Underdetermined Dyson-Schwinger Equations

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This Letter examines the effectiveness of the Dyson-Schwinger (DS) equations as a calculational tool in quantum field theory. The DS equations are an infinite sequence of coupled equations that are satisfied exactly by the connected Green's functions G_n of the field theory. These equations link lower to higher Green's functions and, if they are truncated, the resulting finite system of equations is underdetermined. The simplest way to solve the underdetermined system is to set all higher Green's function(s) to zero and then to solve the resulting determined system for the first few Green's functions. The G_1 or G_2 so obtained can be compared with exact results in solvable models to see if the accuracy improves for high-order truncations. Five $D = 0$ models are studied: Hermitian ϕ^4 and ϕ^6 and non-Hermitian $i\phi^3$, $-\phi^4$, and $i\phi^5$ theories. The truncated DS equations give a sequence of approximants that converge slowly to a limiting value but this limiting value always *differs* from the exact value by a few percent. More sophisticated truncation schemes based on mean-field-like approximations do not fix this formidable calculational problem.

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The objective in quantum field theory is to calculate *connected Green's functions* $G_n(x_1, \ldots, x_n)$, which contain the physical content of the theory. In principle, the program is to solve the field equations for the field $\phi(x)$ and then to calculate vacuum expectation values of time-ordered products of ϕ: $\gamma_n(x_1, \ldots, x_n) \equiv \langle 0|\mathrm{T}\{\phi(x_1)\ldots\phi(x_n)\}|0\rangle$. The nonconnected Green's functions γ_n are then combined into *cumulants* to get G_n [1].

The Dyson-Schwinger (DS) equations purport to be a way to calculate both the perturbative and nonperturbative behavior of G_n by using c-number functional analysis without resorting to operator theory [2–5]. The procedure is to truncate the infinite system of coupled DS equations to a finite set of coupled equations that would give good approximations to the first few G_n. The problem is that, while the DS equations are satisfied exactly by G_n, the DS equations are an *underdetermined* system; each new equation introduces additional Green's functions, so a truncation of the system contains more Green's functions than equations [6]. A plausible strategy is to close the truncated system by setting the highest Green's function(s) to zero and then to solve the resulting determined system.

Here we study the simplest case: quantum field theory in zero-dimensional spacetime. Successive elimination gives *polynomial* equations for G_1 or G_2. We examine the convergence and accuracy of this procedure as the system of coupled equations increases in size for five $D = 0$ theories, Hermitian quartic and sextic theories, and non-Hermitian $\mathcal{PT}$-symmetric cubic, quartic, and quintic theories [7]. The truncated DS equations provide fair numerical values for the connected Green's functions, but these approximations do not converge to the exact results when they are examined in high order.

The DS equations follow directly on differentiating the functional integral for $Z[J]$ [or $\log(Z[J])$] with respect to J, giving γ_n (or G_n),

$$Z[J] = \int \mathcal{D}\phi \, \exp \int dx\{-\mathcal{L}[\phi(x)] + J(x)\phi(x)\},$$

where $\mathcal{L}$ is the Lagrangian, J is a c-number source, and $Z[0]$ is the Euclidean partition function [8,9].

Hermitian quartic $D = 0$ theory.—The functional integral $Z[J]$ becomes an ordinary integral $Z[J] = \int_{-\infty}^{\infty} d\phi e^{-\mathcal{L}(\phi)}$, where $\mathcal{L}(\phi) = \frac{1}{4}\phi^4 - J\phi$. The exact connected two-point Green's function is

$$G_2 = \int_{-\infty}^{\infty} d\phi \, \phi^2 e^{-\phi^4/4} \Big/ \int_{-\infty}^{\infty} d\phi \, e^{-\phi^4/4} = 2\Gamma\left(\frac{3}{4}\right)\Big/\Gamma\left(\frac{1}{4}\right) = 0.675\,978\,240\ldots. \qquad (1)$$

We impose parity invariance when $J = 0$, so all odd Green's functions vanish. The first nontrivial DS equation reads $G_4 = -3G_2^2 + 1$. Truncating this equation by setting $G_4 = 0$, we obtain the approximate result $G_2 = 1/\sqrt{3} = 0.577\,350\ldots$. In comparison with (1), this result is 14.6% low, which is unimpressive.

The next three DS equations are

$$\begin{aligned} G_6 &= -12G_2G_4 - 6G_2^3, \\ G_8 &= -18G_2G_6 - 30G_4^2 - 60G_2^2G_4, \\ G_{10} &= -24G_2G_8 - 168G_4G_6 - 126G_2^2G_6 - 420G_2G_4^2. \end{aligned} \quad (2)$$

This system is underdetermined; the number of unknowns is always one more than the number of equations. To solve this system we eliminate G_4 by substituting the first equation into the second, we eliminate G_6 by substituting the first two equations into the third, and so on. We obtain G_{2n} as an nth degree polynomial $P_n(G_2)$ (dividing by the coefficient of the highest power of G_2):

$$\begin{aligned} &P_2(x) = x^2 - \frac{1}{3}, \qquad P_3(x) = x^3 - \frac{2}{5}x, \\ &P_4(x) = x^4 - \frac{8}{15}x^2 + \frac{1}{21}, \qquad P_5(x) = x^5 - \frac{2}{3}x^3 + \frac{193}{1890}x. \end{aligned} \quad (3)$$

Closing the truncated DS equations means finding the zeros of these polynomials. The positive roots are plotted in Fig. 1. These roots are real and nondegenerate, and range upwards towards the exact G_2 in (1). We cannot know *a priori* which root best approximates G_2 but the roots become denser at the upper end, so we would guess that the largest root gives the best approximation.

Inaccuracy of DS approximants.—The accuracy of the largest root in Fig. 1 improves slowly and monotonically with the order of the truncation. However, while the sequence of largest roots in Fig. 1 converges as $n \to \infty$, the limiting value is 0.663 488…, which is 1.85% *below* the exact value of G_2 in (1). This discrepancy arises because truncating the DS equations means replacing G_{2n} by 0, but G_{2n} is *not small*. The DS equations are exact, so we can compute G_{2n} by substituting G_2 in (1) into (3). We find that the Green's functions grow rapidly with n:

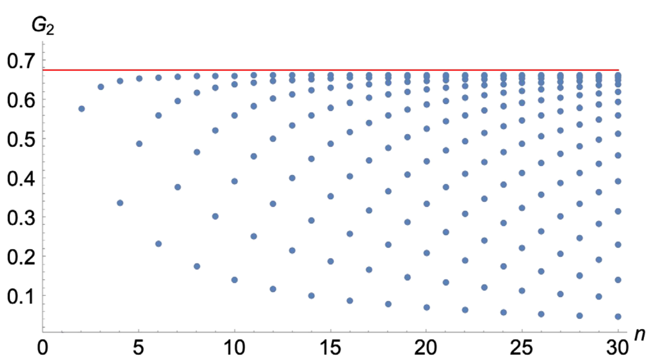

FIG. 1. Positive zeros of $P_n(x)$ in (3) plotted as a function of n up to $n = 30$. The zeros are nondegenerate and range from 0 up to just below the exact value of $G_2 = 0.675\,978\ldots$ (1) (heavy horizontal line).

$G_{20} = -4.2788 \times 10^9$, $G_{22} = 3.0137 \times 10^{11}$. Richardson extrapolation [10] yields the asymptotic behavior of G_{2n}:

$$G_{2n} \sim 2r^{2n}(-1)^{n+1}(2n-1)! \qquad (n \to \infty), \quad (4)$$

where $r = 0.409\,505\,7\ldots$.

Because the DS equations are algebraic when $D = 0$, we can derive this asymptotic behavior analytically: We substitute $G_{2n} = (-1)^{n+1}(2n-1)!g_{2n}$, multiply the $2n$th DS equation by x^{2n}, sum from $n = 1$ to ∞, and define the generating function $u(x) \equiv xg_2 + x^3g_4 + x^5g_6 + \cdots$. The differential equation satisfied by $u(x)$ is nonlinear:

$$u''(x) = 3u'(x)u(x) - u^3(x) - x, \quad (5)$$

where $u(0) = 0$ and $u'(0) = G_2$. We *linearize* (5) by substituting $u(x) = -y'(x)/y(x)$ and get $y'''(x) = xy(x)$, where $y(0) = 1$, $y'(0) = 0$, $y''(0) = -G_2$. The exact solution satisfying these initial conditions is

$$y(x) = \frac{2\sqrt{2}}{\Gamma(1/4)} \int_0^\infty dt\, \cos(xt) e^{-t^4/4}. \quad (6)$$

If $y(x) = 0$, the generating function $u(x)$ becomes infinite, so the smallest value of $|x|$ at which $y(x) = 0$ is the radius of convergence of the series for $u(x)$. A simple plot shows that $y(x)$ vanishes at $x_0 = \pm 2.441\,968\,2\ldots$ [9]. Therefore, $r = 1/x_0 = 0.409\,506\ldots$, which confirms (4).

The asymptotic behavior in (4) indicates that G_{2n} grows *much faster* than the γ_{2n} as $n \to \infty$:

$$\gamma_{2n} = \frac{\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^4/4}}{\int_{-\infty}^{\infty} dx\, e^{-x^4/4}} \sim 2^n \frac{\Gamma(n/2 + 1/4)}{\Gamma(1/4)}.$$

This is astonishing because we get the connected Green's functions by *subtracting* the disconnected parts from γ_{2n}.

Surprisingly, neglecting the huge quantity G_{2n} on the left side of the DS equations (3) still leads to a reasonably accurate result for G_2, as Fig. 1 shows. This is because while the term on the left side is big, the terms on the right are comparably big [9]. We also find that Padé approximants or mean-field-like schemes do *not* improve the convergence. But there *is* a way to get accurate results: Approximating the left side of the DS equations with the asymptotic formula in (4) gives G_2 to high precision (see Fig. 2). This approach works well for $D = 0$ but is difficult to implement if $D > 0$ as the DS equations are coupled nonlinear integral equations instead of algebraic equations.

Non-Hermitian cubic $D = 0$ theory.—The massless Lagrangian $\mathcal{L} = \frac{1}{3}i\phi^3$ defines a non-Hermitian $\mathcal{PT}$-symmetric theory whose one-point Green's function is

$$G_1 = \int dx\, x e^{-ix^3/3} \Big/ \int dx\, e^{-ix^3/3}, \quad (7)$$

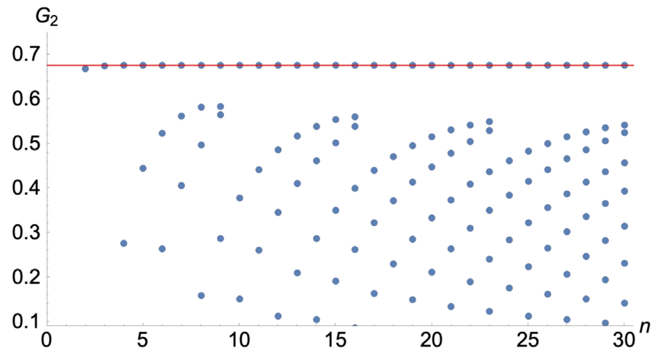

FIG. 2. Dramatic improvement of the results in Fig. 1 obtained by replacing the left side of the DS equations (3) by the asymptotic approximation (4) instead of zero.

where the path of integration terminates in a $\mathcal{PT}$-symmetric pair of Stokes sectors [7], so the exact value of G_1 is $G_1 = -i3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3}) = -0.729\,011\,13\ldots i$.

The first four DS equations are

$$G_2 = -G_1^2, \qquad G_3 = -2G_1G_2 - i,$$
$$G_4 = -2G_2^2 - 2G_1G_3, \qquad G_5 = -6G_2G_3 - 2G_1G_4. \quad (8)$$

To obtain the leading approximation to G_1 we substitute the first equation into the second and truncate by setting $G_3 = 0$. The resulting equation is $G_1^3 = \frac{1}{2}i$ and the solution that is consistent with $\mathcal{PT}$ symmetry is $G_1 = -2^{-1/3}i = -0.793\,700\,53\ldots i$. This result differs by 8.9% from the exact value of G_1.

At higher order we again truncate the system and find the roots of the associated polynomial in G_1. At first, the roots consistent with $\mathcal{PT}$ symmetry obtained by this procedure approach the exact G_1 but unlike the roots for the Hermitian quartic theory, where the approach is monotone (Fig. 1), the approach is oscillatory: for the $n = 4, 5, 6, 7$ truncations the closest roots to the exact G_1 are $-0.693\,361\ldots i$, $-0.746\,900\ldots i$, $-0.712\,564\ldots i$, and $-0.739\,871\ldots i$. However, for $n = 8$ this pattern breaks; the closest root is $-0.712\,368\ldots i$, which is a worse approximation than the $n = 6$ root.

This departure from oscillatory convergence is the first indication of a qualitative change in the approximants. For $n = 10$ the roots closest to G_1 are a pair on either side of the negative-imaginary axis at $-0.717\,367\ldots i \pm 0.016\,050\ldots$. We solve the DS equations up to the 150th truncation and plot in Fig. 3 all roots from $n = 2$ to 150 as dots in the complex plane. These roots become dense on a three-bladed propeller shape, with a small loop at the tip of each blade. The inset shows that dots on the loop surround but do not approach the exact G_1.

The roots in Fig. 3 have threefold symmetry because the truncated DS equations give polynomials having only powers of x^3 (apart from a root at 0). The DS equations depend only *locally* on the integrand of the functional

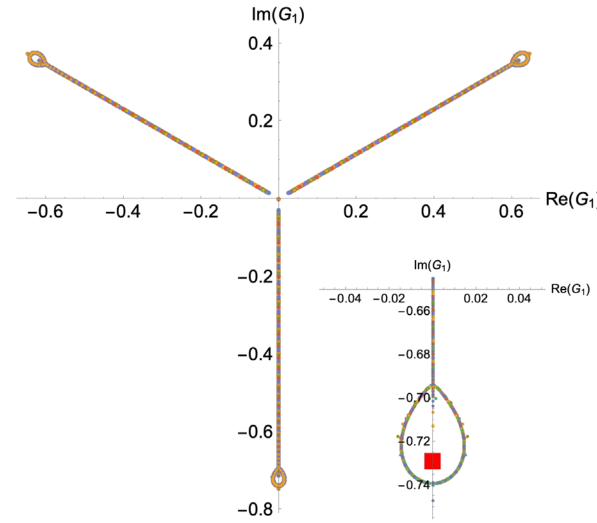

FIG. 3. All solutions of the truncated DS equations (8) for the non-Hermitian cubic theory. Inset: The square indicates the exact $G_1 = -0.729\,011\,13\ldots i$.

integral; they are totally insensitive to the boundary conditions on the functional integrals. There are three pairs of Stokes sectors of angular opening 60° inside of which the integration path in (7) can terminate. These sectors are centered about $\theta_1 = (\pi/2)$, $\theta_2 = -(\pi/6)$, or $\theta_3 = -(5\pi/6)$. If the integration path terminates in the $\mathcal{PT}$-symmetric (2,3) sectors, G_1 is negative imaginary, but if it terminates in the (1,2) or (1,3) sectors, G_1 is complex.

Asymptotic behavior of G_n for large n.—Richardson extrapolation gives the large-n behavior of the exact Green's functions for the cubic theory ($G_{14} = 42\,692.806\,116$, $G_{15} = -255\,589.034\,701\,i$):

$$G_n \sim -(n-1)!r^n(-i)^n \qquad (n \to \infty), \qquad (9)$$

where $r = 0.427\,696\,347\,707\ldots$. Equation (9) is analogous to (4) for the Hermitian quartic theory, and can be confirmed analytically [9].

To calculate r analytically we follow the procedure used above for the Hermitian quartic theory. Define $g_p \equiv -i^n G_p/(p-1)!$ and rewrite the DS equations for the Green's functions G_n as one compact equation:

$$g_p = \frac{1}{p-1}\sum_{k=1}^{p-1} g_k g_{p-k} + \frac{1}{2}\delta_{p,3} \qquad (p \geq 2).$$

Next, multiply by $(p-1)x^p$, sum from $p = 2$ to ∞, and define the generating function $f(x) \equiv \sum_{p=1}^{\infty} x^p g_p$, which obeys the Riccati equation $xf'(x) - f(x) = f^2(x) + x^3$.

Substituting $f(x) = -xu'(x)/u(x)$ linearizes this equation: $u''(x) = -xu(x)$. This is an Airy equation whose general solution is $u(x) = a\mathrm{Ai}(-x) + b\mathrm{Bi}(-x)$. From

$f'(0) = g_1 = -3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})$ we find that a is arbitrary and $b = 0$, so $f(x) = x\mathrm{Ai}'(-x)/\mathrm{Ai}(-x)$.

The power series for the generating function $f(x)$ blows up when the denominator vanishes, when $x = 2.338\,107\,410\,460\ldots$. This is the radius of convergence of the series and its *inverse* is the value of r in (9).

The rapid growth of G_n in (9) explains the slow convergence and inaccurate numerical results obtained by truncating the DS equations (Fig. 3). Once again, using this asymptotic approximation instead of setting $G_n = 0$ gives extremely accurate and rapidly convergent approximations to G_1 [9].

Non-Hermitian quartic $D = 0$ theory.—The Lagrangian $\mathcal{L} = -\frac{1}{4}\phi^4$ defines a non-Hermitian $\mathcal{PT}$-symmetric theory, where

$$G_1 = \frac{\int dx\, x \exp(x^4/4)}{\int dx \exp(x^4/4)} = -\frac{2i\sqrt{\pi}}{\Gamma(1/4)} = -0.977\,741\ldots i,$$

and the path of integration lies inside a $\mathcal{PT}$-symmetric pair of Stokes sectors in the lower-half complex-x plane.

The first three DS equations are

$$\begin{aligned} G_3 &= -G_1^3 - 3G_1G_2, \\ G_4 &= -3G_1G_3 - 3G_2^2 - 3G_1^2G_2 - 1, \\ G_5 &= -3G_1G_4 - 9G_2G_3 - 3G_1^2G_3 - 6G_1G_2^2. \end{aligned} \quad (10)$$

Solving these equations is harder than for the Hermitian quartic or the non-Hermitian cubic theory, as *two* Green's functions must be set to zero to close the system, and *two* coupled polynomials equations must be solved simultaneously. The leading-order truncation leads to $G_1 = -i(3/2)^{1/4} = -1.106\,682\ldots i$, which differs from the exact G_1 above by 13.2%.

This procedure is continued for larger n. The number of roots increases rapidly and the roots have fourfold symmetry in the complex plane. All roots up to $n = 33$ are shown in Fig. 4. There are four concentrations of roots on the axes but $\mathcal{PT}$ symmetry requires that G_1 be negative imaginary. Unlike Fig. 3 the dots are scattered over the complex plane because truncating the DS equations gives two *coupled polynomial equations*.

We can determine the asymptotic behavior of G_n for large n from the DS equations in (10). We find $G_n \sim -i(n-1)!(-i)^n r^n$, where $r = 0.346\,40\ldots$. This result is analogous to the asymptotic behavior in (9).

Quintic and sextic $D = 0$ theories.—The DS equations for the $\mathcal{PT}$-symmetric $D = 0$ Lagrangian $-\frac{1}{5}i\phi^5$ require that *three* higher Green's functions be set to 0 to close the truncated system, leading to three coupled polynomial equations for G_1, G_2, and G_3. Going to the $n = 11$ truncation we see *ten* concentrations of roots in Fig. 5. (The DS equations are insensitive to the choice of Stokes sectors for the functional integral.) There are *two* pairs of $\mathcal{PT}$-symmetric boundary conditions, which give rise to two imaginary values of $G_1 = 0.412\,009\ldots i$ and $G_1 = -1.078\,653\ldots i$ [11], seen on Fig. 5 as heavy dots.

For the sextic case $\mathcal{L} = \frac{1}{6}\phi^6$ we truncate the DS equations and set the *four* highest Green's functions to 0. We must solve four coupled polynomial equations. To reduce the number of solutions we impose parity symmetry, so $G_1 = G_3 = 0$. This eliminates all but three pairs of Stokes sectors. Figure 6 shows three concentrations of roots for G_2 up to the $n = 32$ truncation. The exact values of G_2 (squares) are $6^{1/3}\sqrt{\pi}/\Gamma(1/6) = 0.578\,617\ldots$ and $-0.289\,302\ldots \pm 0.501\,097i$; the error is a few percent.

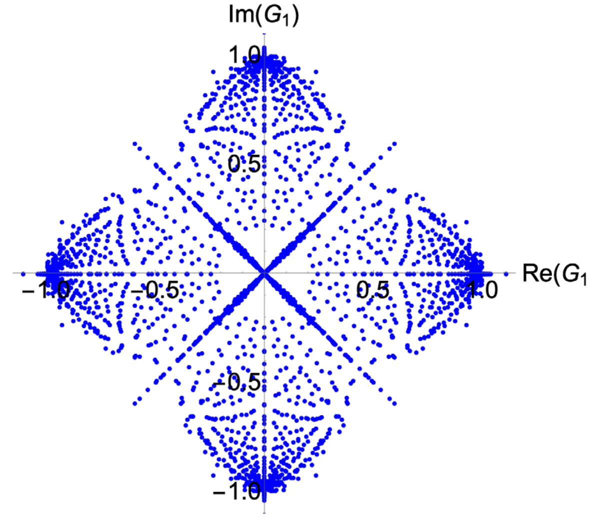

FIG. 4. All roots G_1 up to $n = 33$ plotted as points in the complex plane. The roots exhibit fourfold symmetry but only those on the negative-imaginary axis respect $\mathcal{PT}$ symmetry.

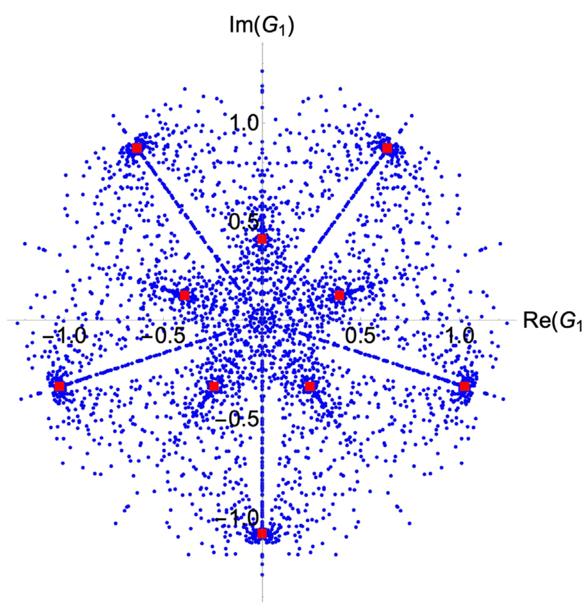

FIG. 5. Solutions to the DS equations for a quintic $D = 0$ field theory. Exact values are denoted by squares.

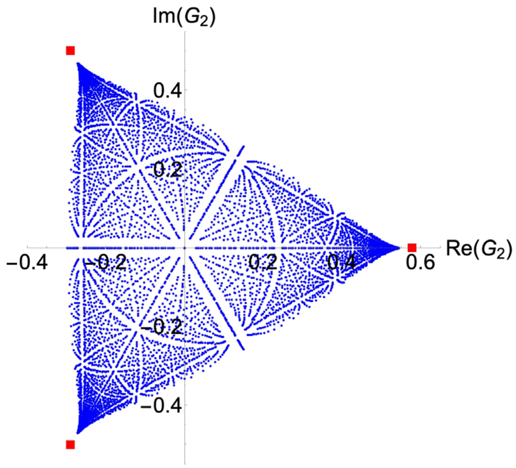

FIG. 6. Sextic case showing three concentrations of parity-symmetric solutions for G_2 differing from the exact values (squares) by a few percent.

Summary.—For five $D = 0$ field theories we have shown that the truncated DS equations yield underdetermined polynomial systems. There is no effective strategy to solve such systems: Closing the systems by setting higher Green's functions to zero gives sequences of approximants that converge to incorrect limiting values. Replacing higher Green's functions with mean-field-like approximations also gives incorrect limiting values, and this approach has the drawback that if $D > 0$, renormalization is required. The one numerically accurate approach is to replace the higher G_n's by their large-n asymptotic behaviors. This is difficult when $D > 0$, but we believe that it may be possible to calculate, and it presents an interesting avenue for further research.

This Letter emphasizes that the DS equations are *local*. Deriving the DS equations assumes only that the functional integrals *exist*; the DS equations are insensitive to which Stokes sectors in function space are used. As a result, the approximants try (but fail) to approach many different limits, most of which are complex [12].

The accuracy of the DS truncations worsens when interaction terms have higher powers of the field because the indeterminacy of the system increases. More Green's functions must be set to 0 to close the truncated system.

For Lagrangians having a weak-coupling constant g we can expand all G_n in the DS equations as series in powers of g. This removes all ambiguities discussed in here and gives the unique weak-coupling expansion for each G_n. However, this merely replicates a Feynman-diagram calculation of the Green's functions and totally ignores the nonperturbative content of the theory.

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[1] We calculate G_n, rather than γ_n, to avoid vacuum divergences. Disconnected contributions to γ_n introduce factors of the spacetime volume, which is infinite.

[2] In the early days of quantum field theory one view was that one could define a field theory as nothing but a set of Feynman rules and thereby evade mathematical issues (S. Coleman, private communication).

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[12] The quantum-mechanics analog of this problem is that the spectrum of a Hamiltonian is undetermined until the boundary conditions on the Schrödinger eigenfunctions are given. For example, the spectrum of the harmonic oscillator Hamiltonian is either positive or negative depending on whether the eigenfunctions vanish as $x \to \pm\infty$ or $x \to \pm i\infty$. See Ref. [5].