Phases of Wilson Lines in Conformal Field Theories

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We study the low-energy limit of Wilson lines (charged impurities) in conformal gauge theories in $2+1$ and $3+1$ dimensions. As a function of the representation of the Wilson line, certain defect operators can become marginal, leading to interesting renormalization group flows and for sufficiently large representations to complete or partial screening by charged fields. This result is universal: in large enough representations, Wilson lines are screened by the charged matter fields. We observe that the onset of the screening instability is associated with fixed-point mergers. We study this phenomenon in a variety of applications. In some cases, the screening of the Wilson lines takes place by dimensional transmutation and the generation of an exponentially large scale. We identify the space of infrared conformal Wilson lines in weakly coupled gauge theories in $3+1$ dimensions and determine the screening cloud due to bosons or fermions. We also study QED in $2+1$ dimensions in the large N_f limit and identify the nontrivial conformal Wilson lines. We briefly discuss ‘t Hooft lines in $3+1$-dimensional gauge theories and find that they are screened in weakly coupled gauge theories with simply connected gauge groups. In non-Abelian gauge theories with S duality, this together with our analysis of the Wilson lines gives a compelling picture for the screening of the line operators as a function of the coupling.

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Introduction.—A natural question in quantum field theory is to understand the space of extended operators. Local operators and their correlation functions have been studied intensely, but relatively little is understood about the space of extended operators.

Relativistic invariance allows us to think of extended operators as either nonlocal operators acting at a given time, or as a modification of the Hamiltonian by an insertion of an impurity in some region of space. Among the possible extended operators, line (one-dimensional) defects provide the simplest examples, as they correspond to a localized impurity in the Hamiltonian picture.

Here we will consider the simplified (yet of great physical importance) scenario when the bulk theory (away from the defect) is a conformal field theory (CFT). Examples of interesting line defects in CFTs include symmetry defects [1–4], spin impurities [5–12], localized external fields [13–17], and ‘t Hooft and Wilson lines in conformal gauge theories.

While the bulk theory is at a fixed point of the renormalization group (RG) flow, in general, a nontrivial RG flow takes place on the defect. One expects on general grounds an infrared fixed point of the line defect, preserving (for a straight or circular defect) the one-dimensional conformal algebra $sl(2,\mathbb{R})$. The infrared fixed point may or may not be trivial. Defect operators are classified according to their $sl(2,\mathbb{R})$ charges. Such a system is commonly referred to as a *defect conformal field theory* (DCFT). See [18] for an introduction to the subject. RG flows on a defect can be triggered by perturbing a DCFT with relevant defect operators. A central question about the dynamics of line defects concerns with their infrared limit, and in particular, if the infrared is screened (i.e., furnishes a trivial DCFT) or not. Under the assumptions of locality and unitarity, RG flows on line defects are constrained by a monotonically decreasing entropy function [19] (in the case of $1+1$ dimensional bulk, see also [20–24]). Another general constraint on RG flows on line defects is due to one-form symmetry: If the line operator is charged under an endable one-form symmetry then it cannot flow to a trivial defect in the infrared (for the definitions and a review, see [25]). Finally, there are constraints on conformal defects due to the bootstrap equations; see [26–29] for recent examples and references.

In this Letter, we focus on a particular class of line operators that naturally exist in conformal gauge theories, i.e., Wilson lines [30]. A Wilson line physically describes the insertion of an (infinitely heavy) charged test particle that moves on a worldline γ:

$$W_R(\gamma) = \mathrm{tr}_R\left[P\exp\left(i\int_\gamma A_\mu dx^\mu\right)\right], \qquad (1)$$

where R is the representation of the gauge group. For a timelike γ, we can think of the Wilson line as a localized charged impurity—it changes the Hamiltonian and the ground state of the system. The definition (1) brings up the following natural question: What is the infrared limit of the Wilson line operator as a function of its representation? In particular, an intriguing property of (1) is that there is no continuous free parameter in the definition of the Wilson line operator. However, as we will argue in this Letter, this does not mean that no RG flow takes place. Physically, this is expected because the electric field sourced by a Wilson line in a sufficiently large representation may destabilize the vacuum. The primary goal of this work is to study this general phenomenon from the viewpoint of defect RG.

We discuss an instability of Wilson lines to screening by charged fields (fermions or bosons) in a variety of physical setups, and relate this to the flow of bilinear operators integrated along the line defect. These bilinear operators are in some sense "missing" from the definition of the Wilson line (1). We calculate the β functions associated with the flow of these new couplings. For small enough charges, the new couplings admit several interesting novel fixed points, while for supercritical charges they flow to large values and dimensional transmutation takes place on the defect. We observe fixed-point mergers taking place at a critical charge. This behavior is reminiscent of how conformality is lost in some QCD-like theories [31–33]. The dimensional transmutation associated with the fixed-point merger implies that the screening cloud is exponentially large.

A brief, qualitative summary of our findings is as follows: We find that, at weak coupling in $3+1$ dimensions, Wilson lines are nontrivial in the infrared for charges (weights) $\lesssim(1/g^2_{\mathrm{YM}})$, while if the charge (weight) exceeds $\sim(1/g^2_{\mathrm{YM}})$ the bilinear operators flow to strong coupling and screen the infrared partially or fully. In $2+1$ dimensions the situation is qualitatively different for weakly coupled charged scalars—a scalar bilinear operator leads to a trivial infrared limit of the Wilson line already for small charges (weights). On the other hand a weakly coupled charged fermion in $2+1$ dimensions does not lead to immediate screening of all Wilson lines.

These instabilities of Wilson lines are general. They were previously discussed for heavy nuclei in QED [34] and for impurities in graphene [35,36]. We will discuss these two cases below in more detail.

We cover several examples in this Letter: (i) Scalar (Fermion) QED$_4$: A relevant bilinear operator must be added to (1) for Wilson lines with charge $|q| > (2\pi/e^2)$ (respectively, $|q| > (4\pi/e^2)$). The coefficient of the bilinear becomes large, and the infrared is qualitatively different from a Coulomb field with q units of charge. It is completely (partially) screened by a condensate cloud, which in some cases is exponentially large. For $|q| < (2\pi/e^2)$ (respectively, $|q| < (4\pi/e^2)$) the bilinear operator is irrelevant but still important to consider, since there are, in general, multiple UV fixed points, which lead to new DCFTs with relevant operators. (ii) Non-Abelian gauge theories in $3+1$ dimensions: As in the QED$_4$ examples, for large enough representations, the infrared limit of the Wilson lines is either completely or partially screened. For subcritical representations, there are potentially several fixed points corresponding to the Wilson line. (iii) QED$_3$ with $2N_f$ Dirac fermions (this is also an important example of a deconfined critical point in condensed matter physics): We find that Wilson lines up to charge $\lesssim 0.56 N_f$ are not screened, while they are screened otherwise. This holds at leading order in the $1/N_f$ expansion.

Finally, we also study similar instabilities for ‘t Hooft lines in four-dimensional conformal gauge theories. Combined with our analysis of the Wilson lines and with non-Abelian electric-magnetic duality, we find a compelling picture for the screening of the line operators as a function of the coupling in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) and in the SU(2) Seiberg-Witten $\mathcal{N}=2$ theory with four fundamental hypers.

An expanded version of this Letter can be found in [37], where the calculations are presented in detail and a few additional examples in $2+1$ dimensions are studied.

Scalar QED$_4$.—We consider massless scalar QED in $3+1$ dimensions with a charge q Wilson line that extends in the time direction at some fixed spatial location $\vec{x}=0$. The action (in mostly minus Minkowski signature) is given by

$$S = \int d^4x\left[-\frac{1}{4e^2}F^2_{\mu\nu} + |D_\mu\phi|^2 - \frac{\lambda}{2}|\phi|^4\right] - q\int dt A_0, \qquad (2)$$

where A_μ is the gauge field, $F_{\mu\nu}$ is the field strength, ϕ is a complex scalar field of charge one, $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, e is the electric charge and λ is a coupling constant. By rescaling $\phi \to \phi/e$ one identifies the following double-scaling limit (Similar double-scaling limits were recently considered in [10,11,38] for different line defects.):

$$e \to 0, \qquad \lambda \to 0, \qquad q \to \infty,$$
$$\frac{\lambda}{e^2} = \text{fixed}, \qquad qe^2 = \text{fixed}, \qquad (3)$$

in which the theory can be treated in the saddle point semiclassical approximation. The generated mass scale associated with QED (i.e., the Landau pole) becomes negligible in this limit and thus one can ignore any RG flow in the bulk and apply the formalism of DCFT. The semiclassical solution yields a Coulomb-potential solution for the gauge field $A_0 = (e^2 q/4\pi r)$ and a vanishing profile for the scalar $\phi = 0$.

For reasons that will soon become clear, let us consider the defect operator $\phi^\dagger\phi$ on the line. Its scaling dimension can be found from the propagator of ϕ fluctuations in the background $A_0 = (e^2 q/4\pi r)$. The defect scaling dimension is inferred from the falloff $\phi \sim r^{(\hat{\Delta}_{\phi^\dagger\phi}/2)-1}$ of spherically symmetric solutions to the linearized equations of motion. One finds (There also exists a second solution for $\hat{\Delta}_{\phi^\dagger\phi}$, whose significance will be explained below.)

$$\hat{\Delta}_{\phi^\dagger\phi} = 1 + \sqrt{1 - \frac{e^4 q^2}{4\pi^2}}. \tag{4}$$

The formula (4) is exact in the limit (3). Note that for $q = 0$, $\hat{\Delta}_{\phi^\dagger\phi} = 2$, which ought to be the case since the bulk and defect scaling dimensions coincide for a trivial defect. For small $e^2 q$, the expression (4) agrees with standard Feynman diagrammatic calculations in perturbation theory.

Let us take $q > 0$ without loss of generality. The scaling dimension (4) implies that for $(e^2 q/2\pi) = 1$ the operator becomes marginal on the defect, while for $(e^2 q/2\pi) > 1$ the scaling dimension becomes complex. Since the bilinear operator is marginal at $(e^2 q/2\pi) = 1$ and slightly irrelevant as we approach $(e^2 q/2\pi) = 1$ from below, we learn that ignoring it in RG flows is inconsistent. In other words, one must consider a more general line defect operator

$$W = P \exp\left(iq \int dt\, A_0 - ig \int dt\, \phi^\dagger \phi\right). \tag{5}$$

In the above, both integrals are evaluated at $\vec{x} = 0$. The parameter g associated with the bilinear line operator has a nontrivial beta function (that can be calculated using methods similar to those given in [39]). The structure of the beta function is shown in Fig. 1. The resulting phase diagram is analogous to the one which is found for

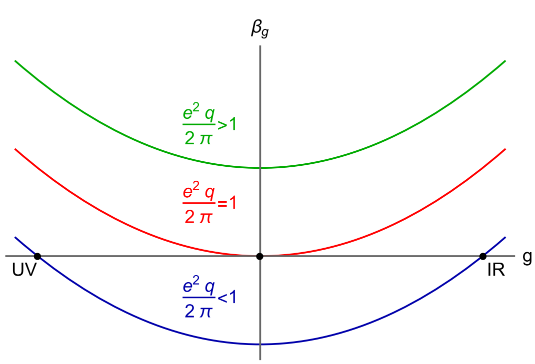

FIG. 1. An illustration of the β function associated with the parameter g in Eq. (5).

double-trace deformations of a theory with an operator in the double-quantization window in AdS/CFT [40–43].

For $(e^2 q/2\pi) < 1$ there are two fixed points, corresponding to two conformal boundary conditions for the scalar near the defect. One of them yields a stable DCFT with no relevant operators, and the other gives an unstable DCFT with one relevant line operator, $\phi^\dagger\phi$. The scaling dimension of the relevant operator is given by

$$\hat{\Delta}_{\phi^\dagger\phi} = 1 - \sqrt{1 - \frac{e^4 q^2}{4\pi^2}}. \tag{6}$$

Starting from the right side of the left-sided fixed point in the blue (bottom) line, one observes a double-trace-like flow from the unstable to the stable DCFT [44]. An analogous RG flow has been recently analyzed in Chern-Simons theories in the ‘t Hooft limit, for Wilson lines in the fundamental representation [47,48].

For $(e^2 q/2\pi) = 1$ the two fixed points merge and the defect operator $\phi^\dagger\phi$ is marginal. For $(e^2 q/2\pi) > 1$ the coupling g flows to $-\infty$ and the infrared has to be analyzed separately. As mentioned in the introduction, the physical behavior described in Fig. 1 is reminiscent of how conformality is believed to be lost in QCD [31]. Here, conformality is lost when $(e^2 q/2\pi) = 1$ in the sense that no DCFTs with finite g exist for $(e^2 q/2\pi) > 1$.

It is of physical interest to analyze the flow when $g \to -\infty$ in order to determine the IR behavior of Wilson lines with sufficiently large charge. Such line operators are only defined with a cutoff r_0, that can be viewed as the nucleus size. In this case one finds that the trivial saddle point where $\phi = 0$ admits a tachyonic instability. The stable saddle point can be obtained numerically. An example is shown in Fig. 2. The electric field starts in the UV (small r) as a Coulomb field and decays until it is completely screened. Accordingly, the scalar profile starts at zero, develops a cloud, and eventually gets screened as well. The integrated charge associated with the scalar condensate is exactly $-q$; i.e., the Wilson line is fully screened. Therefore defects with $(e^2 q/2\pi) > 1$ are trivial DCFTs in the infrared. (In particular, all the bulk one-point correlation functions studied at distances much larger than the size of the cloud vanish.) The same phenomenon and screening mechanism are observed also in the case of $(e^2 q/2\pi) < 1$ if the RG flow starts in the UV from the left side of the unstable fixed point in the blue (bottom) line of Fig. 1.

Analogously to some QCD-like theories, one finds that an exponentially low energy scale is generated when $(e^2 q/2\pi)$ is slightly larger than 1, and dimensional transmutation takes place. This implies that the size of the cloud is exponentially large in units of the cutoff

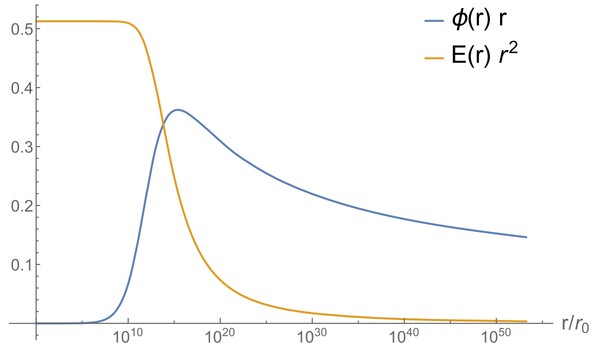

FIG. 2. Plots of the scalar profile (blue) and the electric field (orange) as functions of the distance from the probe charge, all normalized to be dimensionless. The analysis was carried out for $(e^2q/2\pi) = 1.02$ and $\lambda/e^2 = \frac{1}{2}$ by numerically solving the classical equations of motion that follow from (2), with boundary conditions such that the fields decay at infinity, while at a minimal radial position $r = r_0$ we have $F_{0r}|_{r=r_0} = (e^2q/4\pi r^2)$, and $|\phi|^2|_{r=r_0} = 0$, with $\phi \neq 0$ for all $r > r_0$. Different boundary conditions for the scalar field lead to a qualitatively similar plot.

$$R_{\text{cloud}} \sim r_0 \exp\left[\frac{2\pi}{\sqrt{\frac{e^4q^2}{4\pi^2} - 1}}\right]. \tag{7}$$

Equation (7) is derived from the structure of the beta function, analogously to the correlation length in the BKT phase transition [31].

We also comment that while we find two fixed points for $(e^2q/2\pi) < 1$, we do not claim that our analysis of that region is complete. For $(e^2q/2\pi) \leq (\sqrt{3}/2) < 1$ the quartic $|\phi|^4$ becomes relevant in the unstable fixed point, and the dynamics must be reanalyzed. We leave this to future research. In the range $(\sqrt{3}/2) < (e^2q/2\pi) < 1$ the bilinear operator in (5) is the only operator that must be added and our analysis is complete in that regime.

Fermion QED$_4$.—We consider massless fermionic QED in $3+1$ dimensions in the presence of a Wilson line of charge $q > 0$ extending in the time direction. The action is given by

$$S = \frac{1}{e^2}\int d^4x\left[-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}_D \not{D}\psi_D\right] - q\int dtA_0, \tag{8}$$

where ψ_D is a Dirac spinor in four dimensions with charge 1 under the U(1) gauge group. We again work in the semiclassical regime specified by the double-scaling limit in which $e \to 0$, $q \to \infty$, and $e^2q = \text{fixed}$.

The classical saddle point is $A_0 = (e^2q/4\pi r)$, $\psi_D = 0$. Expanding around it we find that there are four spin 1/2 modes that admit two conformal boundary conditions each, for charges such that $(\sqrt{3}/2) < (e^2q/4\pi) < 1$. Criticality occurs for $q = q_c = (4\pi/e^2)$ (which gives $q_c \simeq 137$ in nature [34]). Because of the four independent modes, there are 16 independent bilinear operators that must be taken into account and added to the defect action. These 16 operators can be conveniently classified according to their parity, axial, and SU(2) spin charges; only one bilinear operator preserves all the symmetries.

The resulting phase diagram is a generalization of that shown in Fig. 1. In the subcritical regime, there is one unstable and one stable fixed point which preserve all the symmetries, and various unstable mixed-boundary conditions fixed points breaking some of the symmetries. These are connected by double-trace-like RG flows. Analogously to the flow from the left side of the unstable fixed point in the blue (bottom) line of Fig. 1, there exist also runaway flows for subcritical charges. However, differently from the bosonic case, these flows only lead to screening of up to four units of charge (due to the four independent modes mentioned above). This is a consequence of the Pauli exclusion principle, which forbids the filling of a single state with more than one fermion. Similar to scalar QED$_4$, our analysis of the regime below criticality is not complete, e.g., for $(e^2q/4\pi) < (\sqrt{15}/4)$ a four-fermion term becomes relevant near the unstable fixed points and the dynamics must be reanalyzed.

It is physically interesting to ask what is the deep IR behavior in the supercritical regime when $q > q_c$. Unlike the scalar case, when $q > q_c$ the instability manifests itself in terms of diving states [49]. Physically, these are states that change their nature from particles to holes as we raise q from below to above q_c. Since all hole states must be filled in the ground state, the vacuum develops $q - q_c$ units of screening charge [50]. For $\sqrt{(e^4q^2/16\pi^2) - 1} \ll 1$, each four units of screening charge are localized on successive shells exponentially separated from one another. One may thus account for the backreaction of the Coulomb field perturbatively and compute the radius of the exponentially large fermionic cloud [36]. We find [51]

$$R_{\text{cloud}} \simeq r_0 \exp\left[\frac{2\pi^2}{e^2}\sqrt{\frac{q - q_c}{2q_c}}\right]. \tag{9}$$

As in the scalar case, the screening cloud is exponentially large due to dimensional transmutation.

Non-Abelian gauge theories.—Let us now discuss the implications of our findings for Wilson lines in non-Abelian conformal gauge theories in $3+1$ dimensions. To be concrete, we will refer to either SU(2) gauge theory with maximal $\mathcal{N} = 4$ supersymmetry or the $\mathcal{N} = 2$ SU(2) Seiberg-Witten theory with four fundamental hypers (and for simplicity we take a vanishing θ angle). (The analysis of more general non-Abelian conformal gauge theories is analogous.) Both theories have a coupling constant g_{YM}^2 which can be chosen at will since it is an exactly marginal parameter. We consider a Wilson loop in the $(2s+1)$-dimensional representation of SU(2)

$$W_s = \mathrm{Tr}[Pe^{i\int dx^\mu A_\mu^a T^a}]. \tag{10}$$

This Wilson line is not BPS and it preserves the full continuous flavor symmetry of the model (unlike the BPS lines). Note that BPS lines preserve supersymmetry, which ensures that the ground state has zero energy. Hence there is no instability associated with BPS lines. (At weak coupling, nonsupersymmetric Wilson lines in small representations flow to BPS lines via an SO(6)-breaking deformation [11,52,53].) To analyze the phases of the line operator (10) at weak coupling, we take the double scaling limit $g_{\mathrm{YM}}^2 \to 0$ with $g_{\mathrm{YM}}^2 s = \text{fixed}$ [analogously to (3)]. Repeating the analysis of the fluctuations around this saddle point, we find that a massless scalar field in the spin S representation of the SU(2) gauge group leads to complex scaling dimensions on the line defect for $g_{\mathrm{YM}}^2 sS > 2\pi$. Similar results hold for charged fermions and non-Abelian vector bosons. For $g_{\mathrm{YM}}^2 sS < 2\pi$ several fixed points, as in Fig. 1, exist. Their characterization depends on the matter content.

We see that at weak coupling, there is a finite but large number (of order $1/g_{\mathrm{YM}}^2$) of distinct conformal line defects preserving the full flavor symmetry. As the coupling is increased, presumably only a few Wilson lines remain as nontrivial conformal line operators.

In the $\mathcal{N} = 4$ SYM theory with gauge group SU(2), due to the $\mathbb{Z}_2$ one-form symmetry, the line with $s = 1/2$ cannot be screened and must flow to a conformal line defect. However, in SO(3) gauge theory it is possible that all Wilson lines are in fact screened at strong coupling. In the $\mathcal{N} = 2$ Seiberg-Witten theory there is no one-form symmetry, and it is reasonable to expect that all the Wilson lines are screened at strong coupling.

't Hooft lines.—It is interesting to ask if the expectations of the previous paragraphs are compatible with S duality in these SU(2) gauge theories. To this end we now make some comments about (nonsupersymmetric) 't Hooft lines at weak coupling. In a U(1) gauge theory, a charge q, Lorentz spin ℓ particle moving around magnetic flux n encounters the centrifugal barrier [54]

$$V = \frac{1}{2}|nq|\frac{1-g\ell}{r^2}, \tag{11}$$

where g is the magnetic moment. The formula (11) applies when $|nq|/2 - \ell \geq 0$. For scalars we have a repulsive centrifugal force. For fermions with the standard (weak coupling) magnetic moment $g = 2$ the numerator vanishes and we have no centrifugal barrier, leading to the familiar fermion zero modes, which correspond to marginal bilinear defect operators.

Now let us discuss charged vector bosons. In weakly coupled gauge theories they have $g = 2$. In the case that $|nq| \geq 2$, Eq. (11) is valid and we clearly see that the potential is attractive with coefficient $-\frac{1}{2}|nq|$, which leads to a fall-to-the-center instability of the vector bosons and conjecturally screens the 't Hooft lines. Equivalently, defect operators which are quadratic in the gauge field would have no zero for their beta function and flow to strong coupling, analogously to the behavior of scalars on the green (top) branch of Fig. 1. In $\mathcal{N} = 4$ SYM theory with gauge group SU(2) the minimal monopole has $|n| = 1$ and the vector boson charge is $|q| = 2$. Therefore, even the minimal 't Hooft line is unstable to W-boson condensation at weak coupling, and deep in the infrared it presumably becomes trivial. By contrast, with gauge group $\mathrm{SO}(3)_+$ the charge of the W boson is 1, and hence in the background of the minimal 't Hooft line $|n| = 1$ we have no vector boson instability, and the minimal 't Hooft line should furnish a healthy conformal defect. (It is important that the gauge theory is $\mathrm{SO}(3)_+$ for the minimal 't Hooft line to exist [55].) 't Hooft lines with $|n| > 1$ are all unstable to W-boson condensation, though. These results are consistent with the magnetic one-form symmetries of the SU(2) and $\mathrm{SO}(3)_+$ theories. In summary, recalling that S duality in $\mathcal{N} = 4$ SYM exchanges the SU(2) and $\mathrm{SO}(3)_+$ gauge groups, the absence of 't Hooft lines at weak coupling is precisely dual to our expectations for the screening of Wilson lines as the coupling becomes strong.

For the Seiberg-Witten $\mathcal{N} = 2$ theory with gauge group SU(2) (which is S dual to itself), there are again no 't Hooft lines at weak coupling, and we expect no unscreened Wilson lines at strong coupling either. For more general gauge groups, the labeling of Wilson and 't Hooft lines is explained in [56]. It would be interesting to develop an understanding of which lines are screened as a function of the coupling and the θ angle.

$2+1$ *dimensional critical points.*—The physics of Wilson lines is of interest in $2+1$ dimensions both from the particle theory point of view and also due to the existence of deconfined critical points. We will present here the physics of Wilson lines at the critical point of QED_3 with $2N_f$ charge 1 Dirac fermions. This fixed point is the infrared limit of the Lagrangian

$$\mathcal{L} = \frac{-1}{4e^2}F^2 + i\sum_{a=1}^{2N_f}\bar{\Psi}_a(\partial\!\!\!/ - iA\!\!\!/)\Psi_a. \tag{12}$$

This theory has $\mathrm{U}(1)_T \times \mathrm{SU}(2N_f)$ global symmetry as well as time reversal symmetry. [$\mathrm{U}(1)_T$ stands for the monopole symmetry.] For sufficiently large N_f, (12) flows to an infrared fixed point [57], where the gauge kinetic term is irrelevant. In the presence of a charge q Wilson line, integrating out the fermions, we have the following effective (Euclidean) action

$$S = -2N_f \mathrm{Tr}\log(\partial\!\!\!/ - iA\!\!\!/) + iq\int d\tau A_0. \tag{13}$$

The saddle point in the presence of the Wilson line is fixed up to conformal invariance $F_{\tau i} = iE(x^i/|x|^3)$, where E is some function of q, N_f. Since for large N_f the fermions are approximately free, we can determine when the fermions become unstable by treating them as free fields propagating in the background $F_{\tau i} = iE(x^i/|x|^3)$. This again requires expanding the fermions in fluctuations around the saddle point and reading out the dimension of fermion bilinears from the falloff of the fluctuations. We find that the scaling dimension of fermion bilinears on the line defect is $\Delta = 1 + \sqrt{1 - 4E^2}$ and hence the saddle point $F_{\tau i} = iE(x^i/|x|^3)$ is self-consistent only for $|E| \leq 1/2$. For $q \ll N_f$ we expect $E \ll 1$ and hence one can solve for the saddle point by linearizing the determinant in (13). One finds

$$E = \frac{4q}{\pi N_f} + \mathcal{O}\left(\frac{q^2}{N_f^2}\right). \tag{14}$$

It is more difficult to find the answer for arbitrary $q/N_f \sim 1$. Numerically solving the saddle point equation of (13) we find that $(|q_c|/N_f) \simeq 0.56$, i.e., Wilson lines are not screened for $|q| \leq |q_c|$ (at leading order in $1/N_f$).

The massive phases of QED_3 are $\mathrm{U}(1)_{\pm N_f}$ Chern-Simons theory, which admit N_f lines with nontrivial mutual braiding. It is therefore tempting to assume that the $|q| \leq |q_c|$ conformal lines at the critical point, of which there are (slightly) more than N_f, become the topologically nontrivial lines in the massive phases. Another general lesson from this example is that Wilson lines in $2+1$ dimensional theories with small values of N_f and k would typically have few (or no) conformal Wilson lines. This is analogous to the screening of Wilson lines as the coupling is made strong in $3+1$D.

The critical value $E = 1/2$ is general for weakly coupled fermions in $2+1$ dimensions. In particular it also applies to fermions living on a $2+1$ dimensional plane coupled to a four-dimensional gauge field: a setup which famously describes the low energy limit of graphene [58]. Because of the enhanced Coulomb coupling of these quasiparticles, $E = 1/2$ corresponds to $q \sim 3$ [35,36]; remarkably, this was experimentally confirmed in [59]. Our findings additionally suggest that the charge impurity admits a phase diagram analogous to the one discussed in QED_4, including the existence of new UV fixed points and runaway flows at subcritical q.

As a final comment, we notice that for a weakly coupled charged scalar in a Coulomb field background the trivial saddle point is always unstable in $2+1$ dimensions. This is because the free bulk scaling dimension of the scalar bilinear is 1, hence the scalar sits at the fixed-point merger already at $q = 0$. (Indeed, the bilinear defect perturbation of free field theory is marginally irrelevant [15].)

These facts about conformal lines in bosonic and fermionic $2+1$ dimensional theories could be important for 3D dualities, in the spirit of [47,48]. We leave this subject for the future.

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