


Taming Calabi-Yau Feynman Integrals: The Four-Loop Equal-Mass Banana Integral

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Certain Feynman integrals are associated to Calabi-Yau geometries. We demonstrate how these integrals can be computed with the method of differential equations. The four-loop equal-mass banana integral is the simplest Feynman integral whose geometry is a nontrivial Calabi-Yau manifold. We show that its differential equation can be cast into an ε -factorized form. This allows us to obtain the solution to any desired order in the dimensional regularization parameter ε . The method generalizes to other Calabi-Yau Feynman integrals. Our calculation also shows that the four-loop banana integral is only minimally more complicated than the corresponding Feynman integrals at two or three loops.

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Introduction.—In recent years it has become clear that we may associate to each Feynman integral a geometric object, and that understanding this geometry helps in computing the Feynman integral. Most Feynman integrals that have been computed so far are associated to a sphere. These Feynman integrals evaluate to multiple polylogarithms [1,2]. The next more complicated Feynman integrals are associated to elliptic curves. These evaluate to iterated integrals of modular forms and elliptic polylogarithms [3,4]. These two classes of Feynman integrals (i.e., the ones associated to spheres and elliptic curves) are by now quite well understood and in many examples we are able to transform the differential equation for these Feynman integrals to an ε -factorized form [5]. Examples for the elliptic case can be found in Refs. [6–9]. The differential equation in an ε -factorized form together with values of the Feynman integrals at a boundary point is all that we need: from these data we can easily obtain the analytic solution to any order in the dimensional regularization parameter ε .

We also know that there are Feynman integrals associated to Calabi-Yau manifolds [10]. However, up to now it was not known whether the differential equation for a Feynman integral associated to a nontrivial Calabi-Yau manifold can be transformed to an ε -factorized form. In this Letter we show for the first time that this is indeed possible. This implies that the Feynman integral can be solved to any order in the dimensional regularization parameter ε

(the required boundary values are rather easily obtained). Let us comment on what we mean by a “nontrivial” Calabi-Yau manifold: Calabi-Yau one-folds are elliptic curves, which are well understood. Feynman integrals associated to higher-dimensional Calabi-Yau manifolds can in some cases be related to elliptic curves. This occurs when their Picard-Fuchs operator is the symmetric product of a degree two operator of an elliptic curve [11]. We call such Calabi-Yau manifolds “trivial.” The family of l -loop banana graphs provides examples for Feynman integrals related to Calabi-Yau $(l-1)$ -folds. For this reason it has received significant attention in recent years [12–17]. The three-loop banana integral, a Calabi-Yau twofold, is well known to possess an operator that is a symmetric square [18,19]. It can therefore be treated with methods similar to the elliptic case [20–24]. The first example from the family of banana graphs that is nontrivial in the sense above is therefore the four-loop equal-mass banana integral. It is related to a Calabi-Yau threefold. This integral is, for example, relevant to the process $pp \rightarrow t\bar{t}$ at N⁴LO (i.e. the fourth order beyond the leading one, the abbreviation stands for next-to-next-to-next-to-next-to-leading order).

In this Letter, we show that an ε -factorized differential equation for this Feynman integral exists, give the differential 1-forms appearing in the differential equation, and show how to solve the four-loop equal-mass banana integral to all orders in the dimensional regularization parameter ε . The benefits are threefold. Firstly, we obtain analytic solutions to any order in the dimensional regularization parameter. Secondly, our results give us very fast numerical evaluations. Thirdly, the differential 1-forms define the symbol alphabet for this Feynman integral [see Eq. (26)]. Hence, as a by-product we obtain for the first time the symbol alphabet for a Feynman integral

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associated to a Calabi-Yau threefold, extending recent work on elliptic symbols [25,26].

The method we employ is not specific to this particular Feynman integral, and extends to other Feynman integrals related to Calabi-Yau manifolds. For this Letter we have checked that the same method works for the five-loop and six-loop equal-mass banana integrals. Recently, two publications appeared showing the application of methods similar to the one introduced here to equal-mass banana integrals [27] and ice cone integrals [28] at any loop order. In general, Feynman integrals depend on more than one kinematic variable. Establishing an ε -factorized form for these will require similar steps as going from the equal-mass sunrise (one kinematic variable) to the unequal-mass sunrise (three kinematic variables). The sunrise integrals are related to Calabi-Yau one-folds, and an ε -factorized form exists for both equal and unequal cases. Hence, the results in this Letter open the door for calculating Feynman integrals related to higher-dimensional Calabi-Yau manifolds with more parameters.

We find that there are two new ingredients required for nontrivial Calabi-Yau manifolds. The first one is a change of variables. In the two-loop and three-loop equal-mass banana cases an important step is the change of variables from the dimensionless ratio $m^2/(-p^2)$ to the modular parameter τ , defined as the ratio of the two periods of the elliptic curve. In the four-loop case there is no elliptic curve. However, the above change of variables can be viewed as the mirror map [29–31] for a family of Calabi-Yau manifolds, and this generalizes to the cases of interest.

The second ingredient is the following: The ε -factorized form is achieved by redefining the master integrals. One pattern which emerges is that one of the master integrals of the l -loop banana family in $D = 2 - 2\varepsilon$ space-time dimensions is given by

$$I_2 = \frac{\varepsilon^l}{\omega_1} I_{\underbrace{1 \dots 1}_{l+1}}, \tag{1}$$

where ω_1 is a specific solution of the homogeneous differential equation. With $q = \exp(2\pi i\tau)$ and $\theta = q(d/dq)$, the Picard-Fuchs operators for I_2 at two and three loops are

$$\theta^2 \quad \text{and} \quad \theta^3. \tag{2}$$

At four loops we find

$$\theta^2 \frac{1}{K} \theta^2. \tag{3}$$

The appearance of the new function K is related to the fact that the Picard-Fuchs operator at four loops is not a symmetric product. It also implies that at four loops the first term of I_2 in the ε expansion is not an Eichler

integral. In the literature on Calabi-Yau manifolds the function K is known as the ‘‘Yukawa coupling’’ [32] (see also Refs. [11,33–39]). In general the factorization (or special local normal form) for a Calabi-Yau operator of degree l is [11]

$$\theta^2 \frac{1}{K_1} \theta \frac{1}{K_2} \theta \dots \theta \frac{1}{K_{l-4}} \theta \frac{1}{K_{l-3}} \theta^2, \tag{4}$$

with $K_i = K_{l-2-i}$. This factorization translates into an ansatz for the ε -factorized form [see Eq. (12)]. In the following we discuss concretely the case of the four-loop equal-mass banana integral. It serves as an example where complications due to higher-dimensional Calabi-Yau manifolds occur for the first time. Our main results are the ansatz given in Eq. (12), the differential equation in ε -factorized form given in Eqs. (24) and (25), and the symbol alphabet in Eq. (26).

Notation.—We are interested in the integrals

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = e^{4\gamma_E\varepsilon}(m^2)^{\sum_{j=1}^5 \nu_j - 2D} \int \left(\prod_{a=1}^5 \frac{d^D k_a}{i\pi^{D/2}} \right) \times i\pi^{D/2} \delta^D \left(p - \sum_{b=1}^5 k_b \right) \left(\prod_{c=1}^5 \frac{1}{(-k_c^2 + m^2)^{\nu_c}} \right), \tag{5}$$

where D denotes the number of space-time dimensions, ε the dimensional regularization parameter, and γ_E the Euler-Mascheroni constant. We consider these integrals in $D = 2 - 2\varepsilon$ space-time dimensions. The corresponding Feynman graph is shown in Fig. 1. It is convenient to introduce the dimensionless variable:

$$y = -\frac{m^2}{p^2}. \tag{6}$$

It is well known that this family of Feynman integrals has five master integrals. A possible choice for a basis of master integrals is

$$I_{111110}, I_{111111}, I_{111112}, I_{111113}, I_{111114}. \tag{7}$$

The Feynman integral I_{111110} is a product of four one-loop tadpole integrals and rather simple. We set

$$I_1 = \varepsilon^4 I_{111110} = [e^{\gamma_E\varepsilon} \Gamma(1 + \varepsilon)]^4. \tag{8}$$

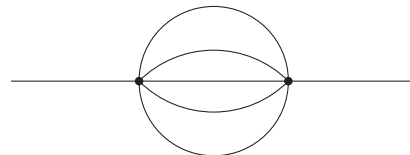


FIG. 1. The four-loop banana graph.

The integral I_1 has uniform transcendental weight, where we assign weight n to the ζ value ζ_n and weight (-1) to ε .

Let us now look at I_{11111} . Useful information can be extracted from the Picard-Fuchs operator [40–43]. For I_{11111} we have the inhomogeneous fourth-order differential equation:

$$L_4 I_{11111} = \frac{120\varepsilon^4}{y^3(1+y)(1+9y)(1+25y)} I_{11110}. \quad (9)$$

The dependence on ε of the fourth-order differential operator L_4 is polynomial. The ε -independent part of L_4 , which we denote by $L_4^{(0)}$, is of particular importance:

$$\begin{aligned} L_4^{(0)} = & \frac{d^4}{dy^4} + 2\left(\frac{1}{y} + \frac{1}{1+y} + \frac{9}{1+9y} + \frac{25}{1+25y}\right) \frac{d^3}{dy^3} \\ & + \frac{(1+98y+1839y^2+3150y^3)}{y^2(1+y)(1+9y)(1+25y)} \frac{d^2}{dy^2} \\ & - \frac{(1+15y)(1-15y-60y^2)}{y^3(1+y)(1+9y)(1+25y)} \frac{d}{dy} \\ & + \frac{1+5y}{y^4(1+y)(1+9y)(1+25y)}. \end{aligned} \quad (10)$$

This operator appears as entry 34 in the table of Ref. [35]. The associated Calabi-Yau manifold has been studied in Ref. [44].

The indicial equation for the operator $L_4^{(0)}$ at the point $y = 0$ is $(\rho - 1)^4 = 0$, showing that $y = 0$ is a point of maximal unipotent monodromy.

Master integrals.—The construction of the master integrals, which put the differential equation into an ε -factorized form, follows with one exception the pattern found at two loops [6] and three loops [24]. Let

$$\begin{aligned} & \omega_1(y), \quad J(y), \quad K(y), \\ & F_{32}(y), \quad F_{42}(y), \quad F_{43}(y), \quad F_{52}(y), \\ & F_{53}(y), \quad F_{54}(y) \end{aligned} \quad (11)$$

be nine *a priori* unknown functions of y . They are however independent of ε . We start from the following ansatz for the master integrals:

$$\begin{aligned} I_1 &= \varepsilon^4 I_{11110}, \\ I_2 &= \frac{\varepsilon^4}{\omega_1} I_{11111}, \\ I_3 &= \frac{J}{\varepsilon} \frac{d}{dy} I_2 + F_{32} I_2, \\ I_4 &= \frac{J}{\varepsilon K} \frac{d}{dy} I_3 + F_{42} I_2 + F_{43} I_3, \\ I_5 &= \frac{J}{\varepsilon} \frac{d}{dy} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4. \end{aligned} \quad (12)$$

The new ingredient at four loops is the appearance of the function K in the definition of I_4 . This ansatz leads to the differential equation

$$dI = AI, \quad (13)$$

where A is a 5×5 matrix. The entries in the first four rows and the entry $A_{5,1}$ are already proportional to ε . The remaining entries in the fifth row may be written as

$$A_{5,k} = \sum_{j=k-5}^1 A_{5,k}^{(j)} \varepsilon^j, \quad k \in \{2, 3, 4, 5\}, \quad (14)$$

where $A_{5,k}^{(j)}$ is independent of ε . We now require that the $A_{5,k}^{(j)}$ with $j < 1$ vanish. This gives ten equations for nine unknown functions. These equations can be satisfied as follows: Let $\omega_1 - \omega_4$ be four independent solutions of $L_4^{(0)} \omega = 0$. From Frobenius it follows that we may write them as

$$\omega_k = \frac{1}{(2\pi i)^{k-1}} \sum_{j=0}^{k-1} \frac{\ln^j y}{j!} \sum_{n=0}^{\infty} a_{k-1-j,n} y^{n+1}. \quad (15)$$

As normalization we choose $a_{0,0} = 1$. We identify the holomorphic solution ω_1 with the one appearing in Eqs. (11) and (12). The first few terms read

$$\omega_1 = y(1 - 5y + 45y^2 - 545y^3) + \mathcal{O}(y^5). \quad (16)$$

We set

$$\tau = \frac{\omega_2}{\omega_1}, \quad q = e^{2\pi i \tau}, \quad (17)$$

and

$$J = \frac{1}{2\pi i} \frac{dy}{d\tau}. \quad (18)$$

Equation (17) defines a change of variables from y to τ (or q). This map is also called the mirror map [45]. We have

$$y = q + 8q^2 + 36q^3 + 168q^4 + \mathcal{O}(q^5). \quad (19)$$

As y goes to zero, τ approaches $i\infty$ and q goes to zero. The function K is given by

$$K = \frac{J^3}{\omega_1^2 y(1+y)(1+9y)(1+25y)}. \quad (20)$$

We define iterated integrals by

$$I(f_1, \dots, f_n; \tau) = \lim_{q_0 \rightarrow 0} R \left[\int_{q_0}^q \frac{dq_1}{q_1} \dots \int_{q_0}^{q_{n-1}} \frac{dq_n}{q_n} f_1(\tau_1) \dots f_n(\tau_n) \right], \quad (21)$$

where R is the operator which removes all $\ln(q_0)$ terms, corresponding to the standard “trailing zero” or “tangential base point” regularization [3,46,47]. As the last argument of all iterated integrals will always be τ , we simply write $I(f_1, \dots, f_n)$ instead of $I(f_1, \dots, f_n; \tau)$. The functions F_{ij} are given by

$$\begin{aligned} F_{32} &= -f_{2,a} - f_{2,b}, \\ F_{43} &= -\frac{1}{K}(f_{2,a} - f_{2,b}), \quad F_{42} = -\frac{1}{K}f_{4,a}, \\ F_{54} &= -f_{2,a} + f_{2,b}, \quad F_{53} = -f_{4,b}, \quad F_{52} = -f_6, \end{aligned} \quad (22)$$

where

$$\begin{aligned} f_{2,a} &= J \left(\frac{5}{2y} - \frac{1}{1+y} - \frac{9}{1+9y} - \frac{25}{1+25y} \right), \\ f_{2,b} &= I(K, h_6), \\ f_{4,a} &= -[I(K, h_6)]^2 - 2KI(1, K, h_6, h_6) - KI(1, h_{8,b}), \\ f_{4,b} &= 4I(1, K, h_6, h_6) + 2I(1, h_{8,b}) + h_4, \\ f_5 &= 120 \frac{J\omega}{y^2}, \\ f_6 &= f_{4,b}I(K, h_6) - 4h_6, \\ f_8 &= f_{4,b}[I(K, h_6)]^2 - 8h_6I(K, h_6) \\ &\quad + K[I(1, h_{8,b}) + 2I(1, K, h_6, h_6)]^2 + h_{8,a}. \end{aligned} \quad (23)$$

The helper functions $h_4, h_6, h_{8,a}$, and $h_{8,b}$ are defined in the Appendix. This completes the definition of the master integrals and it can be verified that with this definition the terms $A_{5,k}^{(j)}$ with $j < 1$ vanish in Eq. (14). Hence, the differential equation is in ϵ -factorized form,

$$dI = 2\pi i \epsilon A_\tau I d\tau, \quad (24)$$

with

$$A_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & f_{2,a} + f_{2,b} & 1 & 0 & 0 \\ 0 & f_{4,a} & f_{2,a} - f_{2,b} & K & 0 \\ 0 & f_6 & f_{4,b} & f_{2,a} - f_{2,b} & 1 \\ f_5 & f_8 & f_6 & f_{4,a} & f_{2,a} + f_{2,b} \end{pmatrix}. \quad (25)$$

This is a differential equation with an alphabet consisting of nine letters:

$$\mathcal{A} = \{1, K, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_5, f_6, f_8\}. \quad (26)$$

We observe the symmetry

$$A_{i,j} = A_{7-j,7-i}, \quad \text{for } i, j \geq 2. \quad (27)$$

We further observe that the q expansions of all entries of the matrix A_τ have integer coefficients (apart from a possible rational prefactor):

$$\begin{aligned} K &= 1 - q + 17q^2 - 253q^3 + 3345q^4 + \mathcal{O}(q^5), \\ f_{2,a} &= \frac{5}{2} - 15q + 167q^2 - 2787q^3 + 40631q^4 + \mathcal{O}(q^5), \\ f_{2,b} &= 7q - 135q^2 + 2275q^3 - 34759q^4 + \mathcal{O}(q^5), \\ f_{4,a} &= -\frac{1}{4}(147q - 3267q^2 + 59943q^3 - 1017027q^4) + \mathcal{O}(q^5), \\ f_{4,b} &= \frac{5}{2} - 52q + 1460q^2 - 33316q^3 + 652212q^4 + \mathcal{O}(q^5), \\ f_5 &= 120(1 + 3q - 27q^2 + 147q^3 - 1467q^4) + \mathcal{O}(q^5), \\ f_6 &= -\frac{1}{2}(21q - 2805q^2 + 108777q^3 - 2772213q^4) + \mathcal{O}(q^5), \\ f_8 &= -\frac{1}{16}(9 - 855q + 7623q^2 + 606789q^3 - 31766841q^4) + \mathcal{O}(q^5). \end{aligned} \quad (28)$$

Results.—We write

$$I_k = \sum_{j=0}^{\infty} I_k^{(j)} \varepsilon^j \quad (29)$$

for the ε expansion of the master integral I_k .

With the differential equation (24) at hand we only need the boundary values as additional input. We choose $y = 0$ as boundary point. The boundary values are easily obtained with the help of the Mellin-Barnes technique. The calculation follows the lines of the corresponding calculation at three loops [22,24]. We need the constant term and all logarithms $\ln(y)$. We obtain for the boundary value of I_2 [15]:

$$I_2|_{y \rightarrow 0} = 5e^{4\gamma_E \varepsilon} \sum_{j=0}^4 \binom{4}{j} (-1)^j y^{j\varepsilon} \times \frac{\Gamma(1+\varepsilon)^{4-j} \Gamma(1-\varepsilon)^{1+j} \Gamma(1+j\varepsilon)}{\Gamma[1-(j+1)\varepsilon]}. \quad (30)$$

Note that Eq. (30) determines the boundary values of all master integrals.

$$I_2 = \varepsilon^4 [I(1, K, 1, f_5) - 80\zeta_3 \ln q] + \varepsilon^5 \{I(1, K, 1, f_2^+, f_5) + I(1, K, f_2^-, 1, f_5) + I(1, f_2^-, K, 1, f_5) + I(f_2^+, 1, K, 1, f_5) - 120\zeta_2 I(1, K, 1) - 40\zeta_3 [13I(1, K) + 2I(1, f_2^-) + 2I(f_2^+, 1)] - 120\zeta_4 \ln q + 80\zeta_2 \zeta_3 - 600\zeta_5\} + \mathcal{O}(\varepsilon^6). \quad (32)$$

From Eq. (28) we obtain the q expansion of all iterated integrals and hence the q expansion of the master integrals. For example, the first few terms of $I_2^{(4)}$ read with $L = \ln q$:

$$I_2^{(4)} = 5L^4 - 80\zeta_3 L + 60L(4-L)q + 15(6 - 34L + 17L^2)q^2 - \frac{10}{9}(123 - 2024L + 1518L^2)q^3 - \frac{25}{8}(851 + 4014L - 4014L^2)q^4 + \mathcal{O}(q^5). \quad (33)$$

In addition, we must compute the value of q from a given value y . In the region $|y| < 1/25$ we may use the series for ω_1 and ω_2 given in Eq. (15) to obtain from a given value of y the corresponding value of q . Note that this is different from the situation at two and three loops, where we may use complete elliptic integrals to obtain the value of q from y .

In the plots, we will use the kinematical variable $x = -1/y = p^2/m^2$ instead of y . The condition $|y| < 1/25$ translates to $|x| > 25$. This is the region where we may evaluate the integral numerically. The correct branch is selected by giving x an infinitesimal positive imaginary part according to Feynman's $i\delta$ prescription. Figures 2 and 3

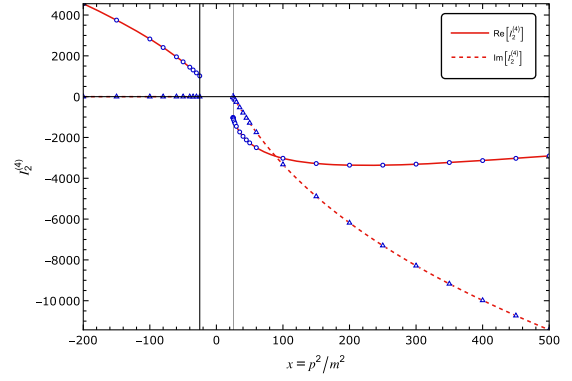


FIG. 2. Comparison of our result for $I_2^{(4)}$ (plotted as curves) with numerical results from pysecdec (points).

With the boundary values we may now obtain analytical results for all master integrals. As an example, we focus on I_2 . With

$$f_2^+ = f_{2,a} + f_{2,b}, \quad f_2^- = f_{2,a} - f_{2,b}, \quad (31)$$

the analytic result for the integral I_2 up to $\mathcal{O}(\varepsilon^5)$ is given by

show the numerical results for the ε^4 term $I_2^{(4)}$ and the ε^5 term $I_2^{(5)}$ of I_2 . We also plotted the results from the program pySecDec [48]. We observe excellent agreement. The CPU time for the numerical evaluation of our result is negligible, a kinematic point by pySecDec takes about 10 min on a desktop.

Numerical evaluations around the singular points $x = 0$, $x = 1$, $x = 9$, and $x = 25$ will be discussed in a longer publication.

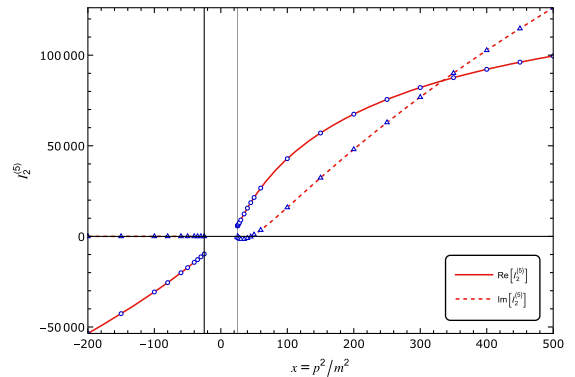


FIG. 3. Comparison of our result for $I_2^{(5)}$ (plotted as curves) with numerical results from pysecdec (points).

Conclusions.—In this Letter, we showed that the differential equation for the four-loop equal-mass banana graph can be cast into an ϵ -factorized form, which allows for a systematic solution to any desired order in the dimensional regularization parameter.

We used the mirror map to define new variables τ and q . The relevant Picard-Fuchs operator is no longer a symmetric product. We showed that this is not an essential complication, it merely introduces the function K .

Our calculation shows that the four-loop equal-mass banana integral is only minimally more complicated than the corresponding Feynman integrals at two or three loops. This is good news, as it opens a path toward Feynman

integrals related to genuinely generic Calabi-Yau manifolds.

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Appendix: Helper functions.—Here, we give the explicit expressions for the helper functions h_4 , h_6 , $h_{8,a}$, and $h_{8,b}$:

$$\begin{aligned}
 h_4 &= \frac{5 + 14y - 569y^2 - 12\,044y^3 + 77\,427y^4 - 302\,850y^5 + 50\,625y^6}{2y(1+y)(1+9y)(1+25y)} \frac{\omega_1^2}{J}, \\
 h_6 &= \left[\frac{7}{y} + \frac{192y}{(1+y)^2} + \frac{64(1-9y)}{(1+9y)^2} - \frac{192(3+25y)}{(1+25y)^2} \right] \omega_1^2, \\
 h_{8,a} &= -(1 + 33y - 577y^2 - 225y^3)(3 - 25y - 187y^2 + 225y^3) \\
 &\quad \times \frac{(3 - 38y - 2167y^2 + 4060y^3 + 307\,893y^4 + 693\,450y^5 - 50\,625y^6)}{16y^3(1+y)^3(1+9y)^3(1+25y)^3} \omega_1^2 J, \\
 h_{8,b} &= \left[-\frac{h_4 J d^2 \omega_1}{\omega_1^2 dy^2} - \left(\frac{d h_4 J}{dy} \frac{1}{\omega_1^2} \right) \left(\frac{d \omega_1}{dy} \right) - \frac{X}{4y^3(1+y)^3(1+9y)^3(1+25y)^3} \omega_1 \right] \omega_1 J, \\
 X &= 10 + 953y + 36\,365y^2 + 624\,143y^3 + 5\,506\,553y^4 + 25\,045\,562y^5 - 43\,328\,414y^6 - 1\,595\,702\,658y^7 \\
 &\quad - 3\,663\,006\,612y^8 + 4\,836\,275\,325y^9 + 29\,858\,270\,625y^{10} + 9\,146\,671\,875y^{11} + 2\,562\,890\,625y^{12}. \tag{A1}
 \end{aligned}$$

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