Shift-Invariant Orders of an Axionlike Particle

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It is generally believed that global symmetries, in particular, axion shift symmetries, can only be approximate. This motivates one to quantify the breaking of the shift invariance that characterizes the flavorful effective couplings of an axionlike particle to standard model fermions. We identify a minimal set of Jarlskog-like flavor invariants that vanish if and only if the axion is shift symmetric. Therefore, they represent order parameters for the breaking of the axion shift symmetry. We illustrate properties of the invariants by matching to a UV model, studying the CP transformation of the invariants, calculating their renormalization group evolution, and investigating similar conditions in the low-energy effective field theory below the electroweak scale. Finally, we discuss the order parameter associated to the nonperturbative shift-breaking induced by the axion-gluons coupling, which is also flavorful.

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Introduction.—Axions [1–12] benefit from an approximate shift symmetry, henceforth referred to as a Peccei-Quinn (PQ) symmetry, rooted in their pseudo–Nambu-Goldstone boson (pNGB) nature. This allows, for instance, QCD axions to receive their mass mostly from QCD or fuzzy axion dark matter to be ultralight. However, there are several reasons to consider some amount of shift breaking. First, quantum gravity objects to exact global symmetries and is expected to generate irreducible corrections to axion potentials and interactions [13–15], whose control in pNGB models goes under the name of axion quality problem. Second, there are cases where shift breaking is a key aspect of model building: for instance, a slight amount of shift breaking is responsible for the scanning of the Higgs mass and the resolution of the hierarchy problem in relaxion models [16]. Therefore, considering shift-breaking axionlike particle (ALP) interactions seems to be necessary to make contact with theory and phenomenology.

Consequently, it is important to clearly pinpoint the presence of physical shift-symmetry-breaking couplings, as well as to quantify their magnitude. To this end, we scrutinize the couplings of axions to standard model (SM) fields. As axions arise in a large class of UV models and—due to their lightness—are omnipresent in processes at all energy scales, an effective field theory (EFT) analysis is the appropriate language to study them in a bottom-up and generic manner. Hence, axion EFTs have been systematically studied since the early days of axion physics [17,18]. In this Letter, we carry on the systematic study of the structural properties of axion EFTs, with a focus on the breaking of axion shift invariance due to the axion couplings to SM fermions.

We work in a nonredundant operator basis that captures the most generic leading-order couplings of a light pseudoscalar a to SM fermions, namely,

$$\mathcal{L} \supset -\frac{a}{f}(\bar{Q}\tilde{Y}_u\tilde{H}u + \bar{Q}\tilde{Y}_d Hd + \bar{L}\tilde{Y}_e He + \text{H.c.}) + \mathcal{O}\left(\frac{1}{f^2}\right), \tag{1}$$

where f is the axion decay constant [we henceforth take $f \gg v$, the electroweak (EW) scale] and $\tilde{Y}_{u,d,e}$ are generic complex matrices in flavor space. The goal of this Letter is to revisit the conditions for the couplings of Eq. (1) to be interpreted as those of a shift-invariant axion and to quantify the deviations from such conditions.

Interactions of shift-invariant axions are commonly described using the following Lagrangian, where the axion shift symmetry $a \to a + \epsilon f$ is manifest:

$$\mathcal{L} \supset \frac{\partial_\mu a}{f}\sum_{\psi\in\text{SM}}\bar{\psi}\, c_\psi \gamma^\mu \psi + \mathcal{O}\left(\frac{1}{f^2}\right). \tag{2}$$

The sum runs over all Weyl fermions of the SM and the c_ψ are Hermitian matrices in flavor space. Then, in order to describe shift-invariant couplings using Eq. (1), one can map the couplings of Eq. (2) onto those of Eq. (1) via field redefinitions [19,20]. This mapping procedure produces $\tilde{Y}$ couplings in Eq. (1) which satisfy [19,20]

$$\tilde{Y}_{u,d} = i(Y_{u,d}c_{u,d} - c_Q Y_{u,d}), \qquad \tilde{Y}_e = i(Y_e c_e - c_L Y_e), \quad (3)$$

where $Y_{u,d,e}$ are the SM Yukawa couplings. Conversely, when the above relations hold for some Hermitian matrices $c_{Q,u,d,L,e}$, Eq. (1) can be turned into Eq. (2) via appropriate field redefinitions [20].

However, these conditions are implicit: given a set of couplings, one has to check whether a set of equations can be solved. In addition, they do not allow one to differentiate between approximate and badly broken shift symmetries, nor to identify a power counting parameter suppressing the breaking. Instead, we will present *explicit* conditions on the axion Yukawa couplings of Eq. (1), which can be directly evaluated given the SM Yukawa couplings. They define quantities that vanish if and only if the axion shift symmetry is preserved and their size quantifies how badly it is broken. Hence, they are order parameters of the breaking of the axion shift symmetry.

This is very similar in spirit to finding the Jarlskog invariant for CP violation in the SM [21,22], or in the SM effective field theory [23], instead of scanning possible field redefinitions that absorb unphysical complex Lagrangian parameters. Hence, it is no surprise that our conditions are expressed in terms of flavor invariants, namely, combinations of Lagrangian parameters that are left unchanged under flavor field redefinitions. This allows us to encode the physical collective effects associated with the presence of the axion shift symmetry.

Flavor-invariant order parameters for the breaking of an axion shift symmetry.—To identify all conditions on $\tilde{Y}_{u,d,e}$, let us first count how many we expect, by comparing the free parameters in the operator bases of Eqs. (1) and (2). We classify couplings according to their behavior under CP: since the axion is a pseudoscalar, real $\tilde{Y}_\psi$ in Eq. (1) and imaginary c_ψ in Eq. (2) are CP odd in the mass basis.

When the axion shift symmetry is broken, we need to use the non-shift-symmetric EFT of Eq. (1), in which the couplings $\tilde{Y}$ are arbitrary 3×3 complex matrices. Instead, in the shift-invariant basis of Eq. (2), there are two Hermitian matrices $c_{L,e}$ in the lepton sector and three in the quark sector $c_{Q,u,d}$. In both cases, two unphysical phases can be removed by the lepton family numbers $\mathrm{U}(1)_{L_i}$, which are symmetries in the SM. Furthermore, a freedom exists in the derivative basis, associated with the addition of the operator $\partial_\mu a J^\mu$, for any (classically) conserved fermionic current of the SM J^μ [24], namely, the baryon number $\mathrm{U}(1)_B$ and the three lepton numbers $\mathrm{U}(1)_{L_i}$. Counting the resulting physical parameters leads us to Table I, where we see that we expect three CP-odd relations in the lepton sector together with nine CP-odd and one CP-even relation in the quark sector that characterize the presence of a shift symmetry in the basis of Eq. (1).

TABLE I. Number of physical coefficients at dimension-five in the EFTs of Eqs. (2) and (1), and numbers of constraints that $\tilde{Y}_{u,d,e}$ need to verify to respect an exact shift invariance.

	Shift-sym. coeff. $c_{Q,u,d,L,e}$		Generic coeff. $\tilde{Y}_{u,d,e}$		No. of constraints	
	CP even	CP odd	CP even	CP odd	CP even	CP odd
Quark sector	17	9	18	18	1	9
Lepton sector	9	4	9	7	0	3

Let us now derive those relations, first for quarks and then for leptons. In order to manipulate physical quantities independent of the flavor basis, we work with flavor invariants, i.e., singlets of the $U(3)^5$ fermion transformations affecting the flavorful couplings in the Lagrangian. As already mentioned, we want to find equivalent conditions to those in Eq. (3). Solving Eq. (3) for $c_{u,d}$ and enforcing Hermiticity, we obtain the following commutator relation:

$$[c_Q, X_x] = i(\tilde{Y}_x Y_x^\dagger + Y_x \tilde{Y}_x^\dagger), \quad (4)$$

with $X_x \equiv Y_x Y_x^\dagger$ and $x = u,\, d$. We can then find flavor-invariant constraints by exploiting well-known commutator identities, for instance, by using that for any two matrices A, B, $\mathrm{Tr}(A^n[A,B]) = 0 \; \forall\, n \in \mathbb{Z}$. In the lepton sector, this already exhausts the number of conditions of Table I, while for the quarks the presence of c_Q in both the up and down sectors demands that we consider more commutator identities to obtain the remaining relations. Eventually, we consider the following minimal set [25] of flavor invariants, linear in $\tilde{Y}_{u,d,e}$:

$$\begin{aligned}
I^{(1,2)}_{u,d,e} &= \mathrm{ReTr}(X^{0,1}_{u,d,e}\tilde{Y}_{u,d,e}Y^\dagger_{u,d,e}), \qquad I^{(3)}_e = \mathrm{ReTr}(X_e^2\tilde{Y}_e Y_e^\dagger), \qquad I^{(3)}_q = \mathrm{ReTr}(X_u^2\tilde{Y}_u Y_u^\dagger + X_d^2\tilde{Y}_d Y_d^\dagger),\\
I^{(1)}_{ud} &= \mathrm{ReTr}(X_d\tilde{Y}_u Y_u^\dagger + X_u\tilde{Y}_d Y_d^\dagger), \qquad I^{(2)}_{ud;u,d} = \mathrm{ReTr}(X^2_{u,d}\tilde{Y}_{d,u}Y^\dagger_{d,u} + \{X_u, X_d\}\tilde{Y}_{u,d}Y^\dagger_{u,d}),\\
I^{(3)}_{ud} &= \mathrm{ReTr}(X_d X_u X_d \tilde{Y}_u Y_u^\dagger + X_u X_d X_u \tilde{Y}_d Y_d^\dagger), \qquad I^{(4)}_{ud} = \mathrm{ImTr}([X_u, X_d]^2([X_d, \tilde{Y}_u Y_u^\dagger] - [X_u, \tilde{Y}_d Y_d^\dagger])).
\end{aligned} \quad (5)$$

These invariants have to vanish for the EFT in the Yukawa basis of Eq. (1) to be shift invariant, which also provides sufficient conditions. This is shown by taking advantage of their linearity in $\tilde{Y}_{u,d,e}$, which allows us to use simple linear algebra: we compute the rank of the matrix that relates the set of invariants $\{I_A\}$ to the entries of $\tilde{Y}_{u,d,e}$ in a given flavor basis. It corresponds to the number of conditions associated with the set of equalities $I_A = 0\,\forall\, A$ and is found to be 13, namely, 10 in the quark sector and 3 in the lepton sector, which agrees with the number of conditions from shift invariance (see Supplemental Material [26]). Therefore, the invariants in Eq. (5) vanish if and only if $\tilde{Y}_{u,d,e}$ describe the couplings of a shift-symmetric axion. We stress that these conditions are algebraic and explicit: given values for $\tilde{Y}_{u,d,e}$, evaluating the invariants suffices to discriminate between shift-invariant or shift-breaking couplings. Let us also emphasize that the five last invariants in Eq. (5) encode *collective* effects, namely, they are associated with the simultaneous presence of both the up- and down-quark sectors.

Examples and properties.—To illustrate the use of our invariants, and confirm that they capture the sources of PQ-breaking and its collective nature, we can match the ALP EFT onto UV models and evaluate them. For definiteness, we focus on the axiflavon-flaxion model [27–30], which realizes the Froggatt-Nielsen and Peccei-Quinn mechanisms through the same spontaneously broken U(1). One introduces a complex scalar flavon ϕ, with the following effective interactions with the SM fields:

$$-\mathcal{L} = \alpha^d_{ij}\left(\frac{\phi}{M}\right)^{q_{Q_i}-q_{d_j}}\bar{Q}_i H d_j + \alpha^u_{ij}\left(\frac{\phi}{M}\right)^{q_{Q_i}-q_{u_j}}\bar{Q}_i \tilde{H} u_j + \alpha^e_{ij}\left(\frac{\phi}{M}\right)^{q_{L_i}-q_{e_j}}\bar{L}_i H e_j + \text{H.c.}, \tag{6}$$

with M as the cutoff of the model and q_i as the charges of the SM fields under the new U(1) ($q_\phi = +1, q_H = 0$). The symmetry is broken by the flavon vacuum expectation value $\langle\phi\rangle = f$, which also sources the hierarchy of the SM Yukawa couplings. Writing $\phi = (1/\sqrt{2})(f + s + ia)$, where s is a massive radial mode and a is the axion field of the theory, one obtains the SM and axion-fermion Yukawa couplings,

$$Y^x_{ij} = \alpha^x_{ij}\left(\frac{f}{\sqrt{2}M}\right)^{q_{x_{L,i}}-q_{x_{R,j}}}, \qquad \tilde{Y}^x_{ij} = (q_{x_{L,i}} - q_{x_{R,j}})Y^x_{ij}. \tag{7}$$

The Lagrangian in Eq. (6) is PQ invariant, hence all couplings in Eq. (7) correspond to a shift-symmetric axion. This is consistent with the fact that our invariants I_A vanish when evaluated on the above couplings. They become nonzero when one introduces a PQ-breaking term. For instance, by adding

$$-\mathcal{L}_{\not{Q}} = \delta_{i1}\delta_{j1}\alpha'\left(\frac{\phi}{M}\right)^{q'_{Q_i}-q'_{u_j}}\bar{Q}_i\tilde{H}u_j + \text{H.c.} \tag{8}$$

to Eq. (6), one finds that all our invariants are proportional to the one quantity that violates the PQ symmetry, $q_{Q_1} - q_{u_1} - [q'_{Q_1} - q'_{u_1}]$. Another illuminating example arises when considering Eq. (6) and changing $q_{Q_1} \to q'_{Q_1}$ in the up-quark coupling only. In this case, the quantity $q'_{Q_1} - q_{Q_1}$ violates the PQ symmetry, but is only resolved by invariants that are sensitive to the collective nature of PQ breaking, namely, those that simultaneously involve $\tilde{Y}_u$ and $\tilde{Y}_d$. Indeed, the change $q_{Q_1} \to q'_{Q_1}$ is a mere relabeling from the perspective of the up-quarks alone, but it breaks PQ symmetry when the down quarks are taken into account.

It is also worth investigating the renormalization group (RG) flow of the set of invariants of Eq. (5), which, as for any complete set of order parameters, should remain closed [31]. Using the ALP EFT RG equations (RGEs) presented in Refs. [19,20], we computed the RG evolution of our set of invariants and checked that it closes on itself, thereby verifying its completeness [34]. The obtained RGEs feature seemingly new flavor-invariant expressions that, however, can be expressed as combinations of our invariants, using the Cayley-Hamilton theorem and related techniques (see, e.g., Refs. [35–38]).

Running to lower energies, one eventually encounters the need to match the SM + ALP EFT onto an EFT below the weak scale, where the generic axion-fermion dimension-five couplings read

$$\mathcal{L} \supset -\frac{a}{f}(\bar{u}_L\tilde{m}_u u_R + \bar{d}_L\tilde{m}_d d_R + \bar{e}_L\tilde{m}_e e_R + \text{H.c.}) \tag{9}$$

for arbitrary complex matrices $\tilde{m}_{u,d,e}$, and the shift-symmetric ones are identical to those of Eq. (2), except that now $\psi \in \{(u,d,e)_{L,R}\}$. Upon matching, one may expect that the conditions that ensure shift invariance also match. However, it turns out that the conditions inherited from the UV are strictly stronger than those that would be derived purely from the IR, since at low energies the up- and down-quark sectors are no longer entangled by electroweak interactions: the conditions for shift invariance in the IR are similar to those of the lepton sector in the UV,

$$I^{(i+1,IR)}_x \equiv \mathrm{Tr}(X_x^{i=0,1,\ldots,N_x-1}\tilde{m}_x m_x^\dagger) = 0, \tag{10}$$

where $x = u,\ d,\ e,\ N_u = 2, N_{d,e} = 3$, the m_x are the ($N_x \times N_x$ complex) mass matrices, $X_x \equiv m_x m_x^\dagger$, and, as announced, there are no longer conditions connecting the up and down sectors. This should not come as a surprise, since both linear and nonlinear realizations of the electroweak symmetry above the weak scale [39,40] match onto the same IR EFT, although the latter allows one to write strictly more dimension-five couplings with a manifest axion shift symmetry than those of Eq. (2). Nevertheless, assuming a matching to a linear phase of the EW symmetry [i.e., using a Higgs doublet H, as in Eq. (2)] and an exact axion shift symmetry, one can derive more conditions at the matching scale. For instance, one finds that

$$I_{ud}^{(1,IR)} \equiv \mathrm{Re}(L_{uddu,prst}^{V1,LL}[(m_d m_d^\dagger)_{rs}(\tilde{m}_u m_u^\dagger)_{tp} + (\tilde{m}_d m_d^\dagger)_{rs}(m_u m_u^\dagger)_{tp}]) = 0 \quad (11)$$

at the matching scale, where $L_{uddu}^{V1,LL}$ refers to the Wilson coefficient of the following four-quark operator:

$$\mathcal{L} \supset -\frac{4}{v^2} V_{\mathrm{CKM},pr} V_{\mathrm{CKM},ts}^{*} \bar{u}_{L,p}\gamma^\mu d_{L,r}\bar{d}_{L,s}\gamma^\mu u_{L,t} \equiv L_{uddu,prst}^{V1,LL}\mathcal{O}_{uddu,prst}^{V1,LL}, \quad (12)$$

where V_{CKM} refers to the Cabibbo–Kobayashi–Maskawa (CKM) matrix. Interestingly, Eq. (11) and all the similar ones at the matching scale remain valid when evolved to lower energies via the RG flow [at the one-loop and $\mathcal{O}(1/(fv^2))$ order] using formulas of Refs. [20,41,42]; i.e., one can show that the RG flow of all invariants that vanish at the matching scale due to the axion shift symmetry is closed. The flow admits as a stable subset the invariants of Eq. (10), consistent with the fact that they represent the order parameters for axion shift breaking at any low energy. In the Supplemental Material [26], we give more details on the RG running of the invariants above and below the EW scale, as well as on the matching to a UV EFT with nonlinearly realized EW symmetry [39,43,44].

Finally, there is a close interplay between leading-order CP violation and shift symmetry in the ALP Lagrangian. All invariants in Eq. (5) but $I_{ud}^{(4)}$ are CP odd, hence CP conservation almost implies axion shift symmetry. Furthermore, the connection holds exactly in the lepton sector of the EFT (or in the IR). Conversely, an exact shift symmetry reduces the number of independent sources of CP violation in the axion EFT, which has an impact on CP-violating observables like electric dipole moments (EDMs), allowing one to distinguish a shift symmetric ALP from a generic one (see below). This connection deserves further investigation, in line with recent works that considered the possibility to reintroduce CP violation through the axion [45,46].

Applications.—Our invariants represent the physical (i.e., invariant under flavor transformations) shift-breaking parameters of the ALP EFT. As such, they can be used to assign and track the different power countings of the shift-breaking and shift-symmetric couplings, which are usually generated at very different scales, in a basis- and model-independent EFT approach. Doing this from the relations in Eq. (3) is not at all obvious. These different power countings are directly imprinted on PQ-breaking observables, which are proportional to our invariants.

In addition, the stability of the invariants and of the associated power countings under RG flow allows us to use them at any energy and to identify their impact on observables. Focusing on indirect probes, which may play an important role in constraining axion couplings, we illustrate this in two examples by exhibiting sum rules valid in the presence of an approximate shift symmetry.

First, we revisit the results of Ref. [45], where the authors map bounds on the neutron EDM d_n and the EDM d_{Hg} of the diamagnetic atom ^{199}Hg, to bounds on ALP couplings, allowing for shift-breaking couplings in the generic basis of Eq. (9). The relevant expressions are given in Ref. [45] in terms of the couplings $\tilde{m}$ of Eq. (9) [47], as well as CP-even and -odd couplings of the axion to gluons and photons, under the assumption that the axion mass is of order a few GeV's so that QCD can be treated perturbatively. Assuming that the axion shift symmetry is approximate, one can use at leading order in the PQ breaking the constraints of Eq. (10) and find that

$$d_{\mathrm{Hg}} \simeq 4 \times 10^{-4} d_n, \quad (13)$$

which is an example of a sum rule between observables following from the axion shift symmetry. Once interpreted in terms of the fundamental parameters $\tilde{m}$, the bounds on d_n and d_{Hg} turn out to be of very similar magnitude [45].

Second, we consider the axion-induced RG running of couplings between SM particles at high energy. Those, assuming no further light degree of freedom, can be captured by the standard model effective field theory (SMEFT). The presence of dimension-five axion couplings induces a RG evolution [48], which deviates from that in the pure SMEFT [49–52], and the invariants in Eq. (5) allow us to immediately identify implications of the axion shift symmetry, by forming flavor-invariant sum rules on the RG evolution of the SMEFT Wilson coefficients that are not sensitive to the axion contribution. Observing RGEs compatible with the SMEFT for the combinations of Wilson coefficients entering those sum rules would then suggest that the axion shift symmetry is at most weakly broken [53]. More precisely, we follow Ref. [48] and define the terms sourcing the deviations from the SMEFT RGEs driven by the ALP as follows:

$$\mu\frac{dC_i^{\mathrm{SMEFT}}}{d\mu} - \gamma_{ji}^{\mathrm{SMEFT}}C_j^{\mathrm{SMEFT}} \equiv \frac{S_i}{(4\pi f)^2}. \quad (14)$$

Then, we contract them appropriately in order to make our invariants appear. For instance, using the explicit expressions of the S_i [48], one finds that

$$\mathrm{ImTr}(X_d X_u X_d S_{uG} Y_u^\dagger + X_u X_d X_u S_{dG} Y_d^\dagger) = \frac{g_3^2}{8\pi^2} C_g I_{ud}^{(3)}, \quad (15)$$

where S_{qG} are the ALP-induced source terms of the dipole operators $\bar{Q}_L \sigma^{\mu\nu} T^a \overset{H}{(\sim)} q_R G_{\mu\nu}^a$ and C_g is the coefficient of the operator

$$-\frac{C_g g_3^2}{16\pi^2}\frac{a}{f}\mathrm{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}), \qquad (16)$$

for G the gluon field strength, $\tilde{G}^{\mu\nu} \equiv (\epsilon^{\mu\nu\rho\sigma}/2)G_{\rho\sigma}$ its dual, and g_3 the $\mathrm{SU}(3)_C$ coupling constant.

Coupling to gluons and nonperturbative shift invariance.—Previously, we have focused on the breaking of shift invariance that arises at the perturbative level. This is, for instance, relevant for interactions that induce axion potentials at the tree or loop levels. We have, however, neglected axion couplings to gauge bosons of the SM and, in particular, to gluons. The latter do not break the shift symmetry at the perturbative level, but they do so *nonperturbatively* [54]. Let us thus add to the Lagrangian of Eq. (1) the term of Eq. (16) and to that of Eq. (2) the same term with $C_g \to C_g^{(s)}$.

As before, we will now look for quantities that must vanish for the shift invariance to hold nonperturbatively. When the axion shift symmetry is exact, it is unbroken at the perturbative level and one can work in the basis of Eq. (2) for axion-fermion couplings, where all terms are unchanged by a shift of a. At the nonperturbative level, one needs to require $I_g \equiv C_g^{(s)} = 0$ to cancel the gluon-induced shift-breaking contributions, for instance, to the axion potential. To identify the nonperturbative order parameter in the most general basis of Eq. (1), we need to account for anomalies when matching from Eq. (2) to Eq. (1),

$$C_g = C_g^{(s)} + \mathrm{Tr}(2c_Q - c_u - c_d). \qquad (17)$$

Eventually, we find a new CP-even condition,

$$I_g = C_g + \mathrm{Im}\mathrm{Tr}(Y_u^{-1}\tilde{Y}_u + Y_d^{-1}\tilde{Y}_d) = 0. \qquad (18)$$

The invariant I_g, which features couplings from the up and down sectors, highlights a new kind of collective breaking at the nonperturbative level, accounting for all contributions to the mixed PQ anomaly with $\mathrm{SU}(3)_C$. In addition, it is valid below the EW scale, after replacing $Y, \tilde{Y} \to m, \tilde{m}$.

Using once more the RGEs of the SM and axion Yukawa couplings [19,20], we can show that all contributions to the running of this invariant cancel at the one-loop level above and below the EW scale [55], $\mu(dI_g/d\mu) = 0$.

Conclusions.—In this Letter, we have investigated the implications of an axion shift symmetry on the dimension-five axion couplings to the SM fermions. In particular, we have found explicit and algebraic conditions implied by the shift symmetry on these couplings, instead of the implicit relations that are well known in the literature. The set of constraints is formulated in a flavor-invariant way and gives necessary and sufficient conditions for shift symmetry to hold, hence yielding a set of 13 order parameters for shift symmetry in the dimension-five axion EFT. Our results make it explicit that the axion shift symmetry is a collective effect. We confirmed these aspects by matching explicit UV scenarios onto the ALP EFT. We stressed that most of the invariants are CP odd and showed that they form a closed set under the RG flow, consistent with the fact that they capture a complete set of order parameters for shift symmetry breaking. We also studied their fate at low energies, below the EW scale, and used the associated constraints to derive sum rules on illustrative observables. Finally, we have extended the discussion to the nonperturbative breaking of the PQ symmetry induced by the axion couplings to gluons. Throughout the Letter, we focused on the constraints at dimension five, i.e., at $\mathcal{O}(1/f)$, but shift symmetry correlates the $\mathcal{O}(1/f^2)$ couplings involving two axion fields and those at $\mathcal{O}(1/f)$. It would be interesting to extend our formalism to these extra conditions.

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[55] We chose a scaling in Eq. (16) similar to that of [19], where C_g already comes with a one-loop factor $g_3^2/(16\pi^2)$. This allowed us to account for the anomalous shift without loop-factor hierarchies in Eq. (17). That also required that we account for anomalous contribution to the running of C_g.