Classical Cost of Transmitting a Qubit

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We consider general prepare-and-measure scenarios in which Alice can transmit qubit states to Bob, who can perform general measurements in the form of positive operator-valued measures (POVMs). We show that the statistics obtained in any such quantum protocol can be simulated by the purely classical means of shared randomness and two bits of communication. Furthermore, we prove that two bits of communication is the minimal cost of a perfect classical simulation. In addition, we apply our methods to Bell scenarios, which extends the well-known Toner and Bacon protocol. In particular, two bits of communication are enough to simulate all quantum correlations associated to arbitrary local POVMs applied to any entangled two-qubit state.

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Introduction.—Quantum resources enable a sender and a receiver to break the limitations of classical communication. When entanglement is available, classical [1–4] as well as quantum communication [5,6] can be boosted beyond purely classical models. A seminal example is dense coding, in which two classical bits can be substituted for a single qubit and shared entanglement [7]. However, entanglement is not necessary for quantum advantages. Communicating an unassisted d-dimensional quantum system frequently outperforms the best conceivable protocols based on a classical d-dimensional system [8–12]; even yielding advantages growing exponentially in d [13,14]. Already in the simplest meaningful scenario, namely, that in which the communication of a bit is substituted for a qubit, sizable advantages are obtained in important tasks like random access coding [15–17]. These qubit advantages propel a variety of quantum information applications [18–22].

It is natural to explore the fundamental limits of quantum over classical advantages. In order to do so, one has to investigate the amount of classical communication required to model the predictions of quantum theory. Previous works consider not only the scenario of sending quantum systems [23–27], but also simulating bipartite [23–34], as well as multipartite entangled quantum systems [35–38]. While such classical simulation of quantum theory is in general challenging, a breakthrough was made by Toner and Bacon [26]. Their protocol shows that any quantum prediction based on standard, projective, measurements on a qubit can be simulated by communicating only two classical bits. However, this does not account for the full power of quantum theory. More precisely, there exists qubit measurements that cannot be reduced to stochastic combinations of projective ones [39]. The most general measurements are known as positive operator-valued measures (POVMs). Physically, they correspond to the receiver interacting the message qubit with a locally prepared auxiliary qubit, and then performing a measurement on the joint system [40]. Such POVMs are even indispensable for important tasks like unambiguous state discrimination [41,42] and hold a key role in many quantum information protocols (see, e.g., [43–51]). Importantly, they also give rise to correlations that cannot be modeled in any qubit experiment based on projective measurements [52–56].

This naturally raises the question of identifying the classical cost of simulating the most general predictions of quantum theory, based on POVMs. In the minimal qubit communication scenario, one may suspect that this cheap price of only two bits is due to the restriction to the, fundamentally binary, projective measurements. In contrast, when measurements are general POVMs, it is even unclear whether the classical simulation cost is finite. Notably, previous work has shown that there exists a classical simulation that requires 5.7 bits of communication on average [23,27]. However, that protocol has a certain probability to fail in each round, leading to an unbounded amount of communication in the worst case.

In this Letter, we explicitly construct a classical protocol that simulates all qubit-based correlations in the prepare-and-measure scenario by using only two bits of communication. Thus, we find that the cost of a classical simulation remains the same when considering the most general class of measurements, although POVMs enable more general quantum correlations than projective measurements. Moreover, we show that two bits is the minimal classical simulation cost, i.e., there exists no classical simulation that uses less communication than our protocol. This is shown through an explicit quantum protocol, based on qubit communication, that eludes simulation with a ternary classical message. Finally, we apply our methods to Bell nonlocality scenarios [57]. We present novel protocols that simulate the statistics of local measurements on entangled qubit pairs.

The prepare-and-measure scenario.—A quantum prepare-and-measure (PM) scenario [see Fig. 1(a)] consists of two steps. First, Alice prepares an arbitrary quantum state of dimension d_Q and sends it to Bob. The state is described by a positive semidefinite $d_Q \times d_Q$ complex matrix $\rho \in \mathcal{L}(\mathbb{C}_{d_Q})$, $\rho \geq 0$ with unit trace $\mathrm{tr}(\rho) = 1$. Second, Bob receives the state and performs an arbitrary quantum measurement on it, obtaining an outcome b. General quantum measurements are described by a POVM, which is a set of operators $\{B_b\}$ that are positive semidefinite, $B_b \geq 0$ and sum to the identity, $\sum_b B_b = \mathbb{1}$. In quantum theory, the probability of outcome b when performing the POVM $\{B_b\}$ on the state ρ is given by Born's rule,

$$p_Q(b|\rho, \{B_b\}) = \mathrm{tr}(\rho B_b). \tag{1}$$

We are interested in constructing classical models for the PM scenario that simulate the predictions of quantum theory, i.e., classical models that reproduce the probability distribution (1). In a classical simulation [see Fig. 1(b)], Alice and Bob may share a random variable λ subject to some probability function $\pi(\lambda)$. This allows them to correlate their classical communication strategies. Alice uses λ and her knowledge of the quantum state ρ to choose a classical message c selected from a d_C-valued alphabet $\{1, \ldots, d_C\}$. Since the selection can be probabilistic, her actions are described by the conditional probability distribution $p_A(c|\rho, \lambda)$. When Bob receives the message, he uses λ and his knowledge of the POVM $\{B_b\}$ to choose his outcome b. Again, this choice can be probabilistic and is therefore described by a conditional probability distribution $p_B(b|\{B_b\}, c, \lambda)$. All together, the correlations obtained from the classical model become

$$p_C(b|\rho, \{B_b\}) = \int_\lambda \mathrm{d}\lambda\, \pi(\lambda) \sum_{c=1}^{d_C} p_A(c|\rho, \lambda) p_B(b|\{B_b\}, c, \lambda). \tag{2}$$

The simulation is successful if, for any choice of d_Q-dimensional states and POVMs, the quantum predictions p_Q can be reproduced with a classical model using messages that attain at most d_C different values. That is, if there exists a d_C and suitable encodings p_A and decodings p_B, such that

$$\forall\ \rho, \{B_b\}:\ p_C(b|\rho, \{B_b\}) = p_Q(b|\rho, \{B_b\}). \tag{3}$$

If this holds, we say that the classical model simulates quantum theory. In particular, we say that the classical simulation is minimal if no classical simulation is possible using a smaller message alphabet size d_C. Furthermore, we remark that for some PM scenarios, shared randomness may be charged as a nonfree resource, leading to different results and problems [17,24,50,55,58–62]. In fact, for the PM scenario we study here, it is known that an infinite amount of shared randomness is required in order to perform the task with finite classical communication [24].

Our focus is on the most fundamental scenario, namely, that based on qubits ($d_Q = 2$). Notice that there exists a trivial classical simulation in which Alice sends the Bloch vector coordinates of her quantum state to Bob. After that, he can classically compute the Born rule and samples his outcome accordingly. However, sending the coordinates requires an infinite amount of communication (d_C unbounded). Whether a classical simulation is possible with a finite value of d_C is much less trivial. Notably, the simulation protocol of Toner and Bacon showed that if we additionally restrict the quantum measurements to be projective, i.e., $B_b^2 = B_b$, a classical simulation with $d_C = 4$ (two bits) is possible [26].

We also remark that here we consider a scenario where Bob does not know Alice's state and Alice does not know Bob's measurement beforehand. This scenario, where Alice and Bob can independently choose between different states and measurements, is even required to provide quantum over classical advantages in several tasks [13–17]. An interesting related scenario is the one where Bob's

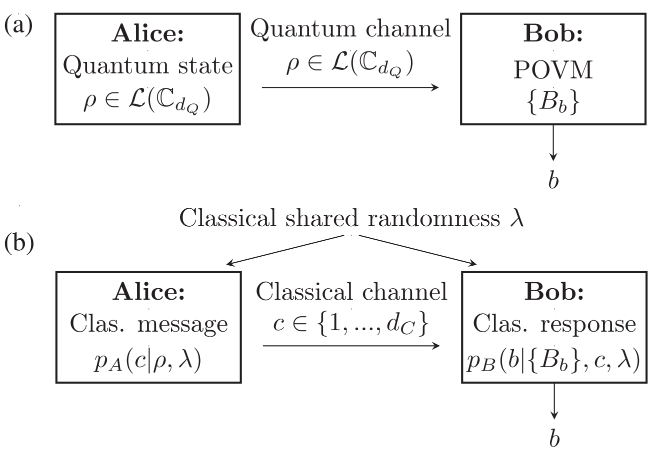

FIG. 1. (a) Quantum PM scenario: Alice sends a d_Q-dimensional state to Bob who performs a POVM to obtain his outcome. (b) Classical PM scenario for simulating the quantum PM scenario: The classical simulation is successful if, for every state and POVM, the probability that Bob outputs b is the same as in the quantum protocol.

measurement is known by Alice, or, equivalently, Bob has only a single choice of measurement. In that case, Frenkel and Weiner [63] proved that, in the presence of shared randomness, a d-dimensional quantum system can always be perfectly simulated by a d-dimensional classical system. This powerful result inspired proposals such as the "no-hypersignaling" principle [64], which is respected by quantum theory. In what follows, we find a minimal classical simulation for general qubit protocols.

Classical simulation protocol.—Qubit states ρ can be represented as $\rho = (\mathbb{1} + \vec{x}\cdot\vec{\sigma})/2$, where $\vec{x}\in\mathbb{R}^3$ is a three-dimensional real vector such that $|\vec{x}|\leq 1$, and $\vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ are the standard Pauli matrices. We may, without loss of generality, restrict ourselves to quantum protocols based on pure states. This corresponds to unit vectors $|\vec{x}| = 1$. Since mixed states are convex combinations of pure states, every classical simulation protocol applicable to pure states can immediately be extended to apply also to mixed states. The classical randomness in the convex combination can simply be absorbed in the shared randomness of the simulation protocol. Similarly, because every qubit POVM can be written as a coarse graining of rank-1 projectors [65], we may restrict ourselves to POVMs proportional to rank-1 projectors. Thus, we write Bob's measurements as $B_b = 2p_b|\vec{y}_b\rangle\langle\vec{y}_b|$, where $p_b\geq 0$, $\sum_b p_b = 1$ and $|\vec{y}_b\rangle\langle\vec{y}_b| = (\mathbb{1} + \vec{y}_b\cdot\vec{\sigma})/2$ for some normalized vector $\vec{y}_b\in\mathbb{R}^3$. In Bloch notation we have

$$\mathrm{tr}(\rho B_b) = p_b(1 + \vec{x}\cdot\vec{y}_b). \tag{4}$$

We now present a classical simulation protocol in which Alice and Bob can perfectly simulate all qubit correlations at the cost of two bits of communication. To this end, it is handy to first define the Heaviside function, defined by $H(z) = 1$ when $z\geq 0$ and $H(z) = 0$ when $z < 0$, as well as the related function $\Theta(z) := z\cdot H(z)$. Consider now the following protocol.

(1) Alice and Bob share two normalized vectors $\vec{\lambda}_1, \vec{\lambda}_2 \in \mathbb{R}^3$, which are uniformly and independently distributed on the unit radius sphere S_2.

(2) Instead of sending a pure qubit $\rho = (\mathbb{1} + \vec{x}\cdot\vec{\sigma})/2$, Alice prepares two bits via the formula $c_1 = H(\vec{x}\cdot\vec{\lambda}_1)$ and $c_2 = H(\vec{x}\cdot\vec{\lambda}_2)$ and sends them to Bob.

(3) Bob flips each vector $\vec{\lambda}_i$ when the corresponding bit c_i is zero. More formally, he sets $\vec{\lambda}_i' := (-1)^{1+c_i}\vec{\lambda}_i$.

(4) Instead of performing a POVM with elements $B_b = 2p_b|\vec{y}_b\rangle\langle\vec{y}_b|$, Bob picks one vector $\vec{y}_b$ from the set $\{\vec{y}_b\}$ according to the probabilities $\{p_b\}$. Then he sets $\vec{\lambda} := \vec{\lambda}_1'$ if $|\vec{\lambda}_1'\cdot\vec{y}_b| \geq |\vec{\lambda}_2'\cdot\vec{y}_b|$ and $\vec{\lambda} := \vec{\lambda}_2'$ otherwise. Finally, Bob outputs b with probability

$$p_B(b|\{B_b\},\vec{\lambda}) = \frac{p_b\Theta(\vec{y}_b\cdot\vec{\lambda})}{\sum_j p_j\Theta(\vec{y}_j\cdot\vec{\lambda})}. \tag{5}$$

The proof that the protocol perfectly reproduces the qubit correlations (4) is given in the Supplemental Material (Sec. I) [66]. A sketch of the first three steps of the protocol is given in Fig. 2. After the third step, the two vectors $\vec{\lambda}_1'$ and $\vec{\lambda}_2'$ are uniformly and independently distributed in the positive hemisphere defined by $\vec{x}$, i.e., their probability densities are $\rho(\vec{\lambda}_i') = H(\vec{x}\cdot\vec{\lambda}_i')/(2\pi)$. As we show, this distribution is enough for Bob to classically reproduce the statistics of every POVM applied to the qubit state associated to $\vec{x}$. Furthermore, in the Supplemental Material (Sec. I) [66] we also present a modified version of that protocol. There, Bob sends first one bit to Alice and then Alice sends one bit back to Bob.

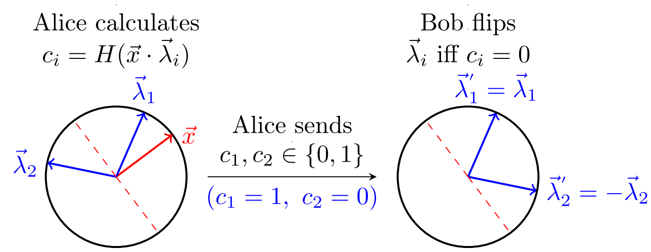

FIG. 2. A two-dimensional illustration of the first three steps in the classical simulation protocol based on two bits.

Two bits are necessary for a classical simulation.—We have shown that two classical bits are sufficient to simulate qubit correlations. We now prove that they are also necessary, i.e., that the above classical simulation protocol is minimal.

To this end, we show that there exists correlations in the qubit PM scenario that cannot be modelled in any classical protocol (2) that uses ternary messages ($d_C = 3$). For this purpose, we consider PM scenarios with a fixed number of inputs for Alice and Bob. Alice selects her input from a set $x\in\{1,\ldots,I_A\}$ and prepares the qubit ρ_x. Bob selects his input from a set $y\in\{1,\ldots,I_B\}$ and performs the two-outcome projective measurement $\{B_{b|y}\}$ with outcomes labelled by $b\in\{1,2\}$. The qubit correlations are then given by $p_Q(b|x,y) = \mathrm{tr}(\rho_x B_{b|y})$. Notice that although Bob could perform POVMs, we are restricting ourselves to projective measurements. These turn out to be sufficient for the proof.

It is key to recognise that the task of deciding whether a given $p(b|x,y)$ admits a classical simulation with a d_C-dimensional message alphabet can be solved by means of linear programming. From the duality theory of linear programming [67], we can obtain a classical dimension witness that certifies that the probabilities $p_Q(b|x,y)$ cannot be simulated by sending classical ternary messages. A classical dimension witness [14,18] is a linear inequality which is respected by *all* classical models in the PM scenario for a given d_C. This can, in general, be written as

$$\sum_{b,x,y}\gamma(b|x,y)p_C(b|x,y) \leq C_d, \tag{6}$$

for some coefficients $\gamma(b|x,y) \in \mathbb{R}$. Here, C_d is the classical bound. A violation of this inequality certifies that no classical model using d_C symbols can simulate $p_Q(b|x,y)$. In the Supplemental Material (Sec. III) [66], we detail these linear programming methods. Inspired by the efficient method to find local bounds of Bell inequalities presented in Ref. [68], we provide a new and efficient algorithm to obtain the classical d_C-dimensional bound $\leq C_d$ for any given set of coefficients $\{\gamma(b,x,y)\}$. Also, drawing inspiration from Ref. [69], we developed computational methods to convert the numerical solutions obtained from standard solvers to rigorous computer-assisted proofs which do not suffer from numerical precision issues due to floating point arithmetic.

In this way, we have obtained several examples of qubit states and measurements that generate quantum correlations $p_Q(b|x,y)$ that do not admit a classical model for $d_C = 3$. An elegant example is obtained from considering $I_A = 6$ states that form an octahedron on the Bloch sphere. They correspond to the eigenstates of the three Pauli operators $(\sigma_X, \sigma_Y, \sigma_Z)$. We let Bob perform $I_B = 24$ different projective measurements. The Bloch vectors of these measurements are oriented such that they point to the vertices of a snub cube [70], which is an Archimedean solid, inscribed in the Bloch sphere. This may be viewed as a PM variant of platonic Bell inequality violations [71]. Specifically, the 24 measurement directions are obtained as follows. Let τ be the one real root of the polynomial $x^3 - x^2 - x - 1$, known as the Tribonacci constant. Take all even (odd) permutations of, $(\pm 1, \pm 1/\tau, \pm\tau)$ and for each permutation, take only the four sign combinations that have an even (odd) number of "+". This gives all vertices of the snub cube. Finally, do a global rotation by 60 degrees in the XY plane, i.e., apply the unitary $U = |0\rangle\langle 0| + e^{(i\pi/3)}|1\rangle\langle 1|$ to all projectors. The linear programming methods reveal that the resulting p_Q has no classical model for $d_C = 3$.

In the Supplemental Material (Sec. III) [66], we discuss a heuristic approach to find states and measurements leading to probabilities which do not admit a classical simulation for $d_C = 3$. Fixing the above six preparations, the sparsest proof we have found uses eleven measurements that correspond to the solution of the Thomson problem [72]. All our computational code is openly available at the online repository [73].

Although no ternary message protocol is sufficient, it may still be that a classical simulation is possible by sending less than two bits on average. For example, Alice may restrict herself to send in some fraction of rounds only a trit, a bit or no communication at all. For the case of sometimes sending a bit or less, we show in the Supplemental Material (Sec. II) [66] that no classical simulation is possible. The reason is closely connected to the zero local weight of the singlet state, also known as the EPR2 decomposition [74,75]. Our argument shows that, if one could simulate qubit correlations by sometimes sending only a bit or less, one could construct a protocol that simulates the singlet state without communication in these rounds. This would induce a local part for the singlet state, which contradicts the EPR2 decomposition.

Simulating nonlocality.—It is straightforward to adapt our classical protocol to simulate the statistics obtained from arbitrary local POVMs on any entangled qudit-qubit state. Indeed, all PM protocols can be adapted to Bell scenarios [23]. For that, Alice chooses her measurement, an arbitrary POVM on a d_Q-dimensional quantum system. Then, she produces an output according to the marginal distribution of her POVM elements and, depending on her outcome, calculates the post-measurement state of Bob's qubit. Finally, she simply uses the classical protocol for the PM scenario to send that qubit state to Bob. Thus, our protocol immediately extends the best previously known one, due to Toner and Bacon [26], to Bell scenarios involving POVMs. At the same time, we use the same amount of classical communication, in fact, two bits.

However, Toner and Bacon also show that only a single bit is necessary to simulate local projective measurements on a qubit pair in the singlet state $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. We can also extend that result by constructing a novel one bit protocol. Here, Alice is restricted to projective measurements with outcomes $a = \pm 1$, but Bob can perform arbitrary POVMs.

(1) Alice and Bob share two normalized vectors $\vec{\lambda}_1', \vec{\lambda}_2 \in \mathbb{R}^3$, which are uniformly distributed on the unit radius sphere S_2.

(2) Instead of performing a projective measurement with projectors $|\pm\vec{x}\rangle\langle\pm\vec{x}| = (\mathbb{1} \pm \vec{x}\cdot\vec{\sigma})/2$, Alice outputs $a = -\mathrm{sgn}(\vec{x}\cdot\vec{\lambda}_1')$ and sends the bit $c = \mathrm{sgn}(\vec{x}\cdot\vec{\lambda}_1')\cdot\mathrm{sgn}(\vec{x}\cdot\vec{\lambda}_2)$ to Bob. Here, $\mathrm{sgn}(z) = 1$ when $z \geq 0$ and $\mathrm{sgn}(z) = -1$ when $z < 0$.

(3) Bob flips the vector $\vec{\lambda}_2$ if and only if $c = -1$. More formally, he sets $\vec{\lambda}_2' \coloneqq c\vec{\lambda}_2$.

(4) Same as "Step 4" in the original prepare-and-measure protocol.

Since $\vec{\lambda}_1'$ is uniformly distributed on S_2, we obtain the correct marginal probabilities $p(a) = 1/2$ for Alice. Furthermore, when Alice outputs $a = +1$, $\vec{\lambda}_1'$ and $\vec{\lambda}_2'$ are distributed on S_2 according to $\rho(\vec{\lambda}_i') = H(-\vec{x}\cdot\vec{\lambda}_i')/(2\pi)$. This corresponds precisely to a classical description of Bob's post-measurement state $-\vec{x}$ (compare with the text below Fig. 2). When Alice outputs $a = -1$, the two vectors are distributed according to $\rho(\vec{\lambda}_i') = H(+\vec{x}\cdot\vec{\lambda}_i')/(2\pi)$, which corresponds to the correct post-measurement state $+\vec{x}$. Therefore, Bob can apply the same response function ("Step 4") as in the original PM protocol, which immediately yields the correct quantum probabilities. Additionally, since singlet correlations have no local part [74,75], one bit of communication is necessary in each round, ensuring the optimality of this protocol. Clearly, this protocol can be easily adapted to any maximally entangled qubit pair by rotating either Alice's or Bob's measurement basis.

Discussion.—We have proven that two bits of communication are necessary and sufficient in order to classically

TABLE I. Comparison between our protocol and the one by Toner and Bacon, previously the best protocol for these scenarios but restricted to only projective measurements (denoted as Proj. in this table) on Bob's side. Our protocols use the same resources, but Bob is allowed to perform POVMs.

Scenario	This Letter	Ref. [26]
PM with qubit	2 bits, POVMs	2 bits, only Proj.
Bell with 2 qubits	2 bits, POVMs	2 bits, only Proj.
Bell with singlet	1 bit, Proj.-POVM	1 bit, Proj.-Proj.

simulate the most general predictions of quantum theory in a qubit prepare-and-measure scenario. Our results also have immediate implications for simulations of nonlocality in scenarios featuring POVMs. In this way, we generalized the well-known protocols of Toner and Bacon [26] from projective measurements to the most general qubit measurements (POVMs). Interestingly, this comes with no increase in the classical cost. See Table I for an overview.

A natural direction is to consider classical simulations for higher-dimensional quantum PM scenarios ($d_Q > 2$), or scenarios involving entanglement. Notably, the latter can sometimes be isomorphic to the former [76]. Although this has received some attention [33,34,38,77], few general results are known. Most notably, it is still an open problem whether a qutrit ($d_Q = 3$) PM scenario can be classically simulated with a finite message alphabet ($d_C < \infty$).

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