Realistic Protocol to Measure Entanglement at Finite Temperatures

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It is desirable to relate entanglement of many-body systems to measurable observables. In systems with a conserved charge, it was recently shown that the number entanglement entropy (NEE)—i.e., the entropy change due to an unselective subsystem charge measurement—is an entanglement monotone. Here we derive finite-temperature equilibrium relations between Rényi moments of the NEE, and multipoint charge correlations. These relations are exemplified in quantum dot systems where the desired charge correlations can be measured via a nearby quantum point contact. In quantum dots recently realizing the multichannel Kondo effect we show that the NEE has a nontrivial universal temperature dependence which is now accessible using the proposed methods.

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Entanglement is a key concept in quantum mechanics, and its measures are being repeatedly used in many areas of physics, including basic quantum theory, quantum information, and many-body physics. For systems at zero temperature or those described by a pure state, one can quantify the entanglement by the system's entanglement entropy. However, the task of measuring entanglement entropy in a many-body system is daunting as, in principle, one needs to measure the full density matrix, requiring access to all degrees of freedom (see, however, Refs. [1,2]). As a result only very few experiments have been able to measure the entanglement entropy in specific systems of cold atomic Bose-Hubbard chains [3,4] and trapped-ion quantum simulators [5,6]. Thus the measurement of entanglement entropy remains one of the outstanding challenges in many-body physics. The task of experimental quantification of entanglement at finite temperatures is even more challenging, as the standard entanglement entropy measure is no longer uniquely defined in this case.

An important step forward was the recognition [7–14] of a relationship between entanglement entropy and fluctuations in condensed matter systems. For example, for a system with a conserved total number of particles $N = N_A + N_B$ a ground state with fluctuating number of particles in subsystem A directly implies a nonlocal quantum correlation with subsystem B. These works focused mainly on generating [7,8], measuring [9,10], and understanding of the scaling of many-body quantum entanglement [11–13]. In fact, for noninteracting fermions, an exact relation between current moments and entanglement entropy was found [10]. Note, however, that in general, such current correlations do not fully capture the charge-neutral contributions [9], often referred to as configuration entropy [15–19].

Yet, while all these previous works considered pure states, general condensed matter systems are described by mixed states, e.g., due to finite temperature. Then, as mentioned above, the entanglement entropy has to be replaced by generalized entanglement measures, such as negativity [20,21] or number entanglement entropy (NEE) [22,23]. However, unlike the case of a pure state, mentioned above, relations between mixed state entanglement quantifiers and charge or current correlations are not known, which makes the task of measuring entanglement at finite temperatures even harder. Previous successful attempts to quantify entanglement for mixed states relied on random unitary measurements [6,24–27], which are hard to apply in general condensed matter systems. Here, we provide exact thermodynamic relations between moments of the NEE and measurable subsystem charge fluctuations. Consequently, measuring these quantities will facilitate obtaining information about entanglement in many-body condensed matter systems at finite temperatures.

The NEE ΔS is defined as the entropy change of a density matrix ρ upon an unselective measurement of N_A [23],

$$\Delta S = -\mathrm{Tr}[\rho_{\hat{N}_A} \log \rho_{\hat{N}_A}] + \mathrm{Tr}[\rho \log \rho], \qquad (1)$$

where $\rho_{\hat{N}_A} = \sum_{N_A} \Pi_{N_A} \rho \Pi_{N_A}$ and Π_{N_A} is a projector to a subspace with subsystem charge N_A. A similar quantity in terms of the reduced density matrix was studied in Ref. [28]. Equivalently, $\rho_{\hat{N}_A}$ is obtained from ρ by annihilating all off-diagonal matrix elements with respect to N_A. When N_A uniquely specifies the subsystem's state, ΔS coincides with the entanglement entropy for a pure state (one can easily generalize NEE to include spin and other globally conserved quantum numbers). Note that even for a mixed state, ΔS does not depend on which of the subsystems one traces over, unlike the standard entanglement entropy.

In fact, our definition in Eq. (1) is a quantifier of coherence [29], associated with a conserved quantity [22]. As the entanglement entropy reflects the coherence in the system, the NEE measures charge coherence and cannot be simply extracted from the charge distribution $P(N_A)$. Instead, as we show below, one has to measure charge correlation functions in order to quantify the NEE in the system.

The difficult task of directly measuring the entanglement entropy or negativity in many-body systems was circumvented in the literature by suggestions to measure instead its first Rényi moments [6,30–34]. We will apply a similar strategy for the NEE. The Rényi moments of ΔS are defined by

$$\Delta S^{(\alpha)} = \frac{1}{1-\alpha}\log\left(\frac{\mathrm{Tr}[\rho^{\alpha}_{\hat{N}_A}]}{\mathrm{Tr}[\rho^{\alpha}]}\right). \tag{2}$$

While the Rényi moments of the NEE do not inherit its monotonicity, for general mixed states, ($\mathrm{Tr}[\rho^{\alpha}] \neq 1$), it can be shown that $\Delta S^{(\alpha)} > 0$ is a sufficient condition for quantum entanglement or quantum coherence [35]. One can formally extract the NEE from its moments (see also Refs. [39,40]) via $\Delta S = \lim_{\alpha\to 1}\Delta S^{(\alpha)}$ [41]. Hereafter we write the NEE as $\Delta S^{(1)}$ for convenience. For pure states $\mathrm{Tr}[\rho^{\alpha}] = 1$ and $\Delta S^{(\alpha)} = [1/(1-\alpha)]\log\sum_{N_A}[P(N_A)]^{\alpha}$ becomes the Rényi entropy of the subsystem charge distribution $P(N_A)$. As mentioned above, for general mixed states, the NEE and its Rényi moments cannot be obtained from $P(N_A)$ alone (see examples in [35]). When the subsystem charge N_A uniquely specifies its state, $\Delta S^{(\alpha)}$ coincides with the usual Rényi entropy.

Typically, Rényi moments are obtained using multiple copy methods [3,30–33,43–45], which are quite difficult to generate experimentally, especially in condensed matter systems. Here, focusing on general thermal states, rather than physically implementing α copies of the system, we demonstrate that the αth Rényi moment of the NEE of a thermal state at temperature T is directly related to a correlation function of the same system at temperature T/α.

Substituting $\rho(T) = Z(T)^{-1}\sum_i e^{-E_i/T}|i\rangle\langle i|$, where $Z(T) = \sum_i e^{-E_i/T}$, and $|i\rangle$ are the eigenstates of H, $H|i\rangle = E_i|i\rangle$, for all the factors of ρ in the numerator in Eq. (2), we find

$$\begin{aligned}\mathrm{Tr}[(\rho_{\hat{N}_A})^{\alpha}] &= \sum_{N_A}\mathrm{Tr}[(\rho\Pi_{N_A})^{\alpha}] \\ &= \sum_{j_1,\dots j_\alpha}\frac{e^{-(E_{j_1}+\cdots+E_{j_\alpha})/T}}{Z(T)^{\alpha}}\langle j_1|\Pi_{N_A}|j_\alpha\rangle \\ &\quad\times\langle j_\alpha|\Pi_{N_A}|j_{\alpha-1}\rangle\dots\langle j_2|\Pi_{N_A}|j_1\rangle. \end{aligned}\tag{3}$$

We now note that this expression is identical up to a multiplicative factor of $Z(T')/Z(T)^{\alpha}$ to the α-point imaginary time Green's function of projection operators,

$$\begin{aligned}&\langle\Pi_{N_A}(-i\tau_{\alpha-1})\Pi_{N_A}(-i\tau_{\alpha-2})\cdots\Pi_{N_A}(-i\tau_1)\Pi_{N_A}(0)\rangle_{T'} \\ &\quad=\sum_{j_1,\dots j_\alpha}\frac{e^{-E_{j_1}/T'}}{Z(T')}\langle j_1|\Pi_{N_A}|j_\alpha\rangle\cdots\langle j_2|\Pi_{N_A}|j_1\rangle \\ &\qquad\times e^{(E_{j_1}-E_{j_\alpha})\tau_{\alpha-1}}e^{(E_{j_\alpha}-E_{j_{\alpha-1}})\tau_{\alpha-2}}\cdots e^{(E_{j_3}-E_{j_2})\tau_1},\end{aligned}\tag{4}$$

evaluated at temperature $T' = T/\alpha$ and imaginary times $\tau_j = j/T$. We also note that the denominator of Eq. (2) can be written as $\mathrm{Tr}[\rho^{\alpha}] = Z(T/\alpha)/Z(T)^{\alpha}$. Combining these results, we obtain

$$\begin{aligned}\Delta S^{(\alpha)} = \frac{1}{1-\alpha}\log\Bigg[\sum_{N_A}\Bigg\langle\Pi_{N_A}\left(\frac{\alpha-1}{iT}\right)\Pi_{N_A}\left(\frac{\alpha-2}{iT}\right)\cdots \\ \times\Pi_{N_A}\left(\frac{1}{iT}\right)\Pi_{N_A}(0)\Bigg\rangle_{T/\alpha}\Bigg].\end{aligned}\tag{5}$$

This equation, which is the main result of this Letter, expresses the Rényi NEE at temperature T in terms of a correlation function at temperature T/α.

In order to demonstrate the power of this relation, let us focus on a many-body system in the Coulomb blockade regime, such as a quantum dot (QD) or a metallic grain with a large charging energy. In this case, the subsystem supports only two charge states, and we provide an explicit protocol to extract the 2nd and 3rd Rényi moments solely from the 2-point charge correlation function of the QD. These relations allow us to measure an entanglement quantifier in mesoscopic systems at finite temperatures.

Consider a subsystem A with only two charge states, denoted for $N_A = 0, 1$. In this case the projection operators become linear in N_A, $\Pi_{N_A=1} = \hat{N}_A$, $\Pi_{N_A=0} = 1-\hat{N}_A$. Thus Eq. (5) becomes a sum of q-point ($q \leq \alpha$) charge correlators.

Notably, the 2nd and 3rd Rényi moments of the NEE in Eq. (5) can be written explicitly in terms of $\langle\hat{N}_A\rangle_{T/\alpha}$, the occupation number of A and $\langle\hat{N}_A(-i(\alpha-1)/T)\hat{N}_A(0)\rangle_{T/\alpha}$, the two-point imaginary time charge correlator. Furthermore, at thermal equilibrium the imaginary time correlator $\langle\hat{N}_A(-i\tau)\hat{N}_A(0)\rangle_T$ can be related with the Fourier transform of the real-time correlator, defined as

$$\chi(\omega, T) = \int dt e^{i\omega t}\langle\hat{N}_A(t)\hat{N}_A(0)\rangle_T. \tag{6}$$

This is the charge noise of the QD. As a result Eq. (5) becomes [35]

$$\Delta S^{(2)} = -\log\left[1 - 2\langle \hat{N}_A\rangle_{T/2} + 2\int \frac{d\omega}{2\pi}\chi(\omega, T/2)e^{-\frac{\omega}{T}}\right],$$
$$\Delta S^{(3)} = -\frac{1}{2}\log\left[1 - 3\langle \hat{N}_A\rangle_{T/3} + 3\int \frac{d\omega}{2\pi}\chi(\omega, T/3)e^{-\frac{2\omega}{T}}\right]. \tag{7}$$

This is our main result for Coulomb blockaded systems. It allows one to measure Rényi moments of the NEE in thermal states at temperature T, from charge correlations measured at temperature T/α. The advantage of our approach is that these charge correlations have been repeatedly measured in mesoscopic systems using charge sensing techniques (see, e.g., [46–50]). In the Supplemental Material [35] we relate explicitly $\chi(\omega, T)$ to the voltage-dependent noise [51] in a quantum-point-contact charge detector electrostatically coupled to the QD as depicted in Fig. 1(a).

When subsystem A fluctuates between more charge states, the projectors Π_{N_A} become nonlinear functions of the charge operator. For example for three charge states denoted $N_A = 0, 1, 2$ we have $\Pi_{N_A=0} = \frac{1}{2}(\hat{N}_A - 1)(\hat{N}_A - 2)$, and so on. Then in order to measure the NEE, Eq. (5) involves higher-point charge correlators. In the context of full counting statistics [56], higher moments such as the third moment of current correlations $\langle I(t_1)I(t_2)I(t_3)\rangle$ have been measured, see, for example, Refs. [57,58]. Such techniques may allow to extend our results to the case with multiple-charge states, and similarly for higher Rényi moments.

Equations (7) apply generally to Coulomb blockaded systems. The simplest possible example is a spinless double dot system. The entanglement of log 2 due to coherent hopping of an electron between the two dots can be directly computed from the density matrix, or, equivalently, from the charge-charge correlation via Eqs. (7), as demonstrated in the Supplemental Material [35]. However, the power of the relations (7) lies in the prospect of applying them to strongly correlated systems, such as magnetic impurities embedded in a continuum, which can lead to a multitude of Kondo effects and to a competition between spin correlations and screening in multiple-impurity systems (for a review see, e.g., [59]). Here we exemplify the usefulness of these relations for multichannel Kondo systems.

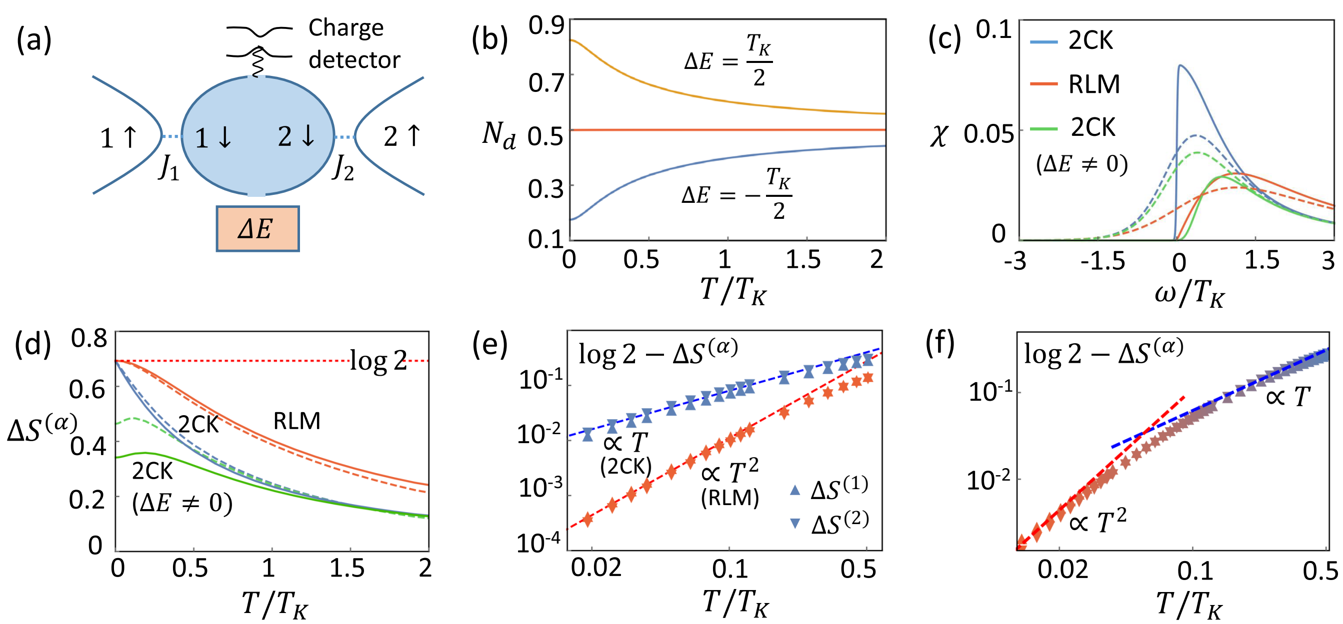

FIG. 1. (a) The charge-multi-channel-Kondo setup [52–55], shown here for two channels (2CK). In each of the channels ($j = 1, 2$) an electron can hop from the metallic dot to its respective lead with matrix elements J_j and thus change the dot occupancy by unity, $N_d = 0 \leftrightarrow 1$, effectively flipping the "impurity spin." The gate detuning from charge degeneracy ΔE serves as an effective magnetic field. The asymmetry $T_- \propto (J_1 - J_2)^2$ allows us to study the crossover between two- and one-channel Kondo models. An additional quantum point contact coupled electrostatically to the dot allows measurements of the dot charge and its fluctuations. (b) The charge $N_d(T)$ and (c) its fluctuations $\chi(\omega)$, required for the evaluation of the Rényi moments $\Delta S^{(\alpha)}$ of the number entanglement entropy (NEE). $N_d(T)$ is shown for $\Delta E = T_K/2$ (yellow), 0 (red), $-T_K/2$ (blue), with $T_- = 0$. $\chi(\omega)$ is shown for $T = 0.01T_K$ (solid lines) and $T = 0.5T_K$ (dashed line) for $T_- = \Delta E = 0$ (2CK), $T_- = T_K$ (resonant level model—RLM) with $\Delta E = 0$, and for $\Delta E = T_K/2$ with $T_- = 0$. In the plot, δ function peaks [35] are not drawn. (d) The Rényi moment of NEE $\Delta S^{(2)}$ obtained from N_d and $\chi(\omega)$ using Eq. (7) (solid lines). The NEE $\Delta S^{(1)}$ is also shown for comparison (dashed lines). We compare the 2CK case, the RLM case, and also the 2CK with finite level detuning ΔE. (e) log − log plot of $\log(2) - \Delta S^{(\alpha)}(T)$ for the 2CK and RLM cases, demonstrating distinct power law behavior; up (down) triangles refer to $\alpha = 1$ ($\alpha = 2$). (f) $\log(2) - \Delta S^{(\alpha)}(T)$ for a small channel anisotropy scale $T_-/T_K = 0.1$, displaying a 1CK–2CK crossover.

The multichannel Kondo (MCK) model describes an impurity spin-1/2 interacting antiferromagnetically with M spinful channels with continuous density of states. For $M = 1$ the impurity is perfectly screened at $T = 0$, while for $M > 1$ the ground state is a non-Fermi liquid, due to overscreening of the impurity by the M channels [60]. In both cases, though, the spin-screening cloud (see, e.g., [61,62]) leads to a zero-temperature entanglement entropy of log 2. Recently, finite-temperature entanglement measures, such as the entanglement of formation (EOF) and negativity ($\mathcal{N}$), have been applied to this model [63,64], demonstrating, for both quantities, a low-temperature scaling behavior of

$$\mathrm{EOF}, \mathcal{N} \simeq \log 2 - a_M T^{2\Delta_M}, \tag{8}$$

where a_M is an M-dependent numerical coefficient and Δ_M is the scaling dimension of the impurity spin operator in the M-channel problem. As determined by the conformal field theory solution [65], $\Delta_M = 2/(2+M)$ for $M \geq 2$, and for the single channel case $\Delta_1 = 1$. Unfortunately, these entanglement measures, while calculable, are practically impossible to measure (as they require measurements of all elements of the density matrix). Below we demonstrate that the entanglement measure NEE for the charge-Kondo model, and its Rényi moments, which can be directly measured by applying Eqs. (7), obey the same scaling relation as Eq. (8), and thus allow direct measurement of the finite-temperature entanglement and the scaling dimensions of MCK systems.

Unlike the commonly observed one-channel Kondo effect, signatures of a two-channel spin-Kondo behavior in quantum dots have only been reported in [66,67]. A major step forward has been recently taken in Refs. [54,55] which utilized a mapping between the two charge states of a metallic dot $N_d = 0$, 1 and spin $S_z = \pm 1/2$ [52,53] to demonstrate scaling and crossover between single and multichannel $M = 1$, 2, 3 Kondo effects [here the number of channels is controlled by the number of one-dimensional leads attached to the dot, see Fig. 1(a)]. Since, in this case of a charge-Kondo effect, the role of spin is played by the deviation of the occupation from half-filling, then charge correlations in the charge-Kondo system map onto spin correlation in spin-Kondo systems, thus allowing us to extract the finite temperature entanglement measure NEE for non-Fermi liquid MCK systems by utilizing Eqs. (7). In this case the NEE reduces at $T = 0$ to the full entanglement entropy.

To be specific, the charge-MCK Hamiltonian is given by [52,53]

$$H = \sum_{j=1}^{M} J_j \sum_{k,k'} (c^\dagger_{j\uparrow k} c_{j\downarrow k'} S_- + c^\dagger_{j\downarrow k} c_{j\uparrow k'} S_+) + \sum_{j,k,\sigma} k c^\dagger_{j\sigma k} c_{j\sigma k} + \Delta E S_z. \tag{9}$$

The system is depicted in Fig. 1(a) for the case $M = 2$. Here J_j is the tunneling between the jth lead and the dot, $c_{j\sigma k}$ and $c^\dagger_{j\sigma k}$ denote annihilation and creation operators, respectively, for electrons with momentum k in the leads for $\sigma = \uparrow$ or in the dot for $\sigma = \downarrow$. Here the spin flip operator changes the charge state of the dot, $S_+|0\rangle = |1\rangle$, and $S_z = (|1\rangle\langle 1| - |0\rangle\langle 0|)/2 = \hat{N}_d - 1/2$, where $|0\rangle$ and $|1\rangle$ are the empty and occupied dot states, and $\hat{N}_d$ is the dot number operator. ΔE is a gate voltage detuning away from the charge degeneracy point, which corresponds to a magnetic field in the conventional MCK model. The electrons are assumed to be spinless, e.g., due to a large magnetic field. The Hamiltonian (9) is identical to the spin-MCK Hamiltonian, where the role of the local spin is played by the occupation states of the metallic dot.

Consider now the NEE between the QD and the leads. Using Eqs. (7), which expresses the NEE as a correlation function, we find for the second Rényi NEE

$$\Delta S^{(2)} = -\log\left[\frac{1}{2} + 2\left\langle S_z\left(\frac{1}{iT}\right) S_z(0)\right\rangle_{T/2}\right]. \tag{10}$$

The second term is a two-point correlation function of the impurity spin operator in imaginary time. Using the general conformal field theory correlation function $\langle \mathcal{O}(-i\tau)\mathcal{O}(0)\rangle_T = (\pi T/\sin \pi\tau T)^{2\Delta_\mathcal{O}}$ for an operator $\mathcal{O}$ with scaling dimension $\Delta_\mathcal{O}$, we obtain [35]

$$\left\langle S_z\left(\frac{1}{iT}\right) S_z(0)\right\rangle_{T/2} \propto \left(\frac{\pi T/2}{\sin(\frac{\pi}{T}T/2)}\right)^{2\Delta_M} \sim T^{2\Delta_M}. \tag{11}$$

Furthermore, we find that at low temperature the NEE $\Delta S^{(1)}$ is also dominated by the same two point correlator [35].

Accordingly, the entanglement measure NEE and its Rényi moments obey the exact same scaling behavior as EOF and $\mathcal{N}$ in Eq. (8). For example, for the 1CK case we obtain a quadratic temperature dependence, $\Delta S^{(\alpha)}_{1\,\mathrm{CK}} \simeq \log 2 - a_{1\alpha}T^2$, whereas in the 2CK case the dependence is linear, $\Delta S^{(\alpha)}_{2\,\mathrm{CK}} \simeq \log 2 - a_{2\alpha}T$, where now the nonuniversal coefficients $a_{M\alpha}$ depend also on the Rényi moment α. As pointed out in Refs. [63,64], this linear behavior, corresponding to $\Delta_2 = 1/2$, is an indication of the existence of a Majorana fermion in the ground state.

The temperature dependence obtained in Eq. (11) is based on a general scaling argument. In order to make a more quantitative comparison with experiments for the case of the 1CK and 2CK models, we can use the exact solution of the corresponding low-temperature Hamiltonian. This allows us to obtain results for the temperature behavior along the crossover between 1CK and 2CK behaviors. Also, the same model allows us to provide quantitative results for the simpler resonant level model.

After a renormalization process at low temperatures, the Hamiltonian for the $M = 2$ MCK model can be mapped into a Majorana resonant level model [68,69]

$$H = \sum_k k\gamma_{x,-k}\gamma_{x,k} + k\eta_{x-k}\eta_{x,k} + i\sqrt{\frac{2T_K}{L}}\sum_k \gamma_k \eta_d + i\sqrt{\frac{2T_-}{L}}\sum_k \eta_k \gamma_d + i\Delta E \gamma_d \eta_d. \tag{12}$$

Here T_K is the Kondo energy scale and $T_- \propto (J_1 - J_2)^2$ [70] is an energy scale associated with channel asymmetry. Here γ_d and η_d are local Majorana zero modes, and $\gamma_{x,k}$, $\eta_{x,k}$ represent the mode expansion of a Majorana field. The charge occupation operator of the metallic dot is written in terms of the local Majorana operators as $\hat{N}_d = \frac{1}{2} + i\gamma_d \eta_d \equiv \hat{N}_A$.

This model describes different regimes. The 2CK state corresponds to $\Delta E = T_- = 0$. As long as $\max\{\Delta E, T_-\} \ll T_K$, this model describes the vicinity of the 2CK fixed point. As T_- increases, it faithfully describes the crossover [71,72] from the 2CK to the 1CK fixed point. Additionally, for the special value $T_- = T_K$, this model maps into the noninteracting resonant level model (RLM), which can be realized in a single level spinless QD, with on site energy ΔE and width $\Gamma \equiv T_K$.

This model can be solved exactly [68]. The results for the occupation number N_d and $\chi(\omega)$, which can be measured using the charge detector, were obtained by solving numerically the resulting integral expressions [35] and are shown in Figs. 1(b) and 1(c) for selected model parameters. For $\Delta E = 0$ (either 2CK or RLM), by symmetry $N_d = 1/2$, and the NEE Rényi moments $\Delta S^{(\alpha)}$ can be extracted using Eqs. (7) solely from the charge noise $\chi(\omega)$. We plot in Fig. 1(d) the resulting $\Delta S^{(2)}$, and for comparison [35] the NEE ($= \Delta S^{(1)}$) which is an entanglement monotone [23], with very similar behavior. Specifically, the resulting $\Delta S^{(\alpha)}$ for the 2CK state is in agreement with our field theory scaling results as seen in the log – log plot in Fig. 1(e). For the RLM the temperature scaling is quadratic.

In Fig. 1(f) we show the crossover from 2CK to 1CK behavior which can be observed for a small channel anisotropy T_-. While $\lim_{T\to 0}\Delta S^{(\alpha)}$ remains log 2 for any J_-, since by symmetry $N_d = 1/2$, the temperature dependence becomes quadratic $\log 2 - \Delta S^{(\alpha)} \propto T^2$ for $T < T_-$. For a finite gate voltage detuning from the fixed point ($\Delta E \neq 0$) we have that $\langle N_d \rangle \neq 1/2$ as seen in Fig. 1(b). As a result, $\lim_{T\to 0} \Delta S^{(\alpha)}$ decreases below log 2 as shown in Fig. 1(d). Since ΔE is a relevant perturbation, we observe a nonmonotonic temperature dependence for some range of values of ΔE as shown in Fig. 1(d) (finite ΔE).

To conclude, we proposed an experimental procedure to measure entanglement at finite temperatures. In particular we demonstrated that an entanglement measure—the number entanglement entropy (NEE)—can be obtained solely from the subsystems charge distribution function and correlation functions. We formulated exact thermodynamic relations between moments of the NEE and subsystem charge correlation functions. A similar observation was made for the quantum Fisher information [73]. The setup we propose for measurement of the NEE in quantum dot systems has already been utilized to measure the charge noise, thus we expect that it can be readily extended to measure finite temperature entanglement measures in various systems, including multichannel Kondo systems.

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