Dynamics-Based Entanglement Witnesses for Non-Gaussian States of Harmonic Oscillators

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We introduce a family of entanglement witnesses for continuous variable systems, which rely on the sole assumption that their dynamics is that of coupled harmonic oscillators at the time of the test. Entanglement is inferred from the Tsirelson nonclassicality test on one of the normal modes, without any knowledge about the state of the other mode. In each round, the protocol requires measuring only the sign of one coordinate (e.g., position) at one among several times. This dynamic-based entanglement witness is more akin to a Bell inequality than to an uncertainty relation: in particular, it does not admit false positives from classical theory. Our criterion detects non-Gaussian states, some of which are missed by other criteria.

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Introduction.—Once troublesome to the founders of quantum mechanics [1,2], entanglement is now well established as one of the defining features of quantum theory. While entanglement in discrete systems has gone through much scrutiny [3,4], the field of continuous variable (CV) quantum entanglement has had its challenges outside the Gaussian regime [5].

Early entanglement criteria for CV systems were based on the second-order moments of quadrature distributions [6,7], which are useful for Gaussian states, but ineffective otherwise [8]. However, a computation consisting only of Gaussian states and operations can be simulated efficiently on a classical computer [9]. Similar no-go theorems also exist for error correction [10] and entanglement distillation [11,12], encouraging the development of methods to detect non-Gaussian CV entanglement. Efforts in this direction include criteria based on higher-order moments [13–17] and full probability distribution [18,19] of quadrature measurements. Recent advances include entropic entanglement criteria based on quasiprobability distributions [20,21] and measurement-device-independent criteria [22], both of which require performing some form of partial tomography.

The criteria to be chosen would of course be informed by the type of CV system used. The obvious test bed for CV entanglement are optical modes [23] and vibrational modes of trapped ions [24], but there has also been significant progress for massive oscillators [25]. For many of these CV systems, harmonic dynamics arises naturally in their implementation.

Because of these motivations, we present a criterion to certify the non-Gaussian entanglement of two CV degrees of freedom that exploits the knowledge of the *dynamics* of the systems. We call it a dynamic-based entanglement witness (DEW) [26]. Specifically, we consider degrees of freedom that undergo types of *harmonic* dynamics, and build on a nonclassicality test for a single oscillator [28,29].

A priori, our criterion exhibits two elegant features: the absence of false positives from classical theory, and the fact that only one observable needs to be measured. These features, to be defined precisely below, can be appreciated by contrast with existing entanglement witnesses, based on generalized uncertainty relations [7,13–17]. Consider specifically the criterion by Duan *et al.* [7]. Generalizing the observables defined in the Einstein-Podolsky-Rosen (EPR) argument [1], these authors defined the commuting dimensionless variables $u=|c|\tilde{x}_1+(1/c)\tilde{x}_2$ (with $\tilde{x}_j = x_j\sqrt{m_j\omega_j/\hbar}$) and $v=|c|\tilde{p}_1-(1/c)\tilde{p}_2$ (with $\tilde{p}_j = p_j/\sqrt{m_j\hbar\omega_j}$) for some $c\in\mathbb{R}$. The two subsystems are then entangled if $\langle(\Delta u)^2\rangle+\langle(\Delta v)^2\rangle < c^2+(1/c^2)$. This requires measuring both positions and momenta, and with a precision set by $\hbar$. At the precision of (say) human perception, two springs at equilibrium are described by $\tilde{x}_j=\tilde{p}_j=0$, values which would imply entanglement if plugged naively in the criterion above. Gross though it is, this example shows the danger of false positives.

A posteriori, we find that our criterion detects states with negative Wigner functions (thus, non-Gaussian), some of which are missed by all existing criteria. As mentioned earlier, Gaussian states have limited usefulness in many quantum protocols [9–12]. In this context, our DEW detects resource states for these protocols [30]. Thus, besides being elegant, our DEW is also a useful addition to the existing toolbox.

The single-oscillator protocol.—We review the Tsirelson nonclassicality test [28] following the generalization given in [29]. The assumption is that the physical quantity A_1 is undergoing a uniform precession at pulsation ω, i.e., $A_1(t)=A_1(0)\cos\omega t+A_2(0)\sin\omega t$, where A_2 is another physical quantity. For classical systems, $A_1(t)$ is the value of A_1 at time t; for quantum systems, it is the corresponding observable in the Heisenberg representation. The protocol for the test (which we call the *precession protocol* hereafter) goes as follows. In each round, the sign of A_1 is measured at one of K different times given by $t_k = (k/K)T$, where $K > 0$, $k = 0, 1, \ldots, K-1$, and T is the period of oscillation. After several rounds, one estimates

$$P_K = \frac{1}{K}\sum_{k=0}^{K-1}\left\{\Pr[A_1(t_k) > 0] + \frac{1}{2}\Pr[A_1(t_k) = 0]\right\}, \quad (1)$$

where the second term in the bracket was introduced in [29] to avoid singular behaviors for states with noninfinitesimal concentration on $A_1 = 0$. By inspection [29], the upper bound $P_K \leq \mathbf{P}_K^c$ for a classical theory is easily derived: $\mathbf{P}_K^c = 1/2$ for K even, and

$$\mathbf{P}_K^c = \frac{1}{2}\left(1 + \frac{1}{K}\right) \quad \text{for } K \text{ odd.} \quad (2)$$

Remarkably, in spite of the fact that the precessing dynamics is identical to the classical one, there exist quantum states for which $P_K > \mathbf{P}_K^c$ for any odd $K > 1$.

For the remainder of the Letter, we focus on the *harmonic oscillator*, i.e., a material point, whose time evolution is governed by the Hamiltonian $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$. The pair $(A_1, A_2) = (\tilde{x}, \tilde{p})$ clearly precesses at pulsation ω and thus satisfies the assumption. On a given state ρ, quantum theory predicts $P_K = \mathrm{Tr}(\rho Q_K)$, where

$$Q_K = \frac{1}{K}\sum_{k=0}^{K-1}\mathrm{pos}[X(t_k)], \quad (3)$$

with $\mathrm{pos}(X)$ defined by $\mathrm{pos}(X)|x\rangle = \frac{1}{2}[1 + \mathrm{sgn}(x)]|x\rangle$.

The maximum quantum score $\mathbf{P}_K = \max_{|\psi\rangle} \langle\psi|Q_K|\psi\rangle$ (denoted $\mathbf{P}_K^\infty$ in [29]) is achieved by $|\mathbf{P}_K\rangle$, the eigenstate of Q_K with the largest eigenvalue. Tsirelson proved that $\mathbf{P}_3 \gtrsim 0.709 > \mathbf{P}_3^c = 2/3$ [28]; similar violations are found for all odd K [29]. The violation can be attributed to having suitable patterns in the Wigner function, in particular, suitably distributed negativities. A state with positive Wigner function cannot give any violation.

Entanglement of two harmonic oscillators.—Now, the key insight is that x may be the position of an *effective* oscillator, built out of two (or more) physical ones. We focus on the case of two oscillators, with arbitrary masses and frequencies, and an x-x coupling. The standard decomposition in normal modes yields

$$\begin{aligned} H &= \sum_{j=1}^{2}\left(\frac{1}{2m_j}p_j^2 + \frac{1}{2}m_j\omega_j^2 x_j^2\right) - \frac{1}{2}gx_1x_2 \\ &= \sum_{\sigma\in\{+,-\}}\frac{1}{2\mu}p_\sigma^2 + \frac{1}{2}\mu\omega_\sigma^2 x_\sigma^2, \end{aligned} \quad (4)$$

where $\mu = \sqrt{m_1 m_2}$,

$$\begin{aligned} x_+(t) &= \left(\frac{m_1}{m_2}\right)^{1/4}\cos\theta\, x_1(t) + \left(\frac{m_2}{m_1}\right)^{1/4}\sin\theta\, x_2(t) \\ x_-(t) &= \left(\frac{m_2}{m_1}\right)^{1/4}\cos\theta\, x_2(t) - \left(\frac{m_1}{m_2}\right)^{1/4}\sin\theta\, x_1(t), \end{aligned} \quad (5)$$

with mixing angle $\theta = \arctan 2[g, \mu(\omega_1^2 - \omega_2^2)]/2$, and normal frequencies $\omega_\pm^2 = [(\omega_1^2 + \omega_2^2)/2] \pm \sqrt{[(\omega_1^2 - \omega_2^2)/2]^2 + (g^2/4\mu^2)}$. The time evolution of the $x_\sigma(t)$ is a uniform precession around phase space with the period $T_\sigma = 2\pi/\omega_\sigma$. Therefore, the single-oscillator protocol can be performed for coupled oscillators with different frequencies by measuring the normal modes $x_\sigma(t_k + t_0)$ at times $t_k = (k/K)T_\sigma$ for $k = 0, 1, \ldots, K-1$. There are many ways to estimate P_K for x_σ: in each round, x_σ can be formed from x_1 and x_2 measured separately, addressed directly (e.g., with motional modes of trapped ions), or measured with an interferometer (e.g., with spatial or polarization modes of photons). Notice also that, up to a multiplicative constant, $x_\sigma(t)$ has the same form as u defined in the entanglement criterion by Duan and co-workers [7].

It is important to stress that H describes the dynamics during the certification protocol, not the interaction that prepared the state under study. Thus, all values of the parameters are allowed. In particular, the certification protocol can be performed when $g = 0$ and $\omega_1 = \omega_2$. In this case, θ can take on any value: indeed, for uncoupled oscillators precessing at the same frequency, all linear combinations of x_1 and x_2 are normal modes at that same frequency.

Another point to note is that the dynamics are assumed to be known, in which case only one quadrature x_σ needs to be measured for the protocol. One could of course replace this assumption by taking the quadratures $\{x_\sigma(t_k)\}_{k=0}^{K-1}$ at the different times to be K different settings of the measurement apparatus.

If $P_K > \mathbf{P}_K^c$ for x_σ, the state of that mode has certainly a negative Wigner function. We want to study when one can further infer that the physical subsystems are entangled. This is not straightforward because, by performing the protocol on one of the normal modes, we learn nothing about the state of the other mode: the latter could be very mixed; or the two normal modes may be even entangled. We are going to provide the conditions under which entanglement can indeed be certified.

Results.—For the quantum system, we denote the annihilation operators of the two physical oscillators as $\{a_1, a_2\}$: they are the subsystems whose entanglement we want to certify. As hinted, $x_\sigma(t)$ is the position of an effective oscillator denoted by the annihilation operator a_σ. Specifically, let $\{a_+, a_-\}$ be a new basis of modes, related to the original by the passive transformation

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{6}$$

with $\theta \in [0, \pi/4]$ the mixing angle previously defined. We are going to study the operator Q_K given by Eq. (3) for the position operator $X_\sigma(t) = \sqrt{(\hbar/2\mu\omega_\sigma)}(a_\sigma e^{-i\omega_\sigma t} + a_\sigma^\dagger e^{i\omega_\sigma t})$, where Q_K and X_σ rely implicitly on θ via Eq. (6).

When $\theta = \pi/4$, we have an analytical proof of existence of entanglement for any violation $P_K > \mathbf{P}_K^{(c)}$:

Result.—If the precession protocol is performed with $\theta = \pi/4$, all states that violate the classical bound (2) for a_+ (or a_-) are entangled in $\{a_1, a_2\}$.

Proof.—We show this for a_+ with a proof by contradiction. The proof for a_- proceeds in a similar way.

Take $\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}$ separable in the $\{a_1, a_2\}$ subsystems. Its Wigner function is

$$W_\rho(\alpha_1, \alpha_2) = \sum_k p_k W_{\rho_1}^{(k)}(\alpha_1) W_{\rho_2}^{(k)}(\alpha_2), \tag{7}$$

where $\{\alpha_1, \alpha_2\}$ are the phase-space coordinates in the $\{a_1, a_2\}$ modes. For $\theta = \pi/4$ in Eq. (6), the Wigner function in terms of $\{\alpha_+, \alpha_-\}$, the phase-space coordinates in the $\{a_+, a_-\}$ modes, can be found with a straightforward coordinate transformation:

$$W_\rho(\alpha_+, \alpha_-) = \sum_k p_k W_{\rho_1}^{(k)}\left(\frac{\alpha_+ - \alpha_-}{\sqrt{2}}\right) W_{\rho_2}^{(k)}\left(\frac{\alpha_+ + \alpha_-}{\sqrt{2}}\right). \tag{8}$$

The measurement outcome in the a_+ basis is determined solely by the reduced Wigner function $W_{\mathrm{tr}_-(\rho)}(\alpha_+) = \int d^2\alpha_- W_\rho(\alpha_+, \alpha_-)$. By converting the passive coordinates in the arguments into active transformations of the states, we find

$$\begin{aligned} W_{\mathrm{tr}_-(\rho)}(\alpha_+) &= 2\sum_k p_k \int d^2\gamma\, W_{\tilde{\rho}_1(\alpha_+)}^{(k)}(\gamma) W_{\tilde{\rho}_2(\alpha_+)}^{(k)}(\gamma) \\ &= \frac{2}{\pi}\sum_k p_k \,\mathrm{tr}(\tilde{\rho}_1^{(k)}(\alpha_+)\tilde{\rho}_2^{(k)}(\alpha_+)), \end{aligned} \tag{9}$$

where

$$\tilde{\rho}_1^{(k)}(\alpha_+) = D\left(\frac{1}{\sqrt{2}}\alpha_+\right) e^{-i\pi a^\dagger a} \rho_1^{(k)} e^{i\pi a^\dagger a} D^\dagger\left(\frac{1}{\sqrt{2}}\alpha_+\right)$$
$$\tilde{\rho}_2^{(k)}(\alpha_+) = D\left(-\frac{1}{\sqrt{2}}\alpha_+\right) \rho_2^{(k)} D^\dagger\left(-\frac{1}{\sqrt{2}}\alpha_+\right).$$

and $D(\alpha)$ is the usual displacement operator. Thus $W_{\mathrm{tr}_-(\rho)} \geq 0$, since it is a convex sum of inner products between density operators [31]. However, negativity in the Wigner function of a_+ is necessary for a violation of the classical bound of the precession protocol [29]. Therefore, when $\theta = \pi/4$, *any* violation of the classical bound witnesses entanglement of the $\{a_1, a_2\}$ subsystems. ■

Next, we are going to study in detail the protocol with $K = 3$. For any value of θ and of $P_3 > \mathbf{P}_3^c = 2/3$, we are going to compute a numerical lower bound on the amount of certifiable entanglement of $\{a_1, a_2\}$. We choose the logarithmic negativity $S_N(\rho) = \log \mathrm{tr}|\rho^{\Gamma_2}|$ as the quantifier of entanglement. Since $\min_\rho S_N(\rho) = \log \min_\rho \mathrm{tr}|\rho^{\Gamma_2}|$, and

$$\begin{aligned} &\min_\rho \mathrm{tr}|\rho^{\Gamma_2}| \\ &\text{subject to}\,\mathrm{tr}[\rho Q_3(\theta)] = P_3 \end{aligned} \tag{10}$$

is a minimization of the trace norm under convex constraints, it can be cast as a standard semidefinite program (SDP) when ρ is truncated in the basis of the excitations of a_+ and a_- [36].

We run the SDP for truncation $0 \leq n_+, n_- \leq \mathbf{n} = 11$. Both the form of the SDP and the choice of the truncation are described in [37], and the script used to perform the SDP is available at [45]. The results are plotted in Fig. 1. We do not fully understand the dependence of the logarithmic negativity on θ and P_3 due to the complexity of the states involved. Broadly speaking, what we do observe from Fig. 1(b) is that the certifiable S_N increases with θ and P_3, as shown more explicitly by the line cuts. For fixed values of $P_3 \sim \mathbf{P}_3$, where $\mathbf{P}_3$ is the maximum quantum score under the truncation $\mathbf{n}$, the certifiable S_N increases with θ until a peak around $\theta \gtrsim 3\pi/16$. Afterwards, the entanglement decreases for larger values of θ, although only slightly. In practice, θ is determined by the system, and one would refer to the corresponding line cut in Fig. 1(c). There, for fixed values of θ, we find that the entanglement monotonically increases with P_3.

We already knew that every P_3 certifies entanglement when $\theta = \pi/4$, and the graph indicates that this remains true down to $\theta \simeq \pi/8$. Below this value, one needs a sufficiently large P_3, a low violation of the classical bound being compatible with separable states. When $\theta = 0$, the precession protocol is performed on the first oscillator, and so no amount of violation detects entanglement.

Comparison with other witnesses.—Now we put our DEW in the context of entanglement witnesses for continuous variables (CVs), by comparing it to other criteria.

First of all, our DEW uses *quadrature measurements* in the terminology of quantum optics. Other measurements than quadratures can be used to witness CV entanglement: for instance, one witness in Zhang *et al.* [13] uses local measurements of the generators of SU(N). In fact, any CV entanglement can, in principle, be witnessed by projecting the state into a finite dimensional subspace, then applying techniques to witness entanglement of qudits [46]. Quadrature measurements have the appeal of having a classical analog, are practical in many platforms, and in

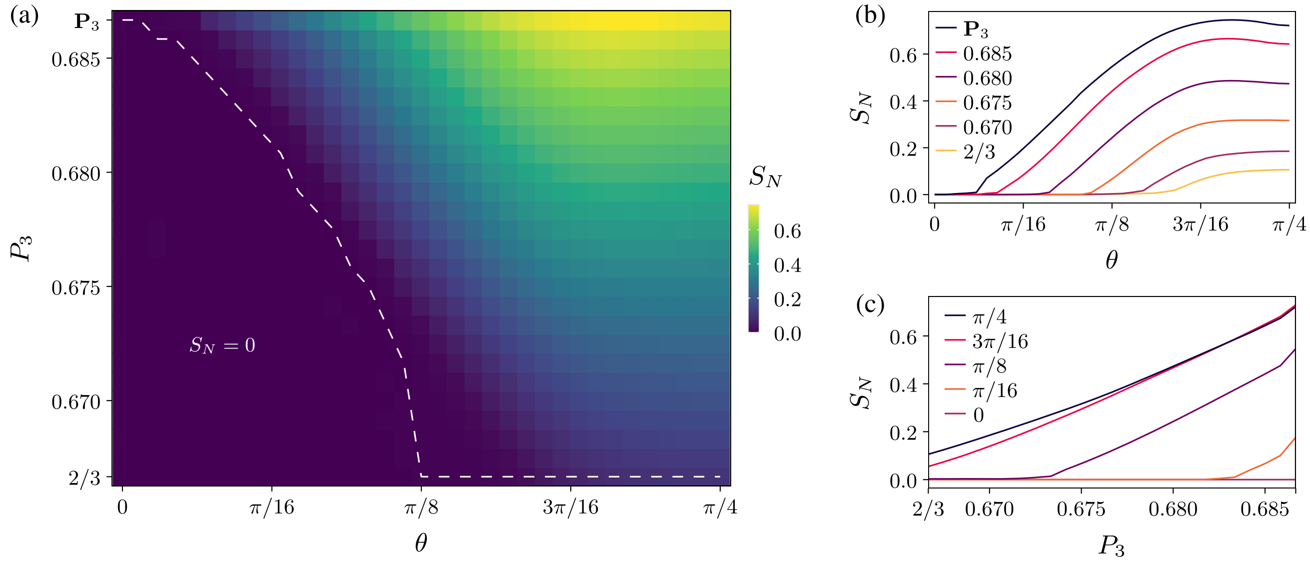

FIG. 1. (a) Heat map of logarithmic negativity as a function of the mixing angle θ and the score P_3. Only the range $0 \le \theta \le \pi/4$ is shown: other values of θ corresponds to this range with a sign change of x_1 and x_2, which can be effected with a local unitary on the $\{a_1, a_2\}$ basis and hence does not affect the amount of entanglement. The dimension of the Hilbert space is truncated at $\mathbf{n} = 11$ for both modes a_+ and a_-, and $\mathbf{P}_3$ is the maximum quantum score under this truncation. The dotted line separates the states with $S_N > 0$ (such that $S_N - \varepsilon > 0$, where $\varepsilon \lesssim 10^{-4}$ is the dual gap of the SDP) and those with $S_N = 0$. (b) Horizontal line cuts of the heat map. For fixed P_3, S_N increases with θ. (c) Vertical line cuts of the heat map. For fixed θ, S_N increases with P_3.

some setups may be even the only available ones at this time (e.g., optomechanical systems with large masses).

As already mentioned in the introduction, the other entanglement witnesses we are aware of are open to *false positives* from classical theory [7,13,14,16,17]. By contrast, in the case of our DEW, poor precision or wrong calibration may prevent the detection of entanglement, but will not lead to false positives. This is similar to what happens with Bell inequalities, where noise and lack of precision may decrease or cancel the violation, but not fake it.

Having mentioned this, it is natural to move on to the comparison in terms of characterization of the devices. Fully device-independent entanglement witnesses (Bell inequalities) that use only quadrature measurements have been hard to find: the few known examples are very specific [47–49]. Recently, a measurement-device-independent criterion was introduced, under the assumption that a trusted source of coherent states is available [22]. Meanwhile, our DEW is *semi-device-independent*: it works under the assumptions that the dynamics is a uniform precession, and that the same quadrature is measured whatever time is picked. Both assumptions are well defined both in classical and in quantum theory. In fact, the notions of "position" and "uniform precession" are operational, and their meaning immediate by everyday experience (what may not be immediate is the identification of a normal mode).

Lastly, let us compare the states that are detected by the existing criteria against those detected by our DEW. Many existing criteria can detect entangled Gaussian states, like the two-mode squeezed state and its limiting case, the EPR state; our DEW misses these states, since their Wigner function is positive. Our DEW is also unsuitable for states with even rotational symmetry, like photon-subtracted or added squeezed number states [50,51] and spin coherent states [15], because they commute with the total parity operator (details in [37]). Conversely, for any $K \ge 3$ odd, consider the states

$$|\{\psi_n\}_{\mathbf{n}}\rangle = \sum_{j=0}^{\mathbf{n}K} |\Psi^j\rangle_1 \otimes |j\rangle_2$$
$$= \left(\sum_{n=0}^{\mathbf{n}} \psi_n |nK\rangle_+\right) \otimes |0\rangle_- \equiv |\Psi_{\mathbf{n}}\rangle_+ \otimes |0\rangle_- \quad (11)$$

with

$$|\Psi^j\rangle = \sum_{n=\lceil j/K \rceil}^{\mathbf{n}} \psi_n \sqrt{\binom{nK}{j}} (\cos\theta)^{nK-j} (\sin\theta)^j |nK - j\rangle,$$

where $\mathbf{n}$ can go to infinity, and the pairs of modes are related by Eq. (6). These states exhibit odd rotational symmetry—making them candidates for a type of bosonic error correcting codes [52]—and are entangled as long as $\theta \bmod \pi/2 \neq 0$ and $|\psi_n| \neq \delta_{n,n_0}$ for one value n_0. All are missed by [7,13,22], and some also by [14,15] (see Ref. [37] for details). Clearly our DEW detects the entanglement of all the states (11) such that $|\Psi_{\mathbf{n}}\rangle$ violates the original precession protocol with K possible probing times. In particular, the eigenstates of Q_K with maximal eigenvalue are of the form $|\Psi_{\mathbf{n}}\rangle$, and so the corresponding $|\{\psi_n\}_{\mathbf{n}}\rangle$ is optimally detected by our DEW. As another example: for $K=3$, $\theta=\pi/4$ and a suitable choice of the ψ_n (see Ref. [37]), the state (11) is the entangled three-level cat state

$$|\Psi(\alpha)\rangle \propto \sum_{k=-1}^{1} |\alpha e^{i2\pi k/3}\rangle_1 \otimes |\alpha e^{i2\pi k/3}\rangle_2.$$

This state is detected by our DEW for $0.88 \lesssim |\alpha| \lesssim 1.23$, while for $1.23 \lesssim |\alpha| \lesssim 1.82$ it is detected by the criteria of [14,15]. Finally, let us notice that one does not need very high excitations: our DEW with $K=3$ detects the state with $(\psi_0, \psi_1, \psi_2) \simeq (0.6172, -0.7017, 0.3450)$, which is a superposition of 0, 3, and 6 excitations in a_+.

More generally, our DEW is not a subset of any member of the family of uncertainty-based entanglement witnesses defined by Shchukin and Vogel [16] and Nha and Zubairy [17], which include [7,14,15] as special cases.

Conclusion.—We have introduced a dynamic-based entanglement witness for two harmonic oscillators. It consists of certifying the quantumness of a normal mode using the Tsirelon protocol [28,29]: the entanglement of the physical oscillators can then be inferred, without having any information about the other normal mode (obviously, having also some information about it can only tighten the lower bounds that we have obtained).

Our criterion detects a different set of states than those captured by previous ones. Also, it does not rely on other features of quantum theory (e.g., some form of uncertainty relations): it only assumes the form of the dynamics. As only straightforward coordinate measurements are used, and false positives from classical theory are excluded, our criterion is useful for objects which are too massive to be fully tomographed. It is timely, as recent advances in optomechanics have allowed for quantum control of objects with masses in the mesoscopic and macroscopic scales [25]. The ability to generate and detect quantum effects in such systems has proven useful in technologies that aim to exploit the effects of quantum mechanics, in particular, quantum sensing [53,54] and metrology [55]. Moreover, it is also important in experimental studies of fundamental physics, in tests of collapse theories [56] and quantum-classical transitions [57].

In order to focus on the essentials of the idea, in this Letter we have kept to the simplest form of dynamics, that of two coupled harmonic oscillators. The underlying Tsirelson protocol can be extended to a wide class of Hamiltonians, possibly with an additional energy constraint [58]. Dynamic-based entanglement certification can also be extended to dissipative dynamics: leaving quantitative estimates for future work, it is clear that one will still be able to certify entanglement provided the relaxation time is long enough. We have also only focused on the bipartite scenario here, but an obvious future direction would be to extend our protocol for the multipartite case. As an early example, we show in [37] that by performing the precession protocol in the $\propto a_1 + a_2 + a_3$ mode, our DEW can detect genuine tripartite entanglement [59].

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[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. **47**, 777 (1935).

[2] E. Schrödinger, Die gegenwärtige situation in der quantenmechanik, Naturwissenschaften **23**, 823 (1935).

[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[4] M. B. Plenio and S. S. Virmani, An introduction to entanglement theory, in *Quantum Information and Coherence*, edited by E. Andersson and P. Öhberg (Springer International Publishing, Cham, 2014), pp. 173–209.

[5] G. Adesso and F. Illuminati, Entanglement in continuous-variable systems: Recent advances and current perspectives, J. Phys. A **40**, 7821 (2007).

[6] R. Simon, Peres-Horodecki Separability Criterion for Continuous Variable Systems, Phys. Rev. Lett. **84**, 2726 (2000).

[7] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Inseparability Criterion for Continuous Variable Systems, Phys. Rev. Lett. **84**, 2722 (2000).

[8] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[9] A. Mari and J. Eisert, Positive Wigner Functions Render Classical Simulation of Quantum Computation Efficient, Phys. Rev. Lett. **109**, 230503 (2012).

[10] J. Niset, J. Fiurášek, and N. J. Cerf, No-Go Theorem for Gaussian Quantum Error Correction, Phys. Rev. Lett. **102**, 120501 (2009).

[11] J. Fiurášek, Gaussian Transformations and Distillation of Entangled Gaussian States, Phys. Rev. Lett. **89**, 137904 (2002).

[12] G. Giedke and J. Ignacio Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A **66**, 032316 (2002).

[13] C.-J. Zhang, H. Nha, Y.-S. Zhang, and G.-C. Guo, Detection of bound entanglement in continuous-variable systems, Phys. Rev. A **82**, 032323 (2010).

[14] M. Hillery and M. S. Zubairy, Entanglement Conditions for Two-Mode States, Phys. Rev. Lett. **96**, 050503 (2006).

[15] H. Nha and J. Kim, Entanglement criteria via the uncertainty relations in su(2) and su(1,1) algebras: Detection of non-Gaussian entangled states, Phys. Rev. A **74**, 012317 (2006).

[16] E. Shchukin and W. Vogel, Inseparability Criteria for Continuous Bipartite Quantum States, Phys. Rev. Lett. **95**, 230502 (2005).

[17] H. Nha and M. S. Zubairy, Uncertainty Inequalities as Entanglement Criteria for Negative Partial-Transpose States, Phys. Rev. Lett. **101**, 130402 (2008).

[18] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho, Entropic Entanglement Criteria for Continuous Variables, Phys. Rev. Lett. **103**, 160505 (2009).

[19] A. Saboia, F. Toscano, and S. P. Walborn, Family of continuous-variable entanglement criteria using general entropy functions, Phys. Rev. A **83**, 032307 (2011).

[20] S. Floerchinger, M. Gärttner, T. Haas, and O. R. Stockdale, Entropic entanglement criteria in phase space, Phys. Rev. A **105**, 012409 (2022).

[21] M. Grttner, T. Haas, and J. Noll, Detecting continuous variable entanglement in phase space with the *q*-distribution, arXiv:2211.17165.

[22] P. Abiuso, S. Bäuml, D. Cavalcanti, and A. Acín, Measurement-Device-Independent Entanglement Detection for Continuous-Variable Systems, Phys. Rev. Lett. **126**, 190502 (2021).

[23] S. L. Braunstein and P. van Loock, Quantum information with continuous variables, Rev. Mod. Phys. **77**, 513 (2005).

[24] R. Blatt and C. F. Roos, Quantum simulations with trapped ions, Nat. Phys. **8**, 277 (2012).

[25] A. K. Sarma, S. Chakraborty, and S. Kalita, Continuous variable quantum entanglement in optomechanical systems: A short review, AVS Quantum Sci. **3**, 015901 (2021).

[26] The name “dynamical entanglement witness” would have been more elegant, but could have been confused with a “witness of *dynamical entanglement*”, which is a notion of entanglement for channels [27]. Here we are concerned with entanglement of states.

[27] G. Gour and C. M. Scandolo, Dynamical Entanglement, Phys. Rev. Lett. **125**, 180505 (2020).

[28] B. Tsirelson, How often is the coordinate of a harmonic oscillator positive?, arXiv:quant-ph/0611147.

[29] L. H. Zaw, C. C. Aw, Z. Lasmar, and V. Scarani, Detecting quantumness in uniform precessions, Phys. Rev. A **106**, 032222 (2022).

[30] F. Albarelli, M. G. Genoni, M. G. A. Paris, and A. Ferraro, Resource theory of quantum non-Gaussianity and Wigner negativity, Phys. Rev. A **98**, 052350 (2018).

[31] It was recently pointed out that Eq. (9) is also known as the *convolution* of two Wigner functions [32,33], which has been used in the past to “smooth out” the negativities of a Wigner function to define a non-negative quasiprobability distribution [34,35]. To our knowledge, this has not been previously exploited to witness entanglement.

[32] R. Werner, Quantum harmonic analysis on phase space, J. Math. Phys. (N.Y.) **25**, 1404 (1984).

[33] S. Becker, N. Datta, L. Lami, and C. Rouzé, Convergence rates for the quantum central limit theorem, Commun. Math. Phys. **383**, 223 (2021).

[34] N. Cartwright, A non-Negative Wigner-type distribution, Physica A (Amsterdam) **83**, 210 (1976).

[35] A. K. Rajagopal, Classical statistics inherent in pure quantum states, Phys. Rev. A **27**, 558 (1983).

[36] L. Vandenberghe and S. Boyd, Semidefinite programming, SIAM Rev. **38**, 49 (1996).

[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.160201 for technical details, some proofs, and case studies, which includes Refs. [38–44].

[38] M. Udell, K. Mohan, D. Zeng, J. Hong, S. Diamond, and S. Boyd, *Convex Optimization in Julia* (IEEE Press, New York, 2014), p. 1828.

[39] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, England, 2004).

[40] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, J. Phys. A **41**, 235303 (2008).

[41] Z. Jiang, M. D. Lang, and C. M. Caves, Mixing nonclassical pure states in a linear-optical network almost always generates modal entanglement, Phys. Rev. A **88**, 044301 (2013).

[42] L. Davidovich, A. Maali, M. Brune, J. M. Raimond, and S. Haroche, Quantum Switches and Nonlocal Microwave Fields, Phys. Rev. Lett. **71**, 2360 (1993).

[43] C. Wang, Y. Y. Gao, P. Reinhold, R. W. Heeres, N. Ofek, K. Chou, C. Axline, M. Reagor, J. Blumoff, K. M. Sliwa, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, A Schrödinger cat living in two boxes, Science **352**, 1087 (2016).

[44] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Optimization of entanglement witnesses, Phys. Rev. A **62**, 052310 (2000).

[45] L. H. Zaw, Script used in: Dynamic-based entanglement witnesses for non-Gaussian states of harmonic oscillators, https://github.com/not-fred/arXiv-2210.10357.

[46] J. Sperling and W. Vogel, Verifying continuous-variable entanglement in finite spaces, Phys. Rev. A **79**, 052313 (2009).

[47] K. Banaszek and K. Wódkiewicz, Testing Quantum Nonlocality in Phase Space, Phys. Rev. Lett. **82**, 2009 (1999).

[48] H. Nha and H. J. Carmichael, Proposed Test of Quantum Nonlocality for Continuous Variables, Phys. Rev. Lett. **93**, 020401 (2004).

[49] R. García-Patrón, J. Fiurášek, N. J. Cerf, J. Wenger, R. Tualle-Brouri, and P. Grangier, Proposal for a Loophole-Free Bell Test Using Homodyne Detection, Phys. Rev. Lett. **93**, 130409 (2004).

[50] A. Ourjoumtsev, A. Dantan, R. Tualle-Brouri, and P. Grangier, Increasing Entanglement Between Gaussian States by Coherent Photon Subtraction, Phys. Rev. Lett. **98**, 030502 (2007).

[51] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, Non-Gaussian quantum states of a multimode light field, Nat. Phys. **16**, 144 (2020).

[52] A. L. Grimsmo, J. Combes, and B. Q. Baragiola, Quantum Computing with Rotation-Symmetric Bosonic Codes, Phys. Rev. X **10**, 011058 (2020).

[53] L. M. de Lpinay, C. F. Ockeloen-Korppi, M. J. Woolley, and M. A. Sillanp, Quantum mechanics-free subsystem with mechanical oscillators, Science **372**, 625 (2021).

[54] C. Gut, K. Winkler, J. Hoelscher-Obermaier, S. G. Hofer, R. M. Nia, N. Walk, A. Steffens, J. Eisert, W. Wieczorek, J. A. Slater, M. Aspelmeyer, and K. Hammerer, Stationary optomechanical entanglement between a mechanical oscillator and its measurement apparatus, Phys. Rev. Res. **2**, 033244 (2020).

[55] H. Kwon, Y. Lim, L. Jiang, H. Jeong, and C. Oh, Quantum Metrological Power of Continuous-Variable Quantum Networks, Phys. Rev. Lett. **128**, 180503 (2022).

[56] J. Zhang, T. Zhang, and J. Li, Probing spontaneous wave-function collapse with entangled levitating nanospheres, Phys. Rev. A **95**, 012141 (2017).

[57] M. Schlosshauer, *Decoherence: And the Quantum-To-Classical Transition*, The Frontiers Collection (Springer, New York, 2007).

[58] L. H. Zaw and V. Scarani, Dynamics-based quantumness certification of continuous variables with generic time-independent Hamiltonians, arXiv:2212.06017.

[59] O. Ghne and G. Tth, Entanglement detection, Phys. Rep. **474**, 1 (2009).

Correction: The two sentences preceding the Conclusion section contained erroneous information and have been removed. A related change has also been made to the Supplemental Material.