\rangle_a$, $|1\rangle_a$ respectively, such that

$$\rho_\theta = \frac{1+\delta}{2}|0\rangle_s\langle 0| + \frac{1-\delta}{2}|0\rangle_a\langle 0|, \qquad (15)$$

$$\dot\rho_\theta = \alpha|0\rangle_s\langle 0| + \beta_s\sigma_x^{(s)} - \alpha|0\rangle_a\langle 0| + \beta_a\sigma_x^{(a)}, \qquad (16)$$

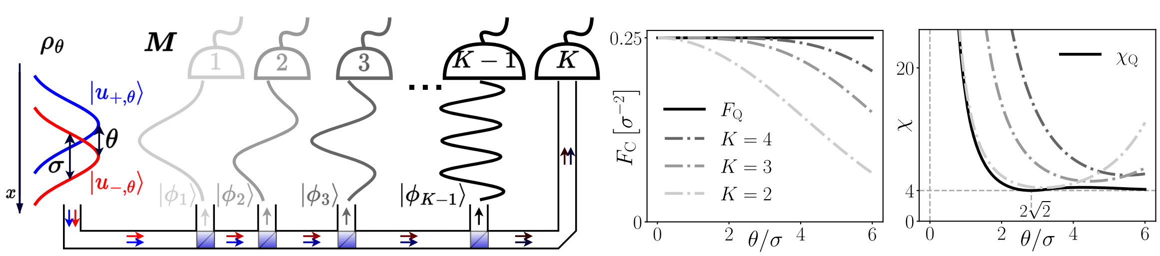

FIG. 2. The single photon state (ρ_θ) in an image plane is the mixture of two Gaussian wave functions of width σ. Their separation θ can be estimated accurately even when $\theta \ll \sigma$ by measuring ρ_θ in the basis of Hermite-Gaussian modes $|\phi_i\rangle$—in practice, it is enough to extract the first few ($K-1$) of these modes. Unfortunately, the presented strategy is very sensitive to measurement noise for small θ, which is reflected by diverging FI MENOS (χ) for $\theta/\sigma \to 0$. No QCRB sat. measurement is free of this issue because χ_Q diverges as well for $\theta \to 0$.

where $\delta = \langle u_{+,\theta} | u_{-,\theta} \rangle$, real constants α, β_a, β_s are specified together with the exact construction of the basis in [19] Section E2. Matrices ρ_θ and $\dot{\rho}_\theta$ are both block diagonal with respect to $\mathcal{H}_s$ and $\mathcal{H}_a$. This means, that the QCRB sat. minimal susceptibility POVM contains only elements acting on $\mathcal{H}_s$ or $\mathcal{H}_a$ (see Ref. [19] Section D for proof). Consider the family of QCRB sat. POVMs

$$\boldsymbol{M}_{\varphi_s,\varphi_a} = \{P^{(s)}_{\varphi_s}, P^{(s)}_{\varphi_s+\pi}, P^{(a)}_{\varphi_a}, P^{(a)}_{\varphi_a+\pi}\}, \tag{17}$$

where $P^{s/a}_{\varphi}$ is a projector on $\cos(\varphi/2)|0\rangle_{s/a} + \sin(\varphi/2)|1\rangle_{s/a}$. We prove ([19] Section E2) that the minimal susceptibility QCRB sat. measurement is of the form $\boldsymbol{M}_{\varphi_s,\varphi_a}$. Then, we obtain χ_Q by minimizing numerically $\chi[\boldsymbol{M}_{\varphi_s,\varphi_a}]$ over φ_s and φ_a, the results are shown in Fig. 2. We observe, that $\chi_Q \to \infty$ when $\theta \to 0$, which means, that no QCRB sat. measurement is robust against noise in the region $\theta \ll \sigma$. Surprisingly, χ_Q does not decrease with θ everywhere—for example, a minimum $\chi = 4$ is achieved in $\theta = 2\sqrt{2}\sigma$. For $\theta \to \infty, \chi_Q \to 4$ again, and then the problem is equivalent to a single source localization both from the point of view of F_Q and χ_Q. Interestingly, it is possible to achieve noise susceptibility smaller than χ_Q, when correlations between subsequent photons are present—see Ref. [19] Section F for a discussion.

Outlook.—Computation of the FI MENOS should be regarded as a natural sanity check whenever any idealized quantum metrological protocol is proposed. If this quantity is large (or divergent) this should ring a bell that the performance of the proposed protocol will be significantly reduced by even a small imperfection in the measurement design. On the contrary, small values indicate that the measurement scheme is robust. The importance of this quantity stems also from the fact, that the FI is a local quantity (computed at single value of parameter) and therefore prone to reveal ephemeral effects that vanish in the presence of even infinitesimal noise—a property that haunts quantum metrology literature a lot. We envisage that our approach may be naturally extended to a multiparameter estimation framework as well as generalized to cover the Bayesian analysis as well. Still, we expect that in these cases it may be much harder or even impossible to obtain the explicit formula for FI MENOS analogous to (11). Moreover, we expect that the role of small measurement disturbances should be less substantial in the Bayesian approach, since such an approach is by construction applicable to more realistic scenarios, when the number of collected data samples is finite, and the protocols are expected to perform well beyond the "local estimation approach".

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[1] J. Preskill, Quantum 2, 79 (2018).
[2] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Phys. 5, 222 (2011).
[3] R. Demkowicz-Dobrzanski, M. Jarzyna, and J. Kołodyński, in *Progress in Optics*, edited by E. Wolf (Elsevier, New York, 2015), Vol. 60, pp. 345–435.
[4] C. L. Degen, F. Reinhard, and P. Cappellaro, Rev. Mod. Phys. 89, 035002 (2017).
[5] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Rev. Mod. Phys. 90, 035005 (2018).
[6] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North Holland, Amsterdam, 1982).
[7] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994).
[8] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nat. Phys. 7, 406 (2011).
[9] R. Demkowicz-Dobrzański, M. Guţă, and J. Kołodyński, Nat. Commun. 3, 1063 (2012).
[10] R. Demkowicz-Dobrzański, J. Czajkowski, and P. Sekatski, Phys. Rev. X 7, 041009 (2017).
[11] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, Nat. Commun. 9, 78 (2018).
[12] S. Zhou and L. Jiang, PRX Quantum 2, 010343 (2021).
[13] A. Datta, L. Zhang, N. Thomas-Peter, U. Dorner, B. J. Smith, and I. A. Walmsley, Phys. Rev. A 83, 063836 (2011).
[14] Y. L. Len, T. Gefen, A. Retzker, and J. Kołodyński, Nat. Commun. 13, 6971 (2022).
[15] P. Skrzypczyk and N. Linden, Phys. Rev. Lett. 122, 140403 (2019).
[16] M. Oszmaniec and T. Biswas, Quantum 3, 133 (2019).
[17] A. W. van der Vaart, *Asymptotic Statistics* (Cambridge University Press, Cambridge, England, 1998).
[18] S. Zhou, C.-L. Zou, and L. Jiang, Quantum Sci. Technol. 5, 025005 (2020).
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.160802 for more information, which includes Refs. [20–22].
[20] T. Dertinger, R. Colyer, G. Iyer, S. Weiss, and J. Enderlein, Proc. Natl. Acad. Sci. U.S.A. 106, 22287 (2009).
[21] S. Kurdziałek and R. Demkowicz-Dobrzański, J. Opt. 23, 075701 (2021).
[22] O. Schwartz and D. Oron, Phys. Rev. A 85, 033812 (2012).
[23] Strictly speaking, we cannot use (2) directly for $\varphi \in \{0, \pi\}$ because then $p_i = 0$ and $\dot{p}_i = 0$ for one term of the sum from (2). If we exclude such terms by convention, we obtain $F_C = 0$, but if we calculate limits $\varphi \to 0$ or $\varphi \to \pi$, we obtain $F_C = 1$.
[24] E. Bettens, D. V. Dyck, A. den Dekker, J. Sijbers, and A. van den Bos, Ultramicroscopy 77, 37 (1999).
[25] M. Tsang, R. Nair, and X.-M. Lu, Phys. Rev. X 6, 031033 (2016).

[26] C. Lupo and S. Pirandola, Phys. Rev. Lett. **117**, 190802 (2016).
[27] M. Tsang, Contemp. Phys. **60**, 279 (2019).
[28] Y. L. Len, C. Datta, M. Parniak, and K. Banaszek, Int. J. Quantum. Inform. **18**, 1941015 (2020).
[29] C. Oh, S. Zhou, Y. Wong, and L. Jiang, Phys. Rev. Lett. **126**, 120502 (2021).
[30] J. O. de Almeida, J. Kołodyński, C. Hirche, M. Lewenstein, and M. Skotiniotis, Phys. Rev. A **103**, 022406 (2021).
[31] S. Kurdzialek, Quantum **6**, 697 (2022).