Relaxation of Phonons in the Lieb-Liniger Gas by Dynamical Refermionization

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Motivated by recent experiments, we investigate the Lieb-Liniger gas initially prepared in an out-of-equilibrium state that is Gaussian in terms of the phonons, namely whose density matrix is the exponential of an operator quadratic in terms of phonon creation and annihilation operators. Because the phonons are not exact eigenstates of the Hamiltonian, the gas relaxes to a stationary state at very long times whose phonon population is *a priori* different from the initial one. Thanks to integrability, that stationary state needs not be a thermal state. Using the Bethe-ansatz mapping between the exact eigenstates of the Lieb-Liniger Hamiltonian and those of a noninteracting Fermi gas and bosonization techniques we completely characterize the stationary state of the gas after relaxation and compute its phonon population distribution. We apply our results to the case where the initial state is an excited coherent state for a single phonon mode, and we compare them to exact results obtained in the hard-core limit.

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Introduction.—Phonons are a central concept in the field of quantum gases. They are quantized sound waves, or collective phase-density excitations, that arise in low-energy and long-wave-length description of quantum gases, e.g., in Bogoliubov theory of Bose-Einstein condensates in $D \geq 2$ spatial dimensions [1], or, in 1D, in Bogoliubov theory for quasicondensates [2,3] and more generally in Luttinger liquid theory [4–6]. Phonons are routinely used to analyze experiments with out-of-equilibrium quantum gases, such as the dynamics generated by a quench of the interaction strength in a 1D Bose gas [7], or by its splitting into two parallel clouds [8,9], or by a quench of the external potential [10]. In general, the description in terms of phonons accounts remarkably well for the observed short time dynamics [9,11]. Crucially, the out-of-equilibrium states produced in these experimental setups are phononic Gaussian states with expectation values of phononic operators obeying Wick's theorem. This is because they are obtained from a thermal equilibrium state, which is itself described by a Gaussian density matrix, by acting on it with linear or quadratic combinations of density and/or phase field.

However, although phonons are exact eigenstates of the effective low-energy Hamiltonian, they are only approximations of the true eigenstates of the microscopic Hamiltonian. Thus, phonons have finite lifetime [12–16] and, at long times, the phonon distribution should evolve. In ergodic systems, this evolution would consist in the relaxation toward thermal equilibrium. But what if the microscopic system is integrable? In this Letter, we investigate the integrable Lieb-Liniger model of 1D Bosons with contact repulsive interactions [17]. We express the phonons in terms of the true, i.e., infinite-lifetime, quasiparticles of the Lieb-Liniger model. We can then characterize the final stationary state of the system after relaxation and we relate the phonon mode occupations to the ones in the initial state.

Sketch of the main result.—The Hamiltonian of the Lieb-Liniger model is

$$H = \int_0^L dx\, \Psi^\dagger \left(-\frac{1}{2}\partial_x^2 + \frac{c}{2}\Psi^\dagger\Psi \right)\Psi, \tag{1}$$

with second-quantized bosonic operators obeying commutation relations $[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$. Here, $c > 0$ is the repulsion strength, L is the length of the system (we use periodic boundary conditions) and we use units such that $\hbar = m = 1$. For N atoms the average density is $\rho_0 = N/L$. The density fluctuation and current operators are

$$\begin{aligned} \delta\rho(x) &= \Psi^\dagger(x)\Psi(x) - \rho_0, \\ J(x) &= \frac{i}{2}\{[\partial_x \Psi^\dagger(x)]\Psi(x) - \Psi^\dagger(x)[\partial_x \Psi(x)]\}. \end{aligned} \tag{2}$$

At low temperature, it is customary to think of low-energy and long wavelength excitations above the ground state as quantized sound waves (phonons) that move to the right or to the left at the sound velocity v. The chiral combinations

$$J_{R/L}(x) = \frac{1}{2}[v\,\delta\rho(x) \pm J(x)] \tag{3}$$

are the currents carried by right-moving (R) or left-moving (L) quasiparticles, with Fourier modes

$$J_R(x) = \frac{v\sqrt{K}}{L}\sum_{n>0}\sqrt{n}\left(e^{i\frac{2\pi nx}{L}}A_{R,n} + e^{-i\frac{2\pi nx}{L}}A^\dagger_{R,n}\right), \tag{4}$$

with $n > 0$ (a similar definition holds for $A_{L,n}$). Here, $K = \pi\rho_0/v$ is the Luttinger parameter. Acting with $A^\dagger_{R/L,n}$ on the ground state $|0\rangle$, one generates R/L phonons. We stress that the excited states generated this way are only *approximations* of the true eigenstates of the Lieb-Liniger Hamiltonian (1). The lifetime of phonons may be large, but it is not infinite, so the phonon population will evolve, until it ultimately reaches a stationary value, governed by the true eigenstates of the Lieb-Liniger model.

Our main result is a general formula that relates the phonon population at infinite time to the one in the initial state. The latter is assumed to be Gaussian in terms of phonons, such that correlation functions of products of operators $J_{R/L}(x)$ reduces to sums of products of one- and two-point correlation functions by Wick's theorem. In the special case of a translation-invariant initial state $\langle.\rangle_0$, populated by R phonons, parametrized by a single function $g(x)$ via [18]

$$\langle J_R(x)J_R(y)\rangle_0 = -\frac{\rho_0 v}{4\pi}\partial_x^2 g(x-y), \tag{5}$$

and $\langle J_R(x)\rangle_0 = 0$, our result is that the two-point function ultimately relaxes to

$$\langle J_R(x)J_R(y)\rangle_\infty = \frac{\pi\rho_0 v}{L^2}\exp[2g(x-y)]. \tag{6}$$

Thus, the phonon population evolves, unless the function $g(x)$ satisfies $\partial_x^2 g = -(4\pi^2/L^2)e^{2g}$. One solution to this equation is the thermal distribution at inverse temperature $\beta \ll L/v$, for which $g(x-y) = -\log\{(2\beta v/iL)\sinh([\pi(x-y)]/\beta v)\}$ [19]. So the thermally occupied phonon modes will not evolve, but more general initial states will show a relaxation phenomenon.

In the rest of this Letter we derive Eq. (6) and its generalization to initial Gaussian phononic states not necessarily translationally invariant. We compare our predictions to exact numerical results obtained in the hard-core (Tonks-Girardeau) limit for a state with a single phononic mode initially displaced. We conclude by discussing perspectives for experimental observation of the evolution of phonon populations.

Eigenstates of the Lieb-Liniger model and Bethe fermions.—For even N ((resp. odd N)), an N-particle eigenstate of (1) is specified by an ordered set of half-integers or integers $I_1 < I_2 < \cdots < I_N$ which uniquely determines the set of N rapidities $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ via the Bethe equations [17,20,21]

$$\lambda_a + \frac{1}{L}\sum_{b=1}^{N}\arctan\left(\frac{\lambda_a - \lambda_b}{c}\right) = \frac{2\pi}{L}I_a. \tag{7}$$

The energy of that eigenstate is $E_{\{\lambda_a\}} = \sum_a \lambda_a^2/2$ and the corresponding wave function is $\langle\text{vacuum}|\prod_{j=1}^N \Psi(x_j)|\{\lambda_a\}\rangle \propto \sum_\sigma \mathcal{A}_\sigma e^{i\sum_a \lambda_{\sigma(a)}x_a}$, where the sum is over all permutations σ of N elements and $\mathcal{A}_\sigma = \prod_{a>b}[1 - ic\,\mathrm{sgn}(x_a - x_b)/(\lambda_{\sigma(a)} - \lambda_{\sigma(b)})]$. In the following, we assume N even. The ground state corresponds to densely packed Bethe half-integers $\{I_a^{(0)}\} = \{-[(N-1)/2], -[(N-1)/2]+1, \ldots, [(N-1)/2]\}$.

Equation (7) provides a one-to-one mapping between the eigenstates $|\{I_a\}\rangle$ of the Lieb-Liniger model and the eigenstates of N noninteracting fermions with momenta $2\pi I_a/L$, $a = 1, \ldots, N$. This mapping preserves the total momentum $P_{\{\lambda_a\}} = \sum_a \lambda_a = (2\pi/L)\sum_a I_a$. It is natural to introduce fermion operators b_J that act on the normalized eigenstate $|\{I_a\}\rangle$ by removing a Bethe half-integer J from the set $\{I_a\}$, if it is present, and by annihilating the state otherwise. Conversely, the operator $b_J^\dagger$ inserts J in the set $\{I_a\}$ unless it is already present. The eigenstate corresponding to the modified state is then multiplied by $(-1)^{n_{I_a<J}}$, where $n_{I_a<J}$ is the number of elements of $\{I_a\}$ smaller than J, to enforce the correct anticommutation relations for the "Bethe fermion" operators $b_J^\dagger/b_J$ [22].

All eigenstates of (1) with a total atom number N are generated by acting on the ground state with an equal number of Bethe fermion creation and annihilation operators. In particular, the low energy states are obtained by acting with creation or annihilation operators close to the R or L Fermi points. For a half-integer l, we define the operators

$$c^\dagger_{R,l} = b^\dagger_{\frac{N}{2}+l}, \qquad c^\dagger_{L,l} = b^\dagger_{-\frac{N}{2}-l}. \tag{8}$$

The low energy eigenstates $|\psi\rangle$ with q R excitations are of the form

$$|\psi\rangle = \prod_{i=1}^{q} c^\dagger_{R,l_i}\prod_{j=1}^{q} c_{R,m_j}|0\rangle, \tag{9}$$

for sets of half-integers $l_j > 0$ and $m_j < 0$, with $q \ll N$, and $|l_j|, |m_j| \ll N$. To lighten our formulas, we consider eigenstates with R excitations only; it is straightforward to generalize our results to include also L excitations. The energy of the low-energy eigenstate (9) is $E = \sum_{j=1}^q[\epsilon(l_j) - \epsilon(m_j)]$ with the "dressed" energy [23] given approximately by the quadratic dispersion relation

$\epsilon(l) = (2\pi v l/L) + (1/2m^*)(2\pi l/L)^2 + O(1/L^3)$ with the effective mass $m^* = [1 + (\rho_0/v)\partial v/\partial\rho_0]$ [24–26].

Phonons.—We will make extensive use of the following simple formula for the matrix elements of $A^\dagger_{R,n}$ [Eq. (4)] between two low-energy states of the form (9) in the thermodynamic limit,

$$\langle\psi_2|A^\dagger_{R,n}|\psi_1\rangle \underset{N\to\infty}{=} \frac{1}{\sqrt{n}}\langle\psi_2|\sum_{l\in\mathbb{Z}+\frac{1}{2}} c^\dagger_{R,n+l}c_{R,l}|\psi_1\rangle. \tag{10}$$

Equation (10) follows from known results about form factors of the density operator in the Lieb-Liniger model, see Refs. [27–30] and Supplemental Material [31]. It shows that a phonon created by $A^\dagger_{R,n}$ is a coherent superposition of Bethe fermion particle-hole pairs and that phonons are obtained by bosonization of the Bethe fermions [19,37]. This implies that $A_{R,n}$ and $A^\dagger_{R,n}$ satisfy bosonic canonical commutation rules [19,37]

$$[A_{R,n}, A^\dagger_{R,n'}] = \delta_{n,n'}. \tag{11}$$

The introduction of the Bethe fermions to describe low-energy eigenstates of the Lieb-Liniger model, and the identification of the phonons as the Bosons obtained by bosonization of these Fermions, are the key ingredients of our derivation.

Bosonization allows us to invert Eq. (10) and represent the Bethe fermion operators $c^\dagger_R(x) = \sum_l e^{-i(2\pi l x/L)} c^\dagger_{R,l}/\sqrt{L}$ as

$$\langle\psi_2|c^\dagger_R(x)c_R(y)|\psi_1\rangle = \langle\psi_2|:e^{-i\varphi_R(x)}::e^{i\varphi_R(y)}:|\psi_1\rangle/L, \tag{12}$$

where the notation :.: denotes normal ordering and $\varphi_R(x) = -i\sum_{n>0}(e^{i2\pi nx/L}A^\dagger_{R,n} - e^{-i2\pi nx/L}A_{R,n})/\sqrt{n}$ is the chiral field, related to the chiral current by $J_R(x) = v\sqrt{K}/(2\pi)\partial_x\varphi_R(x)$. The bosonization formulas require that $\langle\varphi_R(x)\varphi_R(y)\rangle$ has the same short-distance logarithmic divergence as the one in the ground state, $\langle\varphi_R(x)\varphi_R(y)\rangle = -\log(2\pi(x-y+i\epsilon)/iL)$ as $y\to x$ [38], which implies that the phonon population $\langle A^\dagger_{R,n}A_{R,n}\rangle$ decays at least exponentially with n.

Initial state preparation and short time dynamics.—For short times the nonlinearity of the fermionic spectrum has a small effect and, as one restricts to low-energy and long wavelength states, one can approximate the Lieb-Liniger Hamiltonian, Eq. (1), by the Luttinger liquid Hamiltonian

$$H \simeq v\sum_{n>0}\frac{2\pi n}{L}(A^\dagger_{R,n}A_{R,n} + A^\dagger_{L,n}A_{L,n}). \tag{13}$$

This Hamiltonian permits efficient calculation of equal-time correlation functions at thermal equilibrium [39,40]. As explained in the introduction, it also describes successfully several experiments probing out-of-equilibrium dynamics [7–9]. In those experiments, the initial state is Gaussian in terms of phononic operators which motivates our choice to consider an initial phononic Gaussian state. The latter is characterized by the one- and (connected) two-point correlations functions of the chiral currents, that we parametrize in terms of functions $f(x)$ and $g(x,y)$ as

$$\langle J_R(x)\rangle_0 = v\frac{\sqrt{K}}{2\pi}\partial_x f_R(x), \tag{14}$$

$$\langle J_R(x)J_R(y)\rangle_0^{\mathrm{conn}} = v^2\frac{K}{(2\pi)^2}\partial_x\partial_y g_{RR}(x,y), \tag{15}$$

and similarly for $\langle J_L(x)\rangle_0$ and $\langle J_L(x)J_L(y)\rangle_0^{\mathrm{conn}}$, as well as for the possible cross correlation $\langle J_R(x)J_L(y)\rangle_0^{\mathrm{conn}}$. Here, $\langle J_R(x)J_R(y)\rangle^{\mathrm{conn}} = \langle J_R(x)J_R(y)\rangle - \langle J_R(x)\rangle\langle J_R(y)\rangle$. Higher order correlation functions are obtained from those by Wick's theorem for the phononic operators.

Long time dynamics and relaxation.—The key point of this Letter is that the phononic states are not eigenstates of the Lieb-Liniger Hamiltonian and therefore are not well adapted to study the long time evolution. This is clearly seen by examining the phase difference accumulated between different particle-hole states entering a phononic excitation given by the right-hand side of Eq. (10): one can estimate the relevant timescale for the dephasing of a single phonon with momentum n as $t_{\mathrm{deph}} = \hbar m^*(L/2\pi n)^2$.

The long-time behavior of the Lieb-Liniger gas is now well established [41,42]. The system shows a relaxation phenomenon: as long as local observables are concerned, the density matrix at long times is obtained from the initial one by retaining only its diagonal elements in the Bethe-ansatz eigenbasis. Moreover, according to the generalized eigenstate thermalization hypothesis [43–45] which states that all eigenstates are locally identical provided they have the same coarse grained rapidity distribution $\rho(\lambda) = (1/L)\sum_a\delta(\lambda-\lambda_a)$, all diagonal density matrix sufficiently peaked around the correct rapidity distribution [43–45] are acceptable. In this Letter, we choose the Gaussian density matrix [46,47]

$$\hat{\rho}_\infty \propto \exp\left(\sum_I \beta_I b^\dagger_I b_I\right), \tag{16}$$

where the distribution of Bethe half-integers imposed by the Lagrange multipliers β_I ensures the correct distribution of rapidities [48]. A commonly used alternative is the generalized Gibbs ensemble (GGE) in terms of the rapidity distribution. Both ensembles are equally valid as long as local quantities are concerned [49].

Extracting the numbers β_I, or equivalently the expectations $\langle b^\dagger_I b_I\rangle$ from the correlation functions Eqs. (14) and (15) which parametrize the initial phononic Gaussian state, is, generally speaking, an excruciating task. However, for an initial state in the low energy sector, only fermionic states close to the Fermi points are affected and calculation of $\langle b_I^\dagger b_I \rangle$, which reduces to finding the distributions $\langle c_{R,n}^\dagger c_{R,n} \rangle$, $\langle c_{L,n}^\dagger c_{L,n} \rangle$, is a much easier task as it can be done by using bosonization.

To do this, we concentrate on the right movers and introduce $G_R(\xi)$ defined by

$$G_R(\xi) = \frac{1}{L} \sum_l e^{-i2\pi l \xi/L} \langle c_{R,l}^\dagger c_{R,l} \rangle, \tag{17}$$

which can be rewritten as the spatially averaged fermion two-point correlation function

$$G_R(\xi) = (1/L) \int du \langle c_R^\dagger(u + \xi/2) c_R(u - \xi/2) \rangle. \tag{18}$$

The crucial observation is that since G_R is time independent it can be evaluated using the initial state. Since the latter is a phononic Gaussian state, one can use Wick's theorem for φ_R in Eq. (12) to evaluate of the two-point fermionic correlation function. One obtains, using $\langle \varphi_R(x) \rangle_0 = f_R(x)$ and $\langle \varphi_R(x) \varphi_R(y) \rangle_0^{\mathrm{conn}} = g_{RR}(x, y)$,

$$\langle c_R^\dagger(x) c_R(y) \rangle_0 = \frac{1}{L} \exp\{-i[f_R(x) - f_R(y)]\} \times \exp\left[g_{RR}(x, y) - \frac{1}{2} g_{RR}^{\mathrm{reg}}(x) - \frac{1}{2} g_{RR}^{\mathrm{reg}}(y)\right], \tag{19}$$

where $g_{RR}^{\mathrm{reg}}(x) = \lim_{y \to x} g_{RR}(x, y) + \log[2\pi(y - x)/(iL)]$ is independent of the short distance cutoff ϵ. The function $G_R(\xi)$ is obtained by injecting Eq. (19) into Eq. (18). The population of the Bethe fermions, which entirely characterizes the state after relaxation, is then computed inverting Eq. (17). We dub this crucial intermediate result "dynamical refermionization."

Consequence: Relaxation of phonon population.—Mean values of products of phononic operators after relaxation are computed expressing them in terms of fermionic operators thanks to Eq. (10), and using Wick's theorem, valid for the fermionic Gaussian density matrix Eq. (16). In particular, to compute $\langle J_R(x) J_R(y) \rangle_\infty$, we use the relation $J_R(x) = (v\sqrt{K}/L) \sum_l \sum_{n \neq 0} e^{i2\pi n x/L} c_{R,l+n}^\dagger c_{R,l}$, obtained injecting Eq. (10) into (4). This gives, for $x \neq y$,

$$\langle J_R(x) J_R(y) \rangle_\infty = -K v^2 G_R(x - y) G_R(y - x). \tag{20}$$

Also $\langle J_R(x) \rangle_\infty = 0$ due to translational invariance. The phonon population reads

$$\langle A_{R,n}^\dagger A_{R,n} \rangle_\infty = \frac{1}{n} \sum_l \langle c_{R,l+n}^\dagger c_{R,l+n} \rangle (1 - \langle c_{R,l}^\dagger c_{R,l} \rangle). \tag{21}$$

Note that one should consider its weighted sum over a small but nonvanishing width in $k = 2\pi n/L$, to ensure that the quantity is local so Eq. (16) applies.

Equations (20) and (21) constitute the main result of this Letter. The translation-invariant case, Eq. (6), announced earlier, is obtained by using $f_R(x) = 0$, $g_{RR}(x, y) = g(x - y)$, $g_{\mathrm{RR}}^{\mathrm{reg}}(0) = 0$. We stress that the relaxed state of the system is no longer Gaussian in terms of the phonons. Higher order phononic correlation functions require more calculations—one should express them in terms of the fermions, and then use Wick's theorem for fermions.

Example: Application to the case of a single excited phononic mode.—Let us consider the situation where the initial state is obtained from the ground state by a displacement of an R phonon: $f_R(x) = A \cos(k_0 x)$ with the amplitude A and the wave vector k_0, while keeping $g_{RR}(x, y)$ equal to its ground-state value. Figure 1 shows the Bethe fermion distribution obtained from dynamical refermionization, Eqs. (17)–(19). For small amplitudes A, one observes plateaus of width k_0 which reflect the quantization of phonons [31]. As A increases, more plateaus appear, and for large A it becomes the smooth profile $\langle c_{R,n}^\dagger c_{R,n} \rangle = (1/\pi) \arccos[2\pi n/(L A k_0)]$ expected semiclassically [31]. The bottom row of Fig. 1 shows the energy of each phononic mode after relaxation. The difference of the distributions of R and L phonons is a strong signature of the nonthermal nature of the relaxed system. Within the space of R phonons, redistribution of energy among phonons is found to be very efficient: the relaxed distribution is close, albeit not identical, to that expected for a thermal state. We compare the dynamical refermionization predictions to exact results in the asymptotic regime of hard-core bosons ($c \to \infty$): in this regime, the Hamiltonian in terms of Bethe fermions is that of noninteracting fermions and the current operator J_R is equal to that for the Bethe fermions, which enables exact calculations. The initial state is obtained as the ground state of the Hamiltonian $H + (A k_0/\sqrt{K}) \int dx J_R(x) \sin(k_0 x)$. As seen in Fig. 1, results are in excellent agreement with the predictions of dynamical refermionization.

Experimental perspectives.—Our predictions can be tested in cold atom experiments, where initial out-of-equilibrium states can be generated in various ways. By quenching the longitudinal potential from a long-wavelength sinusoidal potential to a flat potential [10], one produces displaced phononic states, corresponding to nonvanishing functions f_R, f_L. A quench of the interaction strength will produce two-modes squeezed phononic states [7], a situation which corresponds to $f_R = f_L = 0$, but to modified functions g_{RR}, g_{LL}, g_{RL}. Alternatively, modulating the coupling constant with time will parametrically excite only part of the phononic spectrum [50]. In the above scenarios the prepared initial state is symmetric under the exchange of R and L phonons. To break this symmetry, one could expose the gas to a potential $V(x) = V_0 \cos(k_0 x - v k t)$ for some short time duration: then only the R phonons would be resonantly excited.

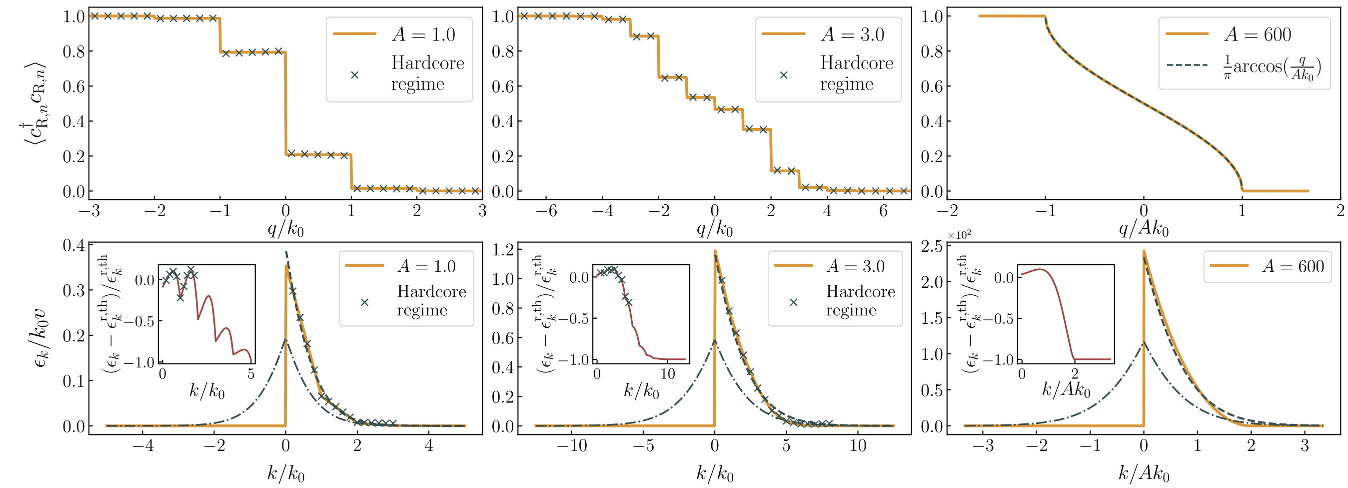

FIG. 1. Relaxation of a single displaced phononic mode, characterized by $f_R(x) = A\cos(k_0 x)$. Solid yellow lines are obtained by dynamical refermionization, i.e., Eqs. (17)–(19), (21) in the thermodynamic limit $L \to \infty$. Top: Bethe-fermions occupations in the vicinity of the Fermi sea right-border. Bottom: energy in each phononic mode after relaxation, $\epsilon_k = |k| v \langle A^\dagger_{R,n} A_{R,n}\rangle_\infty$, where $k = 2n\pi/L$ (respectively, $-2n\pi/L$) for right movers (respectively, left movers). The dash-dotted black lines are the results expected if the system were relaxing to thermal equilibrium. Dashed line shows the distribution $\epsilon_k^{r,\mathrm{th}}$ corresponding to thermal redistribution of the initial energy within the right-mover phonons only and the insets highlight its difference with the dynamical refermionization results. Crosses are exact numerical results obtained in the limiting case of hard-core bosons, performed for 100 atoms (see text), using $k_0 = 5 \times 2\pi/L$ in the left figure (respectively, $k_0 = 2 \times 2\pi/L$ in the central figure) for $A = 1.0$ (respectively, $A = 3.0$), and the relaxed phonon population is computed in the fermionic diagonal ensemble. Those numerical results are performed for a lattice gas of 1000 sites, which represents faithfully the continuous gas only for small k.

To probe the phonon distribution after relaxation, one possibility is to measure the *in situ* long-wavelength density fluctuations [51] and access $\delta\rho(x) = [J_R(x) + J_L(x)]/v$, see Eq. (2). Alternatively, one can use the density ripple techniques to probe the long wavelength phase fluctuations [7,52], whose gradient is the velocity field proportional to $J(x) = J_R(x) - J_L(x)$. The above methods, however, do not discriminate between right and left movers. In order to probe selectively R phonons, one needs to probe the dynamics, for instance using sequences of nondestructive images [53]. Note that the results of this Letter concern only the long wavelength fluctuations in the system: they exclude predictions concerning short range correlations.

At very long times, one expects integrability breaking perturbations to bring the system to a thermal equilibrium. However, such perturbations can be weak enough to have negligible effect during the relaxation time of the Lieb-Liniger phonons, as is observed experimentally in Ref. [10] and modeled recently in Ref. [54].

Interestingly, the occupation of Bethe fermions $\langle c^\dagger_{R,n} c_{R,n}\rangle$—which, in this Letter, is used as an intermediate result—could also be measured experimentally. To measure it, one could first perform an adiabatic increase of the repulsion strength c, which preserves the distribution of Bethe fermions [55], until the hard-core regime is reached. In this regime the distribution of Bethe fermions is the same as the rapidity distribution, which can be measured by a 1D expansion [56,57].

Prospects.—This work calls for further investigations in several directions. First, one could investigate higher order functions of the chiral currents or of the phonons populations to show that the relaxed state is non-Gaussian with respect to the phonons. Second, our predictions call for numerical studies of relaxation in the Lieb-Liniger model away from the strongly interacting regime. Finally, as discussed above, the predictions of this Letter are to be confirmed experimentally.

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