Noninvertible Time-Reversal Symmetry

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In gauge theory, it is commonly stated that time-reversal symmetry only exists at $\theta = 0$ or π for a 2π-periodic θ angle. In this Letter, we point out that in both the free Maxwell theory and massive QED, there is a noninvertible time-reversal symmetry at every rational θ angle, i.e., $\theta = \pi p/N$. The noninvertible time-reversal symmetry is implemented by a conserved, antilinear operator without an inverse. It is a composition of the naive time-reversal transformation and a fractional quantum Hall state. We also find similar noninvertible time-reversal symmetries in non-Abelian gauge theories, including the $\mathcal{N} = 4$ SU(2) super Yang-Mills theory along the locus $|\tau| = 1$ on the conformal manifold.

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Introduction.—In gauge theory with a 2π-periodic θ angle, there can be a manifest time-reversal symmetry at $\theta = 0$. At $\theta = \pi$, there is a slightly more subtle time-reversal symmetry, which is a composition of the naive time-reversal transformation with $\theta \mapsto \theta + 2\pi$. In both cases, the time-reversal symmetry is implemented by an antiunitary operator T obeying $\mathsf{T}^\dagger \times \mathsf{T} = 1$. The relation between T and $\mathsf{T}^\dagger$ depends on the details of the quantum system. Different time-reversal algebra, e.g., $\mathsf{T}^2 = 1$, $\mathsf{T}^2 = (-1)^F$, etc., can be used to place the quantum field theory (QFT) on manifolds with different tangential structures. In this Letter, we discuss more subtle time-reversal symmetries at other values of θ.

Our discussion follows closely the recent developments of a novel kind of generalized global symmetry, the *noninvertible symmetry*. (See Refs. [1,2] for reviews of generalized global symmetries.) It is implemented by conserved operators, or more generally, topological defects 3]] in a relativistic setting, that do not obey a group multiplication law. Yet, they are invariants under renormalization group flows and lead to nontrivial selection rules as well as constraints on the low-energy phase diagram. Recently, based on earlier works in $1+1$ dimensions [4–21], a large class of noninvertible symmetries was discovered in many familiar quantum systems in general spacetime dimensions [22–40], including the standard model [32,33]. (See also [11,41–46] for discussions of noninvertible higher-form symmetries and [47,48] for related discussions under the name of algebraic higher symmetry.) The key for constructing some of these symmetries is to compose a duality transformation (or more generally, an isomorphism of the quantum system) with the gauging of a discrete higher-form symmetry. See also [22,49–58] for lattice realizations of noninvertible symmetries.

In this Letter, we extend this construction to time-reversal symmetries. This was first discussed in [24], and we further generalize it to the free Maxwell theory, massive QED, $\mathcal{N} = 4$ super Yang-Mills theory, and invertible one-form symmetry-protected topological (SPT) phases. We also discuss the possible generalized anomalies of noninvertible time-reversal symmetries.

In Maxwell theory and massive QED, we find that for any rational θ angle, (More precisely, by a rational θ angle we mean θ/π is a rational number. We hope this slight abuse of terminology does not cause any confusion.) the composition of the naive time-reversal transformation and a fractional quantum Hall state leads to a conserved, antilinear operator. However, since the fractional quantum Hall state is a noninvertible phase, the resulting operator is not anti-unitary and it does not have an inverse—it is a noninvertible time-reversal symmetry. More specifically, the product $\mathsf{T}^\dagger \times \mathsf{T} = \mathcal{C} \neq 1$ is a nontrivial *condensation operator* [25,30,48,59–63] (see also [23,24]), which plays a pivotal role in the recent developments of noninvertible symmetry. By composing the noninvertible time-reversal symmetry with the $\mathtt{CPT}$ transformation, which exists in every relativistic QFT, we can also obtain a noninvertible $\mathtt{CP}$ symmetry.

It is surprising that Abelian gauge theory is almost always time-reversal invariant in the space of the θ angle. Indeed, the free Maxwell theory and QED has a noninvertible time-reversal symmetry for a dense subset of the possible values for the θ-angle, in addition to the invertible ones at $\theta = 0, \pi$. This is perhaps related to the reason why the θ angle for an Abelian gauge group is often overlooked, say, in the discussion of the strong CP problem. [Another reason for this is that there is no nontrivial instanton configuration in noncompact space for $U(1)$ because $\pi_3[U(1)] = 0$.] Nonetheless, the θ angle of an Abelian gauge theory (say the massive QED for the real world) is generally a physical CP-violating parameter in the Lagrangian that cannot be removed by any field redefinition.

This Letter is accompanied by Supplemental Material [64], which includes Refs. [65–71], where we provide alternative derivations of noninvertible time-reversal symmetries via half gauging higher-form symmetries and explain the relation to mixed anomalies. We also review the condensation operators and discuss noninvertible time-reversal symmetries in pure $PSU(N)$ Yang-Mills theories there.

Free Maxwell theory.—Consider the free Maxwell theory of a dynamical U(1) gauge field A without any matter fields:

$$\mathcal{L}_{\mathcal{Q}_\tau} = -\frac{1}{2e^2} F \wedge \star F + \frac{\theta}{8\pi^2} F \wedge F, \tag{1}$$

where $F = dA$ [72]. The theory is parameterized by the complexified coupling constant $\tau = (\theta/2\pi) + (2\pi i/e^2) \in \mathbb{H}$ where $\mathbb{H}$ denotes the upper half-plane. We will denote the free Maxwell theory at a given value of τ as $\mathcal{Q}_\tau$ [73].

The Maxwell theory has the $SL(2,\mathbb{Z})$ duality generated by $\mathbb{S}$ and $\mathbb{T}$. They act on the complexified coupling as $\mathbb{S}: \tau \mapsto -1/\tau$ and $\mathbb{T}: \tau \mapsto \tau + 1$. Although $\mathbb{S}^2 = -1 \in SL(2,\mathbb{Z})$ acts trivially on τ, it acts nontrivially on operators as the $\mathbb{Z}_2^{(0)}$ charge conjugation symmetry $A \mapsto -A$, which exists at every value of τ. Inequivalent Maxwell theories are thus labeled by elements in the fundamental domain $\mathcal{F} \equiv \mathbb{H}/PSL(2,\mathbb{Z})$.

At a generic value of the complexified coupling τ, we define a naive time-reversal transformation

$$K:\ \tau \mapsto -\bar{\tau}, \qquad t \mapsto -t, \tag{2}$$

where t is the (Lorentzian) time coordinate. We emphasize that K is generally not a symmetry of the Maxwell theory $\mathcal{Q}_\tau$ at a generic τ, but it maps $\mathcal{Q}_\tau$ to its orientation reversal $\mathcal{Q}_{-\bar{\tau}}$; that is, it flips the sign of θ. We choose K to act on the gauge field A as $A_0(t,\vec{x}) \mapsto -A_0(-t,\vec{x})$ and $A_i(t,\vec{x}) \mapsto A_i(-t,\vec{x})$.

We first review the invertible time-reversal symmetries of the Maxwell theory, which are well known in the literature. See, for instance, [74,75]. The Maxwell theory has an invertible time-reversal symmetry along the following loci inside the fundamental domain $\mathcal{F}$:

$$\theta = 0 \quad \text{or} \quad \theta = \pi \quad \text{or} \quad |\tau| = 1. \tag{3}$$

The nature of the time-reversal symmetries in different cases are slightly different. At $\theta = 0$, the naive transformation K is a symmetry, which we denote as $\mathsf{T}^{\theta=0}$. At $\theta = \pi$, the theory is left invariant under the $\mathbb{T}K$ transformation, and we denote the corresponding time-reversal symmetry as $\mathsf{T}^{\theta=\pi}$. Finally, when $|\tau| = 1$, the theory is left invariant under the $\mathbb{S}K$ transformation. This is because the $\mathbb{S}$ transformation when $|\tau| = 1$ leaves the electric coupling constant e invariant but flips the sign of the θ-angle. Therefore, we have a time-reversal symmetry $\mathsf{T}^{|\tau|=1}$ associated with $\mathbb{S}K$. All these time-reversal symmetries are invertible and square to the identity, $(\mathsf{T}^{\theta=0})^2 = (\mathsf{T}^{\theta=\pi})^2 = (\mathsf{T}^{|\tau|=1})^2 = 1$.

At special values of τ, we have enhanced symmetries. At $\tau = i$, the $\mathbb{S}$ transformation combines with the invertible time-reversal symmetry $\mathsf{T}^{\theta=0}$ to generate the dihedral group of order 8, D_8^{T}. Similarly, at $\tau = e^{\pi i/3}$, the $\mathbb{T}\mathbb{S}^{-1}$ transformation and the time-reversal symmetry $\mathsf{T}^{\theta=\pi}$ generate the dihedral group of order 12, D_{12}^{T}.

We now turn to the new noninvertible time-reversal symmetries in Maxwell theory. We claim that for every rational value of the θ angle,

$$\theta = \pi p/N, \tag{4}$$

the Maxwell theory has a noninvertible time-reversal symmetry. Here N and p are arbitrary integers satisfying $N > 1$, $-N < p < N$, and $\gcd(p,N) = 1$.

We first note that there is a topological interface $\mathsf{I}_{2\pi p/N}$ separating the Maxwell theory at θ and $\theta - 2\pi p/N$ (both sharing the same electric coupling e). For simplicity, we start with $p = 1$, then the world volume Lagrangian was derived in [32]:

$$\mathsf{I}_{\frac{2\pi}{N}} \equiv \exp\left[i \oint_M \left(\frac{N}{4\pi} a \wedge da + \frac{1}{2\pi} a \wedge dA\right)\right]. \tag{5}$$

Here M is a three-manifold where the interface is supported on and a is a dynamical one-form gauge field that only lives on M. This can be viewed as a $\nu = 1/N$ fractional quantum Hall state on M for the bulk electromagnetic gauge field A. We omit the path integral over a in the definition of the interface $\mathsf{I}_{2\pi/N}$. To see that this is the correct topological interface, we insert it along M: $x = 0$, and assume that the θ angles in the $x < 0$ and $x > 0$ regions are θ_- and θ_+, respectively. The equations of motion of a on M and the bulk electromagnetic gauge field A, respectively, give $N da + F = 0$ and $(1/2\pi)(\theta_+ - \theta_-)F = da$. Combining the two equations, we find $\theta_+ - \theta_- = -2\pi/N$, so (5) is indeed the world-volume Lagrangian for the topological interface separating the Maxwell theory with θ and $\theta - 2\pi/N$.

Starting at $\theta = \pi/N$, we can compose the naive time-reversal interface K (which separates a QFT with its orientation reversal) with the topological interface $\mathsf{I}_{2\pi/N}$, which maps the θ angle back to itself:

$$\theta = \pi/N \xrightarrow{\mathsf{I}_{\frac{2\pi}{N}}} \theta = -\pi/N \xrightarrow{K} \theta = \pi/N. \tag{6}$$

We therefore conclude that

$$\mathsf{T}^{\theta=\frac{\pi}{N}} \equiv K \circ \mathsf{I}_{\frac{2\pi}{N}} \tag{7}$$

is an orientation-reversing, topological defect in Maxwell theory at $\theta = \pi/N$. When we choose M to be the whole space at a given time, $\mathsf{T}^{\theta=\pi/N}$ becomes an antilinear, conserved operator that flips the sign of the Lorentzian time. Thus, it is a time-reversal symmetry. Intuitively, $\mathsf{T}^{\theta=\pi/N}$ is a composition of the naive time-reversal transformation and a fractional quantum Hall state (5).

Interestingly, the antilinear operator $\mathsf{T}^{\theta=\pi/N}$ is not anti-unitary. To see this, we compute

$$\begin{aligned}(\mathsf{T}^{\theta=\frac{\pi}{N}})^\dagger \times \mathsf{T}^{\theta=\frac{\pi}{N}} &= (\mathsf{I}_{\frac{2\pi}{N}})^\dagger \times \mathsf{I}_{\frac{2\pi}{N}} \\ &= \exp\left[i\oint_M \left(\frac{N}{4\pi} a da - \frac{N}{4\pi}\bar{a} d\bar{a} + \frac{1}{2\pi}(a-\bar{a}) dA\right)\right] \equiv \mathcal{C}^{(N)}.\end{aligned} \tag{8}$$

Here $(\mathsf{I}_{2\pi p/N})^\dagger$ is the orientation reversal of the interface $\mathsf{I}_{2\pi/N}$, and $\bar{a}$ is the dynamical one-form gauge field living on $(\mathsf{I}_{2\pi/N})^\dagger$. The operator $\mathcal{C}^{(N)}$ is known as the *condensation operator*, see Refs. [25,30,32] and Supplemental Material [64] for more details. The important point is that $\mathcal{C}^{(N)}$ is not a trivial operator; it acts nontrivially on the ‘t Hooft lines [32]. Thus, the time-reversal symmetry $\mathsf{T}^{\theta=\pi/N}$ is noninvertible, and, in particular, is not implemented by an anti-unitary operator.

For $\theta = \pi p/N$ at a more general p, we generalize the topological interface (5) to

$$\mathsf{I}_{\frac{2\pi p}{N}} \equiv \exp\left[i\oint_M \mathcal{A}^{N,p}[dA/N]\right], \tag{9}$$

where $\mathcal{A}^{N,p}[B]$ is the $2+1$D minimal $\mathbb{Z}_N$ TQFT [76] (see also [77–79]) coupled to the $\mathbb{Z}_N^{(1)}$ background two-form gauge field B. It describes a $\nu = p/N$ fractional quantum Hall state. The noninvertible time-reversal symmetry at $\theta = \pi p/N$ is defined as $\mathsf{T}^{\theta=\pi p/N} \equiv K \circ \mathsf{I}_{2\pi p/N}$. Following a similar calculation in [32], we find that it obeys the noninvertible fusion rule $(\mathsf{T}^{\theta=\pi p/N})^\dagger \times \mathsf{T}^{\theta=\pi p/N} = \mathcal{C}^{(N)}$.

At the intersection between $\theta = \pi p/N$ and $|\tau| = 1$, i.e., at $\tau = (p/2N) + i\sqrt{1-(p^2/4N^2)}$, there is an additional noninvertible, linear symmetry, which we explain in detail in the Supplemental Material [64].

We conclude that the free Maxwell theory is time-reversal invariant at $\theta = \pi p/N$. The notion of naturalness [80] becomes rather exotic as we vary the θ angle: at every rational θ-angle, there is a different noninvertible time-reversal symmetry. The situation is somewhat similar to the $1+1$D compact boson conformal field theory, where at every rational radius square R^2 there is a different enhanced chiral algebra.

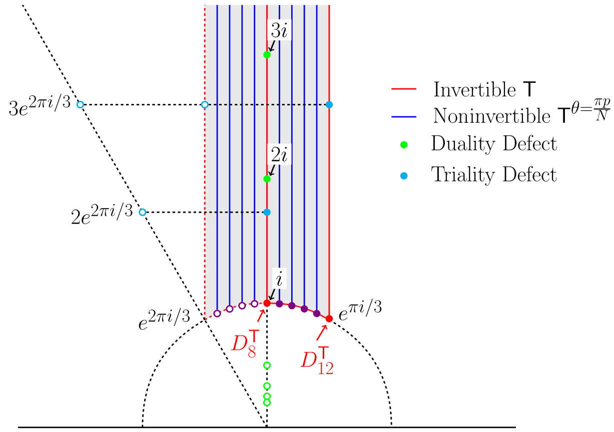

FIG. 1. Some (anti)linear (non)invertible symmetries of the Maxwell theory. The locus where the theory has an invertible time-reversal symmetry is indicated by red lines. At $\tau = i$ and at $\tau = e^{i\pi/3}$, we have enhanced symmetries D_8^{T} and D_{12}^{T}, respectively. At every rational value of the θ-angle with $\theta = \pi p/N$, there is a noninvertible time-reversal symmetry $\mathsf{T}^{\theta=\pi p/N}$, which is indicated by the vertical blue lines. Some of the linear noninvertible symmetries of the Maxwell theory are also indicated. At the green dots, i.e., at $\tau = iN$, the theory realizes a linear, noninvertible duality defect [23]. At the cyan dots, i.e., at $\tau = Ne^{2\pi i/3}$, the theory realizes a linear, noninvertible triality defect [30].

In Fig. 1, we summarize the invertible and noninvertible time-reversal symmetries as well as a few previously known noninvertible linear symmetries in the Maxwell theory across the fundamental domain.

Massive QED.—Consider QED with a single Dirac fermion with mass m. The Lagrangian is given by

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi + m\bar{\Psi}\Psi + \frac{\theta}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}. \tag{10}$$

Using a chiral rotation, we can take $m > 0$ to be a positive real constant, so that the θ angle is physical. Alternatively, we can set the θ angle to zero and work with a complex mass term $me^{-i\theta}$ whose phase is physical.

At $\theta = 0$ and $\theta = \pi$, massive QED has an invertible time-reversal symmetry K and $\mathbb{T}K$, respectively. The choice of K is not unique: one can compose K with a unitary $\mathbb{Z}_2^{(0)}$ symmetry to obtain another time-reversal symmetry. We will not commit to any particular choice of K.

In addition, we claim that the theory has a noninvertible time-reversal symmetry at $\theta = \pi p/N$, similar to the free

Maxwell theory case. It was realized in [32,33] that the classical axial $U(1)_A$ symmetry of massless QED is not completely broken by the ABJ anomaly, but it turns into a linear, noninvertible symmetry $\mathcal{D}_{p/N}$ labeled by the rational numbers p/N. In addition, the massless QED has an invertible time-reversal symmetry K. (When $m = 0$, the θ angle is not physical and can be set to zero by an axial rotation.)

Now we turn on a mass term $me^{-i\theta}$. For a generic value of θ, the mass deformation breaks both the invertible time-reversal symmetry K and the noninvertible symmetry $\mathcal{D}_{p/N}$. However, at a rational value of $\theta = \pi p/N$, the composition of K and $\mathcal{D}_{p/N}$ is preserved by the mass deformation $me^{-i\pi p/N}$. The composed operator implements a noninvertible time-reversal symmetry

$$\mathsf{T}^{\theta=\frac{\pi p}{N}} = K \circ \mathcal{D}_{\frac{p}{N}}. \tag{11}$$

In particular, if we set $p = N = 1$, then $\mathcal{D}_{p/N}$ becomes an invertible defect shifting the θ angle by -2π, that is, it reduces to the $\mathbb{T}^{-1}$ transformation. Correspondingly, the invertible time-reversal symmetry at $\theta = \pi$ is associated with the $K\mathbb{T}^{-1} = \mathbb{T}K$ transformation.

Using $\mathcal{D}_{p/N} \times \mathcal{D}^\dagger_{p/N} = \mathcal{D}^\dagger_{p/N} \times \mathcal{D}_{p/N} = \mathcal{C}^{(N)}$ from [32], we immediately recover the noninvertible fusion rule. In Fig. 2, we summarize the invertible and noninvertible symmetries of massive QED across the complex mass plane.

$\mathcal{N} = 4$ Super Yang-Mills theory.—Next, we discuss noninvertible time-reversal symmetries in the $3+1$D $\mathcal{N} = 4$ SU(2) super Yang-Mills theory and relate them to various linear and antilinear noninvertible symmetries in [23,24,30,31]. For simplicity, we will assume the spacetime manifold to be spin and in particular oriented.

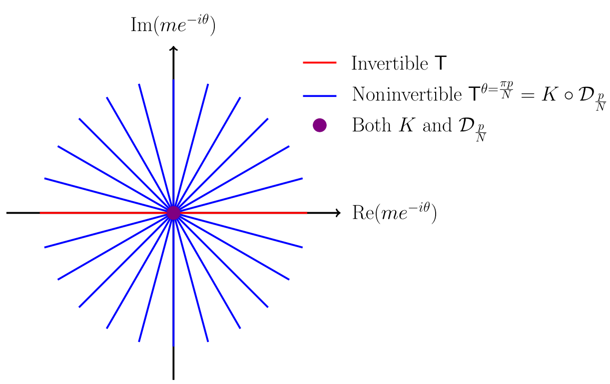

FIG. 2. Invertible and noninvertible time-reversal symmetries of QED on the complex mass plane. If the mass $me^{-i\theta}$ is real (i.e., if $\theta = 0, \pi$), the theory has an invertible time-reversal symmetry. When the phase of the mass term is rational, i.e., $\theta = \pi p/N$, we have the noninvertible time-reversal symmetry generated by $\mathsf{T}^{\theta=\pi p/N}$. At zero mass, we have a linear noninvertible symmetry $\mathcal{D}_{p/N}$ for every rational number p/N and the invertible time-reversal symmetry.

The $\mathcal{N} = 4$ SU(2) super Yang-Mills theory is parametrized by a complexified coupling τ. Similar to the free Maxwell theory, it enjoys an $SL(2,\mathbb{Z})$ electromagnetic duality, but the details differ as we discuss below. While the $\mathbb{T}: \tau \mapsto \tau + 1$ transformation is an exact duality of the SU(2) theory, the $\mathbb{S}: \tau \mapsto -1/\tau$ transformation maps the latter to a theory with an SO(3) gauge group. The $SL(2,\mathbb{Z})$ duality transformation acts nontrivially on the spectrum of line operators [81–83]. The SU(2) theory has a $\mathbb{Z}_2^{(1)}$ center one-form symmetry. Below, S stands for the operation of gauging a $\mathbb{Z}_2^{(1)}$ one-form symmetry, and T stands for the operation of stacking a $\mathbb{Z}_2^{(1)}$ one-form SPT phase. See the Supplemental Material [64] for more details.

Let us start with the antilinear, invertible symmetries of the SU(2) theory. Along $\theta = 0$, there is an invertible time-reversal symmetry $\mathsf{T}^{\theta=0}$ associated with the naive time-reversal transform K [84]. Along $\theta = \pi$, the coupling constant τ is invariant under the $\mathbb{T}K$ transformation. If we further track the dependence on the coupling to the background gauge field of the $\mathbb{Z}_2^{(1)}$ one-form symmetry, then the SU(2) theory at $\theta = \pi$ is invariant under $T\mathbb{T}K = \mathbb{T}TK$ [31]. We denote the corresponding invertible time-reversal symmetry by $\mathsf{T}^{\theta=\pi}$. Similarly, at $\theta = -\pi$, we have an invertible time-reversal symmetry $\mathsf{T}^{\theta=-\pi}$ that is associated with the transformation $\mathbb{T}^{-1}TK$.

Next, we review the linear, noninvertible symmetries of the SU(2) theory. It was shown in [24,30,31] that at $\tau = i$, there is a duality defect $\mathcal{D}_2$ associated with the transformation $\mathbb{S}S$. Importantly, while this duality defect acts invertibly on the local operators as a $\mathbb{Z}_4^{(0)}$ symmetry, it acts noninvertibly on the line operators. Similarly, at $\tau = e^{2\pi i/3}$, there is a triality defect $\mathcal{D}_3$ associated with the transformation $\mathbb{S}\mathbb{T}ST$ [30,31]. At these two special points, we can compose the duality and triality defects with the invertible time-reversal symmetries to obtain noninvertible time-reversal symmetries, i.e., $\mathcal{D}_2 \circ \mathsf{T}^{\theta=0}$ at $\tau = i$ and $\mathcal{D}_3 \circ \mathsf{T}^{\theta=-\pi}$ at $\tau = e^{2\pi i/3}$.

We now move on to the more general noninvertible time-reversal symmetry along the locus $|\tau| = 1$. While the transformation $\mathbb{S}K$ leaves τ invariant, unlike the Maxwell theory, it maps the SU(2) theory to an SO(3) theory. To remedy this, we compose the above transformation with the S transformation, which gauges the $\mathbb{Z}_2^{(1)}$ one-form symmetry. The composite transformation $\mathbb{S}SK$ then leaves the SU(2) theory at any point along $|\tau| = 1$ invariant. We denote the corresponding antilinear symmetry by $\mathsf{T}^{|\tau|=1}$. Since $\mathsf{T}^{|\tau|=1}$ involves the S transformation, it is noninvertible. We have

$$(\mathsf{T}^{|\tau|=1})^\dagger \times \mathsf{T}^{|\tau|=1} = \mathcal{C}_0^{(2)} \equiv \frac{1}{|H^0(M,\mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M,\mathbb{Z}_2)} \eta(\Sigma), \tag{12}$$

where $\eta(\Sigma)$ is the $\mathbb{Z}_2^{(1)}$ one-form symmetry operator on a two-surface Σ, and M is the three-manifold on which $\mathsf{T}^{|\tau|=1}$ is supported. $\mathcal{C}_0^{(2)}$ is the untwisted condensation defect from one-gauging the $\mathbb{Z}_2^{(1)}$ one-form center symmetry on M (see Supplemental Material [64] for more details on condensation defects).

At $\tau = i$, the noninvertible time-reversal symmetry $\mathsf{T}^{|\tau|=1}$ is a composition of the duality defect $\mathcal{D}_2$ (which is associated with $\mathbb{S}S$) and the invertible time-reversal symmetry $\mathsf{T}^{\theta=0}$ (which is associated with K). Similarly, at $\tau = e^{2\pi i/3}$, $\mathsf{T}^{|\tau|=1}$ is the composition of the triality defect $\mathcal{D}_3$ (which is associated with $\mathbb{S}\mathbb{T}ST$) and the invertible time-reversal symmetry $\mathsf{T}^{\theta=-\pi}$ (which is associated with $\mathbb{T}^{-1}TK$).

Finally, we note that the locus $|\tau| = 1$ of the SU(2) theory is mapped to $\theta = \pi$ of the $\mathrm{SO}(3)_-$ theory under the $\mathbb{T}\mathbb{S}\mathbb{T}$ duality transformation (up to a counterterm of $\mathbb{Z}_2^{(1)}$ which can be fixed by a T transformation). In [24], it was found that the $\mathrm{SO}(3)_-$ pure Yang-Mills theory has a noninvertible time-reversal symmetry at $\theta = \pi$. [Recall that the θ angle is 4π periodic for the SO(3) gauge group.] When embedded into the $\mathcal{N} = 4$ $\mathrm{SO}(3)_-$ theory, this noninvertible time-reversal symmetry is related to our $\mathsf{T}^{|\tau|=1}$ of the SU(2) theory by an electromagnetic duality transformation.

See Table I for the summary. Intuitively, the $\mathbb{Z}_2^{(1)}$ one-form symmetry of the $\mathrm{SO}(3)_-$ theory makes it possible to define a noninvertible time-reversal symmetry at half of the naive allowed values of θ, i.e., $\theta \in 2\pi\mathbb{Z}$. In contrast, the $\mathrm{U}(1)^{(1)}$ magnetic one-form symmetry cures the time-reversal symmetry at every rational θ angle for the Abelian gauge theory.

Generalized anomalies and trivially gapped phases.— Are our noninvertible time-reversal symmetries anomalous? While we do not know the general definition of anomalies for a noninvertible global symmetry, one can alternatively ask if the global symmetry is compatible with a trivially gapped phase. Indeed, one of the most important consequences of a ‘t Hooft anomaly for an ordinary global symmetry is that it cannot be matched by a trivially gapped phase.

For the noninvertible time-reversal symmetries $\mathsf{T}^{\theta=\pi p/N}$ in the free Maxwell theory and massive QED, there is a simple argument that they are compatible with a trivially gapped phase. In this sense they do not have a generalized anomaly. This is to be contrasted with the linear, noninvertible symmetries of the Maxwell theory, which are generally incompatible with a trivially gapped phase [23,30].

The proof proceeds as follows. We start with the free Maxwell theory, and couple it to a charge 1 complex scalar field. This coupling does not break the magnetic one-form symmetry, and the noninvertible time-reversal symmetry $\mathsf{T}^{\theta=\pi p/N}$ is preserved by the coupling to this complex scalar field. Then we can turn on a Higgs potential, and drive the whole system to a trivially gapped phase while preserving $\mathsf{T}^{\theta=\pi p/N}$. This proves that the noninvertible time-reversal symmetries in the Maxwell theory are compatible with a trivially gapped phase.

Below, we explicitly find trivially gapped quantum field theories which realize our noninvertible time-reversal symmetries. This serves as a consistency check that these symmetries in the Maxwell theory and QED are nonanomalous.

Our noninvertible time-reversal symmetries always arise from the invariance of a QFT under the $CKS = SK$ transformation (modulo duality transformations), provided we choose the appropriate counterterm of the background gauge field for the $\mathbb{Z}_N^{(1)}$ one-form symmetry. Here, C is the “charge conjugation” operation which flips the sign of the background gauge field, and S is the operation of gauging the $\mathbb{Z}_N^{(1)}$ one-form symmetry.

A trivially gapped phase which realizes the noninvertible time-reversal symmetry is represented by a $\mathbb{Z}_N^{(1)}$ one-form SPT phase, that is invariant under the CKS transformation. We will consider oriented SPT phases, both bosonic and fermionic. Our analysis is an antilinear version of the ones in [23,30].

For odd N, the possible SPT phases are labeled by an integer $q \sim q + N$, where the partition function is given by $\exp\left[(2\pi i q/N)\int B\cup B\right]$. Here and below, B is a background gauge field for the $\mathbb{Z}_N^{(1)}$ one-form symmetry, whereas b denotes a dynamical gauge field. If we apply the S transformation, the resulting theory is again invertible if and only if $\gcd(q, N) = 1$. Under this condition, we obtain

$$\sum_b \exp\left[\int\left(\frac{2\pi i q}{N} b\cup b + \frac{2\pi i k}{N} b\cup B\right)\right] = \exp\left[\frac{2\pi i}{N}\left(-\frac{k^2}{4q}\right)\int B\cup B\right], \tag{13}$$

TABLE I. Some (anti)linear (non)invertible symmetries of the $\mathcal{N} = 4$ SU(2) super Yang-Mills theory. Here K is the naive time-reversal transformation, $\mathbb{S}$, $\mathbb{T}$ generate the electromagnetic duality, and S, T generate the (projective) $SL(2,\mathbb{Z}_2)$ transformation via gauging the $\mathbb{Z}_2^{(1)}$ center one-form symmetry.

Complexified coupling τ	Symmetry	Invertible?	Linear?
$\theta = 0$	K	Yes	Antilinear
$\theta = \pi$	$\mathbb{T}TK$	Yes	Antilinear
$\|\tau\| = 1$	$\mathbb{S}SK$	No	Antilinear
$\tau = i$	$\mathbb{S}S$	No	Linear
$\tau = e^{2\pi i/3}$	$\mathbb{S}\mathbb{T}ST$	No	Linear

where we are neglecting the overall normalization and the gravitational counterterms.

On the other hand, the K transformation acts as complex conjugation on the partition function. Therefore, under the KS transformation, $q \mapsto +k^2/4q$. The C transformation acts trivially on the SPT phase, since its action is quadratic in B. Thus, the CKS invariance of the SPT phase requires $q = +k^2/4q \mod N$, or, equivalently,

$$4q^2 = k^2 \mod N. \tag{14}$$

Since N is odd, 2 is invertible mod N, and $q = 2^{-1}k \mod N$ is always a solution of the equation. We conclude that the CKS invariance is compatible with a trivially gapped phase for any N and p.

For even N, the possible SPT phases are $\exp[(2\pi i q/2N)\int \mathcal{P}(B)]$, where $\mathcal{P}(B)$ is the Pontryagin square of B. We have $q \sim q + 2N$ for the bosonic case, and $q \sim q + N$ for the fermionic case. In the even N case, the S transformation acts as

$$\sum_b \exp\left[\int\left(\frac{2\pi i q}{2N}\mathcal{P}(b) + \frac{2\pi i k}{N} b \cup B\right)\right] = \exp\left[\frac{2\pi i}{2N}\left(-\frac{k^2}{q}\right)\int \mathcal{P}(B)\right]. \tag{15}$$

Next, under the CKS transformation, $q \mapsto +k^2/q$. Therefore, the condition for the SPT phase to be CKS invariant is

$$q^2 = k^2 \begin{cases} \mod 2N & \text{if bosonic} \\ \mod N & \text{if fermionic} \end{cases}. \tag{16}$$

We see that $q = k$ is always a solution, consistent with the claim that there is no generalized anomaly.

Conclusion and outlook.—We find that the free Maxwell theory and massive QED for the real world at a rational θ angle is time-reversal invariant. The time-reversal symmetry is implemented by a conserved, antilinear operator $\mathsf{T}^{\theta=\pi p/N}$, but it is noninvertible because of the fractional quantum Hall state attached to the operator. By composing with the CPT transformation, the noninvertible time-reversal symmetries that we found also imply the existence of noninvertible CP symmetries in these theories. Similar noninvertible time-reversal symmetries also exist in $\mathcal{N} = 4$ super Yang-Mills theories and in various SPT phases.

In this Letter, we only identify the existence of these new time-reversal symmetries, and defer their applications for future investigations. Below we outline a few interesting future directions: (i) In the context of the strong CP problem, it is natural to wonder if there is a hidden noninvertible T or CP symmetry when the SU(3) θ angle vanishes in the standard model [85]. (ii) It is known that at $\theta = 0$, there are 7 different versions of time-reversal symmetric Maxwell theory, distinguished by the quantum numbers of the Wilson and ‘t Hooft lines [87,88]. It would be interesting to generalize this discussion to Maxwell theory with noninvertible time-reversal symmetry at a rational θ angle. (iii) It would be interesting to compute the partition function of the Maxwell theory at $\theta = \pi p/N$ on an unoriented manifold using our noninvertible time-reversal symmetry, extending the classic results of [89,90]. Similar generalizations of [91,92] can be explored for the $\mathcal{N} = 4$ super Yang-Mills theory along $|\tau| = 1$.

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