

## Antipodal Self-Duality for a Four-Particle Form Factor

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We bootstrap the symbol of the maximal-helicity-violating four-particle form factor for the chiral part of the stress-tensor supermultiplet in planar  $\mathcal{N} = 4$  super-Yang-Mills theory at two loops. When minimally normalized, this symbol involves only 34 letters and obeys the extended Steinmann relations in all partially overlapping three-particle momentum channels. In addition, the remainder function for this form factor exhibits an antipodal self-duality: It is invariant under the combined operation of the antipodal map defined on multiple polylogarithms—which reverses the order of the symbol letters—and a simple kinematic map. This self-duality holds on a four-dimensional parity-preserving kinematic hypersurface. It implies the antipodal duality recently noticed between the three-particle form factor and the six-particle amplitude in this theory.

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*Introduction.*—Symmetries play a central role in modern formulations of fundamental physics, where they reflect simple facts about the world such as conservation laws and how different types of particle interact. However, sometimes new symmetries emerge in our theories whose physical implications are not immediately clear. These discoveries often lead to the development of more powerful mathematical techniques for making predictions and have the potential to guide us to new physical principles.

The planar limit of  $\mathcal{N} = 4$  super-Yang-Mills (SYM) theory has proven to be an especially rewarding place to look for (and exploit) novel symmetries in particle physics. Most famously, scattering amplitudes and form factors in this theory are dual to Wilson loops with lightlike edges [1–13] and, as such, respect a dual conformal symmetry [1,14–17]. These quantities also exhibit interesting number-theoretic symmetries [18], as well as intriguing connections to cluster algebras, tropical fans, and positive geometries [19–43]. While most of these special properties do not directly generalize to nonsupersymmetric theories, their study has still led to significant improvements in our understanding of more general classes of amplitudes, form factors, and Feynman integrals (see, for instance, [36,44–50]).

Much of this recent progress has been fueled by the in-depth study of amplitudes that evaluate to multiple polylogarithms (MPLs) [51–56], which are endowed with a Hopf algebra structure [57–62]. In particular, one part of the Hopf algebra of MPLs—the symbol map—greatly simplifies the study of what sequences of discontinuities appear in polylogarithmic functions. This fact has been leveraged to bootstrap certain amplitudes and form factors in planar  $\mathcal{N} = 4$  SYM theory to extremely high loop orders [27,45,63–74].

In an unexpected recent development, these bootstrap results have revealed a mysterious new duality between the maximally helicity-violating (MHV) six-particle amplitude in parity-preserving kinematics and the MHV three-particle form factor that involves a single insertion of the chiral stress tensor multiplet, which includes the Bogomol’nyi-Prasad-Sommerfield operator  $\text{tr}(\phi^2)$  [75]. Namely, these two quantities are related to each other by a map that appears in the Hopf algebra structure of MPLs: the antipode map. At the level of the symbol, the antipode map simply reverses the order of discontinuities in MPLs. Thus, this duality can be loosely understood as the observation that the three-particle form factor and the MHV six-particle amplitude (in parity-preserving kinematics) encode exactly the same sequences of discontinuities but in the opposite order—after a suitable map between the kinematic variables that describe the two processes. At the moment, no physical argument is known for why this duality should hold, but it has been checked explicitly through at least seven loops [74–76]. Interestingly, two-loop MHV

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amplitudes have also been shown to exhibit a different type of antipodal symmetry in parity-even kinematics, which is conjectured to hold to all particle multiplicity [77].

In this Letter, we describe a more general antipodal duality, which applies to four-particle form factors. Namely, the four-particle form factor is *self-dual* under the action of the antipode in a four-dimensional subspace of its kinematics. In fact, as we will show below, this new duality *implies* the duality between the three-particle form factor and the six-particle amplitude. The reason is that these quantities appear in the double- and triple-collinear limits of the four-particle form factor.

In order to substantiate this claim, we first bootstrap the symbol of the four-particle form factor at two loops. We do this by identifying the letters that appear in the Feynman integrals contributing to this form factor [48,78–80]. We then construct the space of integrable symbols for these letters. Amazingly, the form factor is uniquely identified in this space by just the first-entry condition, invariance under a standard set of discrete symmetries, and the strict (leading-power) double-collinear limits. (This rigidity is reminiscent of the seven-particle amplitude, whose three-loop MHV symbol could be bootstrapped with only mild collinear information [81].) Three other limits—triple-collinear limits, the recently computed limit of lightlike operator momenta [82], and the near-collinear limit—serve as cross-checks on our result.

Multiple normalizations are needed to expose the different properties of this four-particle form factor. In one normalization, the symbol of the form factor obeys the extended Steinmann relations (defined below) in all partially overlapping three-particle momentum channels. In another normalization, this form factor is antipodally self-dual. We describe these normalizations, as well as the self-duality map, in more detail below. We provide the symbol in both normalizations in the ancillary files for [83], which also describe its various discrete symmetries and kinematic limits.

*The bootstrap.*—We begin our bootstrap by removing the infrared divergences and the MHV tree-level prefactor from the four-particle MHV form factor [84]  $\mathcal{F}_4^{\text{MHV}}$ :

$$\mathcal{F}_4^{\text{MHV}} = \mathcal{F}_4^{\text{min}} \times F_4, \quad (1)$$

where

$$\mathcal{F}_4^{\text{min}} = \mathcal{F}_4^{\text{MHV,tree}} \times \exp \left[ -\frac{g^2}{e^2} \sum_{i=1}^4 \left( \frac{\mu^2}{-s_{i,i+1}} \right)^\epsilon \right]. \quad (2)$$

Here, the 't Hooft coupling is  $g^2 = N_c g_{\text{YM}}^2 / (16\pi^2)$ . We have omitted contributions proportional to transcendental constants, because they vanish at the level of the symbol at which we are working. Our main objective is to calculate  $F_4$ , which depends on eight dimensionless ratios:

$$u_i = \frac{(p_i + p_{i+1})^2}{q^2}, \quad v_i = \frac{(p_i + p_{i+1} + p_{i+2})^2}{q^2}, \quad (3)$$

where  $i = 1, 2, 3, 4$  and  $q = \sum_{i=1}^4 p_i$  is the (generically off-shell) momentum of the operator insertion. All indices should be understood to be mod 4. Because of momentum conservation and the masslessness of the four particles,  $p_i^2 = 0$ , these variables satisfy three constraints:

$$-u_1 + u_3 + v_4 + v_1 = 1, \quad (4)$$

$$-u_2 + u_4 + v_1 + v_2 = 1, \quad (5)$$

$$-u_3 + u_1 + v_2 + v_3 = 1. \quad (6)$$

Correspondingly, the four-particle form factor depends on five independent variables.

When expanded perturbatively in the coupling,

$$F_4 = 1 + \sum_{L=1}^{\infty} g^{2L} F_4^{(L)}, \quad (7)$$

the  $L$ -loop contribution  $F_4^{(L)}$  is expected to be expressible as a linear combination of MPLs of weight  $2L$  with rational coefficients. The symbol of an MPL can be defined iteratively by its total differential [58]:

$$dG = \sum_{x \in \mathcal{L}} G^x d \ln x \Rightarrow \mathcal{S}(G) = \sum_{x \in \mathcal{L}} \mathcal{S}(G^x) \otimes x, \quad (8)$$

where  $G^x$  are MPLs of one lower weight. The  $d \ln$  arguments  $x$  are referred to as symbol letters, while the total multiplicative span of the letters appearing in an MPL is referred to as its symbol alphabet,  $\mathcal{L}$ . For more background on MPLs and the symbol map, see, for instance, [85].

The symbol of the one-loop form factor is [7]

$$\begin{aligned} \mathcal{S}(F_4^{(1)}) &= 2v_1 \otimes (1 - v_1) + \frac{u_1}{u_2 u_4} \otimes u_1 \\ &+ \frac{u_1}{v_4 v_1} \otimes \frac{u_1 - v_4 v_1}{u_1} + \text{cyclic}, \end{aligned} \quad (9)$$

where the cyclic transformation maps  $u_i \rightarrow u_{i+1}$  and  $v_i \rightarrow v_{i+1}$ . We recall that the two-loop remainder function is related to the form factor itself by

$$R_4^{(2)} = F_4^{(2)} - \frac{1}{2} [F_4^{(1)}]^2, \quad (10)$$

and  $R_4^{(2)}$  has smooth behavior in factorization limits [9,63,65].

In order to bootstrap  $F_4^{(2)}$  or  $R_4^{(2)}$ , we first assemble the alphabet of symbol letters that can appear in the four-particle form factor. Since the Feynman integrals that contribute to this form factor are all known [48,78,79],

this can be done easily. (While the nonplanar double-pentagon Feynman integrals that contribute to this form factor have not yet been published [79], they do not give rise to letters beyond those that appear in the planar pentabox and nonplanar hexabox topologies [80].) Altogether, the relevant Feynman integrals depend on 113 independent letters. Five different square roots appear in this alphabet, one of which involves the dihedrally invariant argument

$$\text{tr}_5 = 4i\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta \quad (11)$$

and four of which are organized into pairs of two-orbits under the action of the dihedral group. (In the notation of Ref. [48], these four additional roots correspond to two orientations of  $\Delta_3$  and two orientations of  $\Sigma_5$ . Further orientations of these roots do not appear because they correspond to different planar orderings.) The dihedral group  $D_4$  is generated by the order-four cyclic transformation and a flip transformation, for example,  $u_1 \leftrightarrow u_4$ ,  $u_2 \leftrightarrow u_3$ , and  $v_1 \leftrightarrow v_3$ . Spacetime parity acts by flipping the sign in front of  $\text{tr}_5$  while leaving the sign in front of the other square roots intact.

We next construct the space of integrable weight-four symbols that draws upon this alphabet. We are interested only in symbols that have the correct branch cuts (satisfy the first-entry condition [86]), which in this context states that only the eight letters  $\{u_i, v_i\}$  can occur in the first entry. We also impose invariance under the dihedral symmetry group  $D_4$ . There are 522 independent symbols  $s_i$  satisfying these conditions, which we use to formulate our initial ansatz for the symbol of the remainder function  $R_4^{(2)}$ :

$$\mathcal{S}(R_4^{(2)}) = \sum_{i=1}^{522} c_i s_i, \quad (12)$$

where the  $c_i$  are undetermined rational coefficients.

To fix the values of the coefficients in Eq. (12), we first require that our ansatz is even under all elements of the algebraic Galois group, which flip the signs in front of each of the five square roots separately. (Note that the square-root signs are arbitrary conventions, on which the amplitude cannot depend. Note also that one of the elements is parity.) This imposes 148 independent conditions on the coefficients in our ansatz. Next, we require that our ansatz for  $R_4^{(2)}$  reduces to the three-particle form factor remainder  $R_3^{(2)}$  [65] when two of the external particles become collinear. In this limit, the 113-letter alphabet involves 25 spurious letters, in addition to the six letters describing the three-particle form factor  $F_3$ . Matching our ansatz onto the correct expression completely fixes the remaining 374 coefficients and, thus, uniquely determines the symbol of  $R_4^{(2)}$  or, equivalently, of  $F_4^{(2)}$ . The sparse systems of linear

TABLE I. Number of free parameters that remain after imposing each constraint in the bootstrap procedure, starting with the 113-letter alphabet. The limits are taken at leading power, except for the FFOPE.

Constraints	Parameters
First entry, integrability, $D_4$ invariance	522
Galois symmetry	374
Strict double-collinear limit $\rightarrow R_3^{(2)}$	0
Strict triple-collinear limit $\rightarrow \hat{R}_6^{(2)}$	0
Lightlike limit	0
FFOPE	0

equations that encode these constraints can be solved efficiently over finite fields using the SPASM software library [87]. The numbers of free parameters at each stage in the calculation are collected in Table I.

Although we started with an initial ansatz of over one hundred letters, our result for the symbol of  $F_4^{(2)}$  involves only 34 letters (notably, all five square roots still appear). As expected, it also obeys the Steinmann relations [88,89], which forbid sequential discontinuities in partially overlapping momentum channels; in the case of massless scattering amplitudes, these relations apply when both channels contain at least three particles [70]. However, while the Steinmann relations put constraints on only the first two entries of the symbol, many amplitudes have been found to obey an *extended* set of Steinmann relations, in which the same constraints hold for all adjacent entries in the symbol [18,72,90,91]. In processes involving one massive and four massless external legs, the (extended) Steinmann relations forbid the letter  $v_i$  from appearing next to  $v_j$  in the symbol when  $j \neq i$ . Notably, the two-loop master integrals that contribute to  $F_4^{(2)}$  obey only the Steinmann relations—not the extended Steinmann relations—between  $v_i$  and  $v_{i+2}$  [48]. However, the extended Steinmann relations are obeyed in *all* channels by  $F_4^{(2)}$ . Similarly, higher-point amplitudes in this theory obey all extended Steinmann relations when normalized minimally (although amplitudes do not respect dual conformal invariance in this normalization) [28,72].

*Special kinematic limits.*—We can check our results in a number of different kinematic limits. First, we consider the lightlike limit, where the operator momentum  $q^2 \rightarrow 0$ . In this limit, the 34 letters that appear in the symbol of  $R_4^{(2)}$  reduce to 13 independent multiplicative combinations. Nine of these combinations match the lightlike letters reported in Ref. [82], while four are spurious and must drop out of  $R_4$ . We have confirmed that our  $R_4^{(2)}$  symbol correctly reproduces the symbol of the lightlike form-factor remainder reported in Ref. [82].

Although our bootstrap procedure made use of information about the strict, or leading power, collinear limit, we can still make nontrivial predictions for the subleading

powers in the expansion of  $F_4^{(2)}$  around this limit. Such terms are predicted by the recently developed form factor operator product expansion (FFOPE) [92–94]. To carry out this cross-check, we rewrite our kinematic variables in terms of the OPE variables  $T$ ,  $T_2$ ,  $S$ ,  $S_2$ , and  $f_2$  [92,95]:

$$\begin{aligned} u_1 &= \frac{T^2 T_2^2}{(T^2 + 1)(S^2 + T^2 + T_2^2 + 1)}, \\ u_2 &= \left[ (1 + T^2) + \frac{S^2 [S_2 T_2 (1 + f_2^2) + f_2 (1 + S_2^2 + T^2 + T_2^2)]}{f_2 S_2^2} \right]^{-1}, \\ u_3 &= \frac{S^2}{(T^2 + 1)(S^2 + T^2 + T_2^2 + 1)}, \\ u_4 &= \frac{S^2 T^2}{S_2^2} u_2, \\ v_1 &= \frac{T_2^2 + 1}{S^2 + T^2 + T_2^2 + 1}, \end{aligned} \quad (13)$$

while  $v_2$ ,  $v_3$ , and  $v_4$  are fixed by the relations (4)–(6). The near-collinear limit in these variables corresponds to an expansion around small values of  $T_2$ , which we have computed by expanding our symbol to  $\mathcal{O}(T^2 T_2^2)$ . We checked this expansion against the predictions made by the FFOPE (using the procedure explained in Ref. [93]) for the  $T^2 T_2 \ln(T)$ ,  $T^2 T_2 \ln(T_2)$ ,  $T^2 T_2$ , and  $T^2 T_2^2 \ln(T_2)$  contributions. [We note that the contributions at  $\mathcal{O}(T^0)$  simply reproduce the OPE expansion of  $\hat{R}_6$ , while the contributions at  $\mathcal{O}(T_2^0)$  reproduce the OPE expansion of  $R_3$ .] Each of these checks was carried out as a series in  $S$  and  $S_2$  and to all available powers of  $f_2$ .

Finally, it can be seen from the OPE [92–107], as well as from arguments based on dual conformal invariance and factorization [9], that the four-particle form factor remainder  $R_4$  must reduce to the six-particle MHV amplitude's remainder  $\hat{R}_6$  as  $T \rightarrow 0$  in the parametrization introduced in Eq. (13). In both cases, the limit can be interpreted as a triple-collinear limit; in the six-particle case, the limit covers all of the dual-conformally invariant phase space, allowing the triple-collinear splitting amplitude's finite part to be identified with  $\hat{R}_6$ . We have checked that this limit is indeed obeyed.

*Antipodal self-duality.*—As discussed in the last section, the four-particle form factor possesses kinematic limits in which it reduces to the three-particle form factor and to the six-particle amplitude. These two quantities were recently discovered to be antipodally dual [75], making it tempting to investigate whether the four-particle form factor could be dual to itself in parity-preserving kinematics, where  $\text{tr}_5$  vanishes. In the OPE variables, this hypersurface simply corresponds to setting  $f_2 = 1$ . In the  $u_i$  and  $v_i$  variables, it requires setting the Gram determinant

$$\begin{aligned} &u_2^2 [u_1^2 - 2u_1(1 + u_3) + (1 - u_3)^2] + [u_1 v_2 - v_1(v_2 - u_3)]^2 \\ &- 2u_2 \{ u_1^2 v_2 + u_1 [u_3(2 - v_1 - v_2) - v_2(1 + v_1)] \\ &+ v_1(1 - u_3)(v_2 - u_3) \} \end{aligned} \quad (14)$$

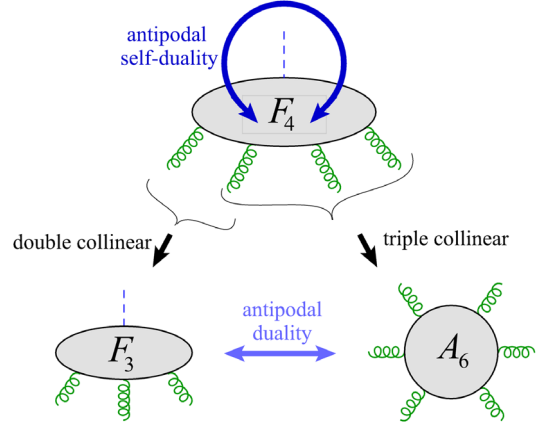


FIG. 1. The four-particle form factor is antipodally self-dual in parity-preserving kinematics. This duality maps the double- and triple-collinear limits of the form factor to each other. Because the four-particle form factor reduces to the three-particle form factor and the six-particle amplitude in these limits, the self-duality of the four-particle form factor implies the duality observed in Ref. [75].

to zero. Note that none of the other four square roots rationalize on this  $\text{tr}_5 = 0$  surface.

Surprisingly, we find that an antipodal self-duality does, in fact, hold on this parity-preserving hypersurface:

$$R_4|_{\text{tr}_5=0} = S(R_4|_{\text{tr}_5=0})|_{u_i, v_i \rightarrow g(u_i), g(v_i)}, \quad (15)$$

where  $S(F)$  denotes the polylogarithmic antipode of  $F$ , which acts at the level of the symbol as [108,109]

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m x_m \otimes \cdots \otimes x_2 \otimes x_1, \quad (16)$$

and the kinematic map is defined by

$$g(u_1) = u_1 \sqrt{\frac{u_2 u_4}{(u_2 - v_1 v_2)(u_4 - v_3 v_4)}}, \quad (17)$$

$$g(v_1) = (1 - v_1) \sqrt{\frac{u_1 u_2}{(u_1 - v_4 v_1)(u_2 - v_1 v_2)}}, \quad (18)$$

plus cyclic images. In the OPE variables, this mapping takes an even simpler form:

$$\begin{aligned} g(T) &= \sqrt{\frac{T_2}{S_2}}, & g(S) &= \sqrt{\frac{1}{T_2 S_2}}, \\ g(T_2) &= \frac{T}{S}, & g(S_2) &= \frac{1}{TS}, \end{aligned} \quad (19)$$

and it is clear that  $g^2 = 1$ . Notably, this map reduces to the duality map described in Ref. [75] upon identifying  $T_2$  and  $S_2$  with the OPE parameters that describe the six-particle, which were denoted  $\hat{T}$  and  $\hat{S}$  in Ref. [75]. This

identification naturally arises in the triple-collinear limit of  $R_4$ , where the form factor reduces to the six-particle remainder  $\hat{R}_6$ , which is mapped to the double-collinear limit of  $R_4$  by Eq. (19). However, the self-duality of  $R_4$  in Eq. (15) holds more generally in the *full* four-dimensional space of parity-preserving kinematics. Figure 1 depicts the relation between the antipodal self-duality of the four-particle form factor and the previously-observed antipodal duality between form factors and amplitudes. [Note from Eqs. (17) and (18) that the lightlike limit  $u_i, v_i \rightarrow \infty$  maps to finite  $u_i, v_i$  under the kinematic map  $g$ , which implies that the lightlike form factor does not exhibit the antipodal self-duality we have found for  $q^2 \neq 0$ .]

While antipodal self-duality (15) holds for the remainder function  $R_4$ , there is an obstruction to it holding for  $F_4$ . Namely, the one-loop form factor  $F_4^{(1)}$  contains final entries that are not in the set  $\{g(u_i), g(v_i)\}$ ,  $i = 1, 2, 3, 4$ , dictated by antipodal self-duality. From the symbol (9) of  $F_4^{(1)}$ , it is easy to see that the final entry sitting behind the first entry  $v_1$  is just  $g(v_1)^2$ . However, the final entry sitting behind  $u_1$  is  $(u_1 - v_4 v_1)/(u_4 u_2)$ , and its logarithm is not a linear combination of  $\{\ln g(u_i), \ln g(v_i)\}$ . Furthermore, it does not seem possible to repair this obstruction by any simple adjustment of the normalization  $F_4^{(1)}$  that is consistent with both the first-entry condition and the Steinmann relations.

*Principle of maximal transcendentality.*—The two-loop three-particle form factor is known to satisfy the principle of maximal transcendentality (PMT) [110–113], meaning that the  $\mathcal{N} = 4$  SYM result for  $\text{tr}(\phi^2)$  matches the maximally transcendental part of the Higgs-to-three-gluon amplitude in pure Yang-Mills theory (or QCD) in the leading large-top-mass limit [operator  $\text{tr}(F^2)$ ] [62,65,114–118], for all gluon helicity configurations. The PMT has recently been extended (in a different way) to the four-gluon form factor of  $\text{tr}(F^3)$  [118]. Here we ask: Can the PMT for  $\text{tr}(\phi^2)$  be extended to any Higgs-to-four-gluon helicity amplitudes? At one loop, the PMT already fails for the color-ordered helicity configurations  $(---+)$  and  $(-+-+)$  and their parity conjugates [119,120], but it works for  $(- - + +)$  and  $(- - - -)$  and their parity conjugates [121,122].

At two loops and leading color, we cannot say much about  $(- - + +)$  currently, because our starting point (12) was dihedrally invariant, and the  $(- - + +)$  form factor (divided by the tree) need not be invariant, although its leading-transcendental part happens to be at one loop. That leaves  $(- - - -)$ , which is dihedrally invariant. The leading-transcendental, weight-four, parts of its double-collinear limits match those in planar  $\mathcal{N} = 4$  SYM, thanks to the PMT holding for the Higgs-to-three-gluon amplitude and for the  $g \rightarrow gg$  splitting amplitude [123]. We found earlier that there is a unique weight-four, dihedrally and parity-invariant function with specified double-collinear limits. Thus,  $R_4^{(2)}$  should also provide the parity-even part of the

two-loop remainder for  $A_4^{(2)}(\phi, 1^-, 2^-, 3^-, 4^-)$ , where  $\phi = H + iA$ , with  $H$  the Higgs boson and  $A$  a pseudoscalar coupling to  $\text{tr}(F\tilde{F})$  [116]. We also find a unique weight-four parity-odd dihedrally invariant function that vanishes in both the double- and triple-collinear limits. We leave to future work the tantalizing questions of whether the coefficient of this parity-odd function vanishes, as suggested by the PMT, and, more generally, how much of the two-loop Higgs-to-four-gluon amplitude in QCD can be bootstrapped.

*Discussion and conclusions.*—In this Letter, we have bootstrapped the two-loop four-particle MHV form factor of the chiral part of the stress-tensor supermultiplet in planar  $\mathcal{N} = 4$  SYM theory and have found that it possesses an antipodal self-duality in parity-preserving kinematics. While we can check this duality only at two loops, the previously identified antipodal duality between  $R_3$  and  $\hat{R}_6$  that it implies has been checked through seven loops (and even eight loops [76]), which strongly indicates that the self-duality of  $R_4$  will hold to all loop orders.

The two-loop four-particle form factor is a highly constrained quantity. It is determined uniquely by its invariance under Galois and dihedral symmetries, as well as its double-collinear limit. Our result passes a wealth of cross-checks, including the comparison of the near-collinear limit to the FFOPE, triple-collinear limits, and the lightlike limit. The stringent constraints also raise the question of whether the form factor can be bootstrapped to higher loop orders. The main uncertainty is whether new letters appear at three loops, beyond the 113 letters that appear in the two-loop integrals contributing to this quantity. While great strides have been made recently toward better characterizing the analytic structure of Feynman integrals from first principles (see, for instance, [124–133]), this question remains hard to address without a direct computation. As we will report elsewhere [134], the answer is that no new letters, beyond the 113, are required to successfully bootstrap the three-loop four-particle MHV form factor.

It is natural to wonder if further antipodal (self-)dualities hold between form factors and amplitudes at higher particle multiplicity. Notably, as one increases the number of scattering particles, form factors contain an increasingly rich pattern of form factors and amplitudes in their (multi) collinear limits, giving rise to many intriguing possibilities. In a similar vein, it would be interesting to calculate the two-loop next-to-MHV form factor for the chiral part of the stress-tensor supermultiplet and to search for further antipodal dualities that involve this quantity.

More generally, it is critical to understand better the physical reason for the antipodal dualities observed in form factors. Our work already suggests that the role of the six-point amplitude may be a red herring, that it participates only because it is also the triple-collinear limit of the four-particle form factor. The simplicity of the map (19) in OPE

variables suggests that the reason for the duality might be related to the flux-tube excitations describing the OPE limit. In this context, it is worth noting that there is a fixed surface for Eq. (19), which contains the limit  $T, T_2 \rightarrow 0$  where small numbers of flux-tube excitations dominate. This fact may make it possible to examine how the duality acts on single flux-tube excitations. That action could be a clue in unraveling the physical origin and, thereby, predicting in advance where else antipodal duality will emerge.

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