Radiation Reaction and Gravitational Waves at Fourth Post-Minkowskian Order

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We obtain the total impulse in the scattering of nonspinning binaries in general relativity at fourth post-Minkowskian order, i.e., $\mathcal{O}(G^4)$, including linear, nonlinear, and hereditary radiation-reaction effects. We derive the total radiated spacetime momentum as well as the associated energy flux. The latter can be used to compute gravitational-wave observables for generic (un)bound orbits. We employ the ("in-in") Schwinger-Keldysh worldline effective field theory framework in combination with modern "multiloop" integration techniques from collider physics. The complete results are in agreement with various partial calculations in the post-Minkowskian and post-Newtonian expansion.

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Introduction.—Waveform models are an essential ingredient in data analysis and characterization of gravitational wave (GW) signals from compact binaries [1]. The level of accuracy plays a critical role, in particular for future detectors such as LISA [2] and ET [3]. In order to benefit the most from the anticipated observational reach [2–9], the modeling of GW sources must therefore continue to develop—both through analytic methodologies [10–18] and numerical simulations [19–21]—in parallel with the expected increase in sensitivity with next-generation GW interferometers.

Motivated by the effective-one-body (EOB) formalism [22–27], the boundary-to-bound (B2B) dictionary between unbound and bound observables [28–30], and benefiting from powerful "multiloop" integration tools [31–61], significant progress has been achieved in recent years in our analytic understanding of (classical) gravitational scattering in the post-Minkowskian (PM) expansion in powers of G (Newton's constant); both via effective field theory (EFT) [62–78] and amplitude-based [79–98] methodologies. The PM regime incorporates an infinite tower of post-Newtonian (PN) corrections at a given order in G that may increase the accuracy of phenomenological waveform models [99,100].

However, despite some notable exceptions [26,73–77,79,84,97,98,101], the majority of the PM computations have so far impacted our knowledge of the *conservative* sector, with potential interactions [66,92] as well as "tail effects" [67,93] known to 4PM order. Yet, until now, complete results had not been obtained at the same level of accuracy. The purpose of this Letter is therefore to report the total change of (mechanical) momentum, a.k.a. the impulse, for the gravitational scattering of nonspinning bodies—including all the hitherto unknown linear, nonlinear and hereditary radiation-reaction *dissipative* effects—at $\mathcal{O}(G^4)$, from which we derive the total radiated spacetime momentum and GW energy flux.

Building on pioneering developments in the PN regime [102–109], the derivation proceeds via the EFT approach in a PM scheme [62], extended in [76] to simultaneously incorporate conservative and dissipative effects via the "in-in" Schwinger-Keldysh formalism [110–114]. As discussed in [76], the in-in impulse resembles the "in-out" counterpart used in the conservative sector [66,67], except for its causal structure which entails the use of retarded Green's functions [76]. After adapting integration tools to our problem, the calculation of the impulse is mapped to a series of "three-loop" mass-independent integrals. As in previous derivations [62,63,66,67], the latter are solved via the methodology of differential equations [32–38]. The relativistic two-body problem is then reduced to obtaining the necessary boundary conditions in the near-static limit. The boundary integrals are computed using the method of regions [45], involving potential (off-shell) and radiation (on-shell) modes [102]. The full solution is thus *bootstrapped* to all orders in the velocities from the same type of calculations needed in the EFT approach with PN sources [102,109,115–118]. As a nontrivial check, by rewriting retarded Green's functions as Feynman propagators plus a *reactive* term [76], we recover the value in [66,67] for the Feynman-only (conservative) part. Agreement is also found in the overlap with various PN derivations [26,27,119–124] and partial PM results [98] obtained using the relations in [22].

The B2B dictionary [28–30] allows us to connect scattering data to observables for bound states via analytic continuation. However, similarly to the lack of periastron advance at 3PM [28,29,63], the symmetries of the problem yield a vanishing coefficient for the radiated energy integrated over a period of ellipticlike motion at 4PM, trivially recovered by the B2B map. Nevertheless, since nonlinear radiation-reaction effects do not contribute to the integrated radiated energy at $\mathcal{O}(G^4)$, we can then derive the GW flux in an adiabatic approximation [30]. This allows for the computation of radiative observables for generic (un)bound orbits through balance equations, as in the EOB approach [22], thus including an infinite series of velocity corrections.

The EFT in-integrand.—Following the Schwinger-Keldysh formalism [110–114] adapted to the EFT approach in [76,104], the effective action is obtained via a *closed-time-path* integral involving a doubling of the metric perturbation ($h^{\pm}_{\mu\nu}$) as well as the worldline ($x^{\alpha}_{a,\pm}$) degrees of freedom, schematically,

$$e^{i\mathcal{S}_{\rm eff}[x_{a,\pm}]} = \int \mathcal{D}h^+\mathcal{D}h^- e^{i\{S_{\rm EH}[h^\pm]+S_{\rm PP}[h^\pm,x_{a,\pm}]\}}, \qquad (1)$$

with $S_{\rm EH}$ and $S_{\rm PP}$ the closed-path version of the Einstein-Hilbert and point-particle worldline actions, respectively. We ignore here spin degrees of freedom and finite-size effects (see [64,65]). We also restrict ourselves to the classical regime and therefore the path integral in (1) is computed in the saddle-point approximation—keeping only connected 'tree-level' Feynman diagrams of the gravitational field(s)—with the compact objects treated as external nonpropagating sources.

In this scenario, the matrix of (causal) propagators is given by the Keldysh representation:

$$K^{AB}(x-y) = i\begin{pmatrix} 0 & -\Delta_{\rm adv}(x-y) \\ -\Delta_{\rm ret}(x-y) & 0 \end{pmatrix}, \qquad (2)$$

with $A, B \in \{+,-\}$ and $\Delta_{\rm ret/adv}$ the standard retarded and advanced Green's functions. The impulse, e.g., for particle 1, then follows from [76]

$$\Delta p_1^\mu = -\eta^{\mu\nu}\int_{-\infty}^{\infty} d\tau_1 \frac{\delta\mathcal{S}_{\rm eff}[x_{a,\pm}]}{\delta x^\nu_{1,-}(\tau_1)}\bigg|_{\rm PL} = \sum_n G^n \Delta^{(n)} p_1^\mu, \qquad (3)$$

to all PM orders, where the subscript "PL" stands for the physical limit: $\{x_{a,-}\to 0, x_{a,+}\to x_a\}$ [103]. As for the conservative sector [66,67], we must also include *iterations* from lower order solutions to the equations of motion. The latter are obtained from the effective action in the same physical limit. The diagrams needed to $\mathcal{O}(G^4)$ are depicted in Fig. 1. Mirror images (not shown) must also be computed. See [76] for details.

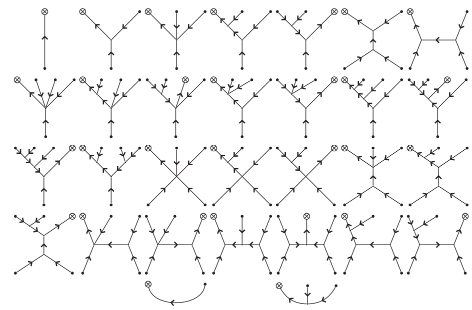

FIG. 1. In-in Feynman topologies to 4PM order. The arrows indicate the flow of (retarded) time. The crosses represent the location of the derivative in the impulse in (3). The last two diagrams are the only "self-energies" needed to 4PM [76].

The impulse is further decomposed into scalar integrals in the perpendicular and longitudinal directions, i.e., for particle 1 (and likewise for particle 2)

$$\Delta^{(n)} p_1^\mu = c_{1b}^{(n)}\frac{\hat b^\mu}{b^n} + \frac{1}{b^n}\sum_a c^{(n)}_{1\check u_a}\check u_a^\mu, \qquad (4)$$

with $b^\mu \equiv b_1^\mu - b_2^\mu$ the impact parameter, $b\equiv\sqrt{-b^\mu b_\mu}$, and $\hat b^\mu \equiv b^\mu/b$. We use the notation [97]

$$\check u_1^\mu \equiv \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1}, \qquad \check u_2^\mu \equiv \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}, \qquad \gamma\equiv u_1\cdot u_2, \qquad (5)$$

with u_a's the incoming velocities, $b\cdot u_a = 0$, $u_a^2 = 1$, and $\check u_a \cdot u_b = \delta_{ab}$. Ignoring absorption, the preservation of the on-shell condition, $p_a^2 = m_a^2$, implies $2p_a\cdot\Delta p_a = -(\Delta p_a)^2$, which serves as a nontrivial consistency check.

Integration.—Similarly to the derivations in [66,67], but incorporating the key distinction between Feynman and retarded propagators, the components of the impulse can be reduced to different families of integrals,

$$\int\prod_{i=1}^{3}\frac{d^d\ell_i}{\pi^{d/2}}\frac{\delta(\ell_i\cdot u_{a_i})}{(\pm\ell_i\cdot u_{\not a_i} - i0)^{n_i}}\prod_{k=1}^{9}\frac{1}{D_k^{\nu_k}}, \qquad (6)$$

restricted by Dirac-δ functions. Following [66,67], the $\ell_{i=1,2,3}$'s are the loop momenta, n_i, ν_k are integers, and $a_i\in\{1,2\}$, with $u_{\not 1} = u_2$, $u_{\not 2} = u_1$. In contrast to the conservative part, the D_k's are now various sets of retarded and advanced propagators, e.g., $\{(\ell^0 \pm i0)^2 - \boldsymbol{\ell}^2, \ldots\}$, consistent with causality. The same constraints as before apply on the external data [63], such that the relevant integrals can only depend on γ. As in our previous calculations [66,67], we conveniently introduce the parameter x, defined through the relation $\gamma \equiv (x^2+1)/2x$ [95], and compute these integrals by using dimensional regularization (in $d \equiv 4 - 2\epsilon$ dimensions) and the method of differential equations [32–38].

The integration problem then resembles the steps already performed for the computation in [66,67], except for a few notable differences. First of all, as before we use integration-by-parts (IBPs) relations [39–44] and reduce (6) to a basis of master integrals. Because of the fewer number of symmetries of the in-in integrand, the algebraic manipulations become a bit more involved than with Feynman-only propagators. But more importantly, the boundary conditions in the near-static limit $\gamma \simeq 1$ must be computed in terms of retarded and advanced Green's functions. For this purpose, we resort to the method of regions, expanding into potential and radiation modes. The potential-only part was obtained in [66], and recovered here from the full solution. For radiation modes, the same type of integrals appearing in PN derivations [109], combined with leftover integrals over potential-only modes at one and two loops, are sufficient to bootstrap the entire answer. See [61] for more details.

Total impulse.—Inputting the values of the boundary master integrals and translating from x to γ space, we find

$$
\begin{aligned}
\frac{c_{1b}^{(4)\,\mathrm{tot}}}{\pi} = & -\frac{3h_1 m_1 m_2 (m_1^3+m_2^3)}{64(\gamma^2-1)^{5/2}} + m_1^2 m_2^2 (m_1+m_2)\Bigg[\frac{21 h_2 \mathrm{E}^2(\frac{\gamma-1}{\gamma+1})}{32(\gamma-1)\sqrt{\gamma^2-1}} + \frac{3h_3 \mathrm{K}^2(\frac{\gamma-1}{\gamma+1})}{16(\gamma^2-1)^{3/2}} - \frac{3h_4 \mathrm{E}(\frac{\gamma-1}{\gamma+1})\mathrm{K}(\frac{\gamma-1}{\gamma+1})}{16(\gamma^2-1)^{3/2}} + \frac{\pi^2 h_5}{8\sqrt{\gamma^2-1}} \\
& + \frac{h_6 \log(\frac{\gamma-1}{2})}{16(\gamma^2-1)^{3/2}} + \frac{3h_7 \mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{(\gamma-1)(\gamma+1)^2} - \frac{3h_7 \mathrm{Li}_2(\frac{\gamma-1}{\gamma+1})}{4(\gamma-1)(\gamma+1)^2}\Bigg] + m_1^3 m_2^2 \Bigg[\frac{h_8}{48(\gamma^2-1)^3} + \frac{\sqrt{\gamma^2-1}\,h_9}{768(\gamma-1)^3\gamma^9(\gamma+1)^4} + \frac{h_{10}\log(\frac{\gamma+1}{2})}{8(\gamma^2-1)^2} \\
& - \frac{h_{11}\log(\frac{\gamma+1}{2})}{32(\gamma^2-1)^{5/2}} + \frac{h_{12}\log(\gamma)}{16(\gamma^2-1)^{5/2}} - \frac{h_{13}\operatorname{arccosh}(\gamma)}{8(\gamma-1)(\gamma+1)^4} + \frac{h_{14}\operatorname{arccosh}(\gamma)}{16(\gamma^2-1)^{7/2}} - \frac{3h_{15}\log(\frac{\gamma+1}{2})\log(\frac{\gamma-1}{\gamma+1})}{8\sqrt{\gamma^2-1}} + \frac{3h_{16}\operatorname{arccosh}(\gamma)\log(\frac{\gamma-1}{\gamma+1})}{16(\gamma^2-1)^2} \\
& - \frac{3h_{17}\mathrm{Li}_2(\frac{\gamma-1}{\gamma+1})}{64\sqrt{\gamma^2-1}} - \frac{3}{32}\sqrt{\gamma^2-1}\,h_{18}\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right)\Bigg] + m_1^2 m_2^3 \Bigg[\frac{3h_{15}\log(\frac{2}{\gamma-1})\log(\frac{\gamma+1}{2})}{8\sqrt{\gamma^2-1}} + \frac{3h_{16}\log(\frac{\gamma-1}{2})\operatorname{arccosh}(\gamma)}{16(\gamma^2-1)^2} + \frac{h_{19}}{48(\gamma^2-1)^3} \\
& + \frac{h_{20}}{192\gamma^7(\gamma^2-1)^{5/2}} + \frac{h_{21}\log(\frac{\gamma+1}{2})}{8(\gamma^2-1)^2} + \frac{h_{22}\log(\frac{\gamma+1}{2})}{16(\gamma^2-1)^{3/2}} + \frac{h_{23}\log(\gamma)}{2(\gamma^2-1)^{3/2}} - \frac{h_{24}\operatorname{arccosh}(\gamma)}{16(\gamma^2-1)^3} + \frac{h_{25}\operatorname{arccosh}(\gamma)}{16(\gamma^2-1)^{7/2}} - \frac{3h_{26}\operatorname{arccosh}^2(\gamma)}{32(\gamma^2-1)^{7/2}} \\
& + \frac{3h_{27}\log^2(\frac{\gamma+1}{2})}{2\sqrt{\gamma^2-1}} + \frac{3h_{28}\log(\frac{\gamma+1}{2})\operatorname{arccosh}(\gamma)}{16(\gamma^2-1)^2} + \frac{h_{29}\mathrm{Li}_2(\frac{1-\gamma}{\gamma+1})}{4\sqrt{\gamma^2-1}} + \frac{3h_{30}\mathrm{Li}_2(\frac{\gamma-1}{\gamma+1})}{8\sqrt{\gamma^2-1}}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
c_{1\check{u}_1}^{(4)\,\mathrm{tot}} = & \frac{9\pi^2 h_{31} m_1 m_2^2 (m_1+m_2)^2}{32(\gamma^2-1)} + \frac{2h_{32} m_1 m_2^2 (m_1^2+m_2^2)}{(\gamma^2-1)^3} + m_1^2 m_2^3\Bigg[\frac{4h_{33}}{3(\gamma^2-1)^3} - \frac{8h_{34}}{3(\gamma^2-1)^{5/2}} \\
& + \frac{8h_{35}\operatorname{arccosh}(\gamma)}{(\gamma^2-1)^3} - \frac{16h_{36}\operatorname{arccosh}(\gamma)}{(\gamma^2-1)^{3/2}}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
c_{1\check{u}_2}^{(4)\,\mathrm{tot}} = & -m_1^4 m_2\left(\frac{9\pi^2 h_{31}}{32(\gamma^2-1)} + \frac{2h_{32}}{(\gamma^2-1)^3}\right) + m_1^3 m_2^2\Bigg[-\frac{4h_{37}}{3(\gamma^2-1)^3} + \frac{h_{38}}{705600\gamma^8(\gamma^2-1)^{5/2}} + \frac{\pi^2 h_{39}}{192(\gamma^2-1)^2} + \frac{h_{40}\operatorname{arccosh}(\gamma)}{6720\gamma^9(\gamma^2-1)^3} \\
& + \frac{32h_{41}\operatorname{arccosh}(\gamma)}{3(\gamma^2-1)^{3/2}} - \frac{8h_{42}\operatorname{arccosh}^2(\gamma)}{(\gamma^2-1)^2} + \frac{32h_{43}\operatorname{arccosh}^2(\gamma)}{(\gamma^2-1)^{7/2}} + \frac{h_{44}\log(2)\operatorname{arccosh}(\gamma)}{8(\gamma^2-1)^2} + \frac{3h_{45}\left[\mathrm{Li}_2(\frac{\gamma-1}{\gamma+1}) - 4\mathrm{Li}_2(\sqrt{\frac{\gamma-1}{\gamma+1}})\right]}{16(\gamma^2-1)^2} \\
& + \frac{3h_{46}\left[\log(\frac{\gamma+1}{2})\operatorname{arccosh}(\gamma) - 2\mathrm{Li}_2(\sqrt{\gamma^2-1}-\gamma)\right]}{8(\gamma^2-1)^2} - \frac{h_{47}\{\mathrm{Li}_2\left[-(\gamma-\sqrt{\gamma^2-1})^2\right] - 2\log(\gamma)\operatorname{arccosh}(\gamma)\}}{16(\gamma^2-1)^2}\Bigg] \\
& + m_1^2 m_2^3\Bigg[-\frac{2h_{48}}{45(\gamma^2-1)^3} + \frac{h_{49}}{1440\gamma^7(\gamma^2-1)^{5/2}} + \frac{\pi^2 h_{50}}{48(\gamma^2-1)^2} + \frac{h_{51}\operatorname{arccosh}(\gamma)}{480\gamma^8(\gamma^2-1)^3} - \frac{16h_{52}\operatorname{arccosh}(\gamma)}{5(\gamma^2-1)^{3/2}} - \frac{16h_{53}\operatorname{arccosh}^2(\gamma)}{(\gamma^2-1)^2}
\end{aligned}
$$

$$-\frac{32h_{54}\text{arccosh}^2(\gamma)}{(\gamma^2-1)^{7/2}}-\frac{h_{55}\log(2)\text{arccosh}(\gamma)}{4(\gamma^2-1)^2}+\frac{h_{56}\left[\text{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)-4\text{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)\right]}{32(\gamma^2-1)^2}$$
$$+\frac{h_{57}\left[\log\left(\frac{2}{\gamma+1}\right)\text{arccosh}(\gamma)+2\text{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)\right]}{4(\gamma^2-1)^2}+\frac{h_{58}\left\{\text{Li}_2\left[-(\gamma-\sqrt{\gamma^2-1})^2\right]-2\log(\gamma)\text{arccosh}(\gamma)\right\}}{8(\gamma^2-1)^2}\Bigg]. \quad (7)$$

See Table I for the list of $h_i(\gamma)$ polynomials.

The impulse for the second particle follows by exchanging $1 \leftrightarrow 2$ in the masses, incoming velocities, and impact parameter. As expected from the calculations in [66,67], the complete results feature dilogarithms $[\text{Li}_2(z)]$, and complete elliptic integrals of the first $[\text{K}(z)]$ and second $[\text{E}(z)]$ kind.

Conservative: As shown explicitly in [76], a (time-symmetric) conservative contribution can be identified by rewriting retarded Green's functions in terms of Feynman propagators plus a reactive term, and keeping the real part of the Feynman-only piece, i.e., $\Delta p^\mu_{\text{cons}} \equiv \mathbb{R}\Delta p^\mu_{\text{F}}$, with imaginary terms cancelling out against counterparts from the reactive terms. Performing these steps in the full in-in integrand, and associated boundary conditions entering in the total impulse, we readily recover the conservative results in [66,67] including potential and radiation-reaction tail effects.

Dissapative: As discussed in [76], the terms stemming off of the mismatch between Feynman and retarded propagators incorporate dissipative effects. Needless to say, these terms can also be read off directly from the total result by subtracting the conservative part. Following the analysis in [66,67], we disentangle the various pieces according to factors of $v_\infty^{-2\epsilon}$, with $v_\infty \equiv \sqrt{\gamma^2-1}$, which signal the presence of an on-shell mode.

Starting with a single radiation mode we encounter *instantaneous* dissipative effects at linear order in the radiation-reaction. The latter are odd under time reversal and contribute to the $\hat{b}$ and $\check{u}_a$ directions. We find agreement in the overlap with known partial results in linear-response theory in the PN literature [26,27,119–123], as well as with the (odd) contribution to the $c^{(4)}_{1b}$ coefficient derived in [27,98]. All of the remaining radiative terms involve two radiation modes. After removing the Feynman-only (radiative) conservative pieces [66,67], the leftovers contain hereditary as well as nonlinear radiation-reaction dissipative effects. The former enters both in the longitudinal and perpendicular directions, whereas the latter contributes only to the total radiated perpendicular momentum at this order [125]. We also find perfect consistency with known nonlinear and hereditary results in the PN expansion [120–124].

See the Supplemental Material [126] and ancillary file for explicit expressions.

Radiated energy and momentum.—From the impulse we derive the change in the mechanical momentum of the system (in the incoming center of mass), which gives us the total radiated momentum, $P^\mu_{\text{rad}} = -(\Delta p^\mu_1 + \Delta p^\mu_2)$. The radiated energy for hyperboliclike motion at 4PM, given by $\Delta E_{\text{hyp}} \equiv P_{\text{rad}} \cdot [(m_1 u_1 + m_2 u_2)/M\Gamma]$ with $\Gamma \equiv (E/M)$ (M, E the total mass and energy and $\nu \equiv (m_1 m_2/M^2)$), then becomes

$$\Delta E^{4\text{PM}}_{\text{hyp}} = -\frac{G^4M^5\nu^2}{b^4\Gamma}\left\{\frac{15\pi^2(\gamma^2-1)(27(\gamma^2-1)h_{31}+2h_{50})+64(45h_{32}-h_{48})}{1440(\gamma^2-1)^3}+\frac{h_{49}}{1440\gamma^7(\gamma^2-1)^{5/2}}\right.$$
$$-\text{arccosh}^2(\gamma)\left(\frac{16h_{53}}{(\gamma^2-1)^2}+\frac{32h_{54}}{(\gamma^2-1)^{7/2}}\right)-\frac{h_{55}\log(2)\text{arccosh}(\gamma)}{4(\gamma^2-1)^2}+\frac{h_{57}\log(\frac{2}{\gamma+1})\text{arccosh}(\gamma)}{4(\gamma^2-1)^2}-\frac{h_{58}\log(\gamma)\text{arccosh}(\gamma)}{4(\gamma^2-1)^2}$$
$$+\text{arccosh}(\gamma)\left(\frac{h_{51}}{480\gamma^8(\gamma^2-1)^3}-\frac{16h_{52}}{5(\gamma^2-1)^{3/2}}\right)-\frac{h_{56}\text{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{8(\gamma^2-1)^2}+\frac{h_{56}\text{Li}_2(\frac{\gamma-1}{\gamma+1})}{32(\gamma^2-1)^2}+\frac{h_{57}\text{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)}{2(\gamma^2-1)^2}$$
$$+\frac{h_{58}\text{Li}_2[-(\gamma-\sqrt{\gamma^2-1})^2]}{8(\gamma^2-1)^2}+\nu\left[\frac{4(-45h_{32}+30h_{33}-30h_{37}+h_{48})}{45(\gamma^2-1)^3}+\frac{\pi^2[54(\gamma^2-1)h_{31}+h_{39}-4h_{50}]}{96(\gamma^2-1)^2}\right.$$
$$-\text{arccosh}^2(\gamma)\left(\frac{16(h_{42}-2h_{53})}{(\gamma^2-1)^2}-\frac{64(h_{43}+h_{54})}{(\gamma^2-1)^{7/2}}\right)+\frac{h_{38}-490\gamma(3840\gamma^7h_{34}+h_{49})}{352800\gamma^8(\gamma^2-1)^{5/2}}$$

TABLE I. Polynomials entering the total impulse as well as conservative and dissipative parts (see the Supplemental Material).

$h_1 = 515\gamma^6 - 1017\gamma^4 + 377\gamma^2 - 3$

$h_2 = 380\gamma^2 + 169$

$h_3 = 1200\gamma^2 + 2095\gamma + 834$

$h_4 = 1200\gamma^3 + 2660\gamma^2 + 2929\gamma + 1183$

$h_5 = -25\gamma^6 + 30\gamma^4 + 60\gamma^3 - 129\gamma^2 + 76\gamma - 12$

$h_6 = 210\gamma^6 - 552\gamma^5 + 339\gamma^4 - 912\gamma^3 + 3148\gamma^2 - 3336\gamma + 1151$

$h_7 = -\gamma(2\gamma^2 - 3)(15\gamma^2 - 15\gamma + 4)$

$h_8 = 420\gamma^9 + 3456\gamma^8 - 1338\gamma^7 - 15822\gamma^6 + 13176\gamma^5 + 9563\gamma^4 - 16658\gamma^3 + 8700\gamma^2 - 496\gamma - 1049$

$h_9 = -22680\gamma^{21} + 11340\gamma^{20} + 116100\gamma^{19} - 34080\gamma^{18} - 216185\gamma^{17} + 74431\gamma^{16} + 232751\gamma^{15} - 304761\gamma^{14} + 333545\gamma^{13} - 32675\gamma^{12} - 500785\gamma^{11} + 535259\gamma^{10} - 181493\gamma^9 + 3259\gamma^8 + 9593\gamma^7 + 9593\gamma^6 - 3457\gamma^5 - 3457\gamma^4 + 885\gamma^3 + 885\gamma^2 - 210\gamma - 210$

$h_{10} = -280\gamma^7 + 50\gamma^6 + 970\gamma^5 + 27\gamma^4 - 1432\gamma^3 + 444\gamma^2 + 366\gamma - 129$

$h_{11} = 2835\gamma^{11} - 10065\gamma^9 - 700\gamma^8 + 13198\gamma^7 + 1818\gamma^6 - 9826\gamma^5 + 5242\gamma^4 + 11391\gamma^3 + 18958\gamma^2 + 10643\gamma + 2074$

$h_{12} = \gamma(945\gamma^{10} - 2955\gamma^8 + 4874\gamma^6 - 5014\gamma^4 + 8077\gamma^2 + 5369)$

$h_{13} = \gamma(280\gamma^7 + 580\gamma^6 + 90\gamma^5 - 856\gamma^4 - 2211\gamma^3 + 1289\gamma^2 + 2169\gamma - 1965)$

$h_{14} = \gamma(2\gamma^2 - 3)(280\gamma^7 - 890\gamma^6 - 610\gamma^5 + 1537\gamma^4 + 380\gamma^3 - 716\gamma^2 - 82\gamma + 85)$

$h_{15} = 35\gamma^4 + 60\gamma^3 - 150\gamma^2 + 76\gamma - 5$

$h_{16} = \gamma(2\gamma^2 - 3)(35\gamma^4 - 30\gamma^2 + 11)$

$h_{17} = 315\gamma^8 - 860\gamma^6 + 690\gamma^4 - 960\gamma^3 + 1732\gamma^2 - 1216\gamma + 299$

$h_{18} = 315\gamma^6 - 145\gamma^4 + 65\gamma^2 + 21$

$h_{19} = 840\gamma^9 + 1932\gamma^8 + 234\gamma^7 - 17562\gamma^6 + 20405\gamma^5 - 2154\gamma^4 - 11744\gamma^3 + 12882\gamma^2 - 4983\gamma + 102$

$h_{20} = 3600\gamma^{16} + 4320\gamma^{15} - 23840\gamma^{14} + 7824\gamma^{13} + 14128\gamma^{12} + 16138\gamma^{11} - 9872\gamma^{10} - 47540\gamma^9 + 63848\gamma^8 - 37478\gamma^7 + 13349\gamma^6 - 1471\gamma^4 + 207\gamma^2 - 45$

$h_{21} = -350\gamma^7 + 1425\gamma^5 - 400\gamma^4 - 1480\gamma^3 + 660\gamma^2 + 285\gamma - 124$

$h_{22} = -300\gamma^7 + 210\gamma^6 + 1112\gamma^5 + 2787\gamma^4 + 2044\gamma^3 + 3692\gamma^2 + 6744\gamma + 1759$

$h_{23} = \gamma(75\gamma^6 - 140\gamma^4 - 283\gamma^2 - 852)$

$h_{24} = \gamma(2\gamma^2 - 3)(210\gamma^6 - 720\gamma^5 + 339\gamma^4 - 576\gamma^3 + 3148\gamma^2 - 3504\gamma + 1151)$

$h_{25} = \gamma(2\gamma^2 - 3)(350\gamma^7 - 960\gamma^6 - 705\gamma^5 + 1632\gamma^4 + 432\gamma^3 - 768\gamma^2 - 93\gamma + 96)$

$h_{26} = \gamma^2(3 - 2\gamma^2)^2(35\gamma^4 - 30\gamma^2 + 11)$

$h_{27} = 15\gamma^3 + 60\gamma^2 + 19\gamma + 8$

$h_{28} = \gamma(70\gamma^6 - 645\gamma^4 + 768\gamma^2 + 63)$

$h_{29} = -75\gamma^6 + 90\gamma^4 + 333\gamma^2 + 60$

$h_{30} = 25\gamma^6 - 30\gamma^4 + 60\gamma^3 + 129\gamma^2 + 76\gamma + 12$

$h_{31} = (1 - 5\gamma^2)^2$

$h_{32} = 80\gamma^8 - 192\gamma^6 + 152\gamma^4 - 44\gamma^2 + 3$

$h_{33} = \gamma(2\gamma^2 - 1)(64\gamma^6 - 216\gamma^4 + 258\gamma^2 - 109)$

$h_{34} = (2\gamma^2 - 1)^3(5\gamma^2 - 8)$

$h_{35} = \gamma(2\gamma^2 - 3)(2\gamma^2 - 1)^3$

$h_{36} = 8\gamma^6 - 28\gamma^4 + 6\gamma^2 + 3$

$h_{37} = \gamma(384\gamma^8 - 1528\gamma^6 + 384\gamma^4 + 2292\gamma^2 - 1535)$

$h_{38} = 393897472\gamma^{16} - 791542442\gamma^{14} - 3429240286\gamma^{12} + 3966858415\gamma^{10} + 767410066\gamma^8 - 21241500\gamma^6 + 7188300\gamma^4 - 1837500\gamma^2 + 385875$

$h_{39} = 1575\gamma^7 - 2700\gamma^6 - 3195\gamma^5 + 3780\gamma^4 + 4993\gamma^3 - 1188\gamma^2 - 1485\gamma + 108$

$h_{40} = -3592192\gamma^{18} + 2662204\gamma^{16} + 46406238\gamma^{14} - 37185456\gamma^{12} - 25426269\gamma^{10} + 222810\gamma^8 - 246540\gamma^6 + 79800\gamma^4 - 19950\gamma^2 + 3675$

$h_{41} = 44\gamma^6 - 32\gamma^4 - 425\gamma^2 - 82$

$h_{42} = \gamma(16\gamma^6 + 24\gamma^4 - 226\gamma^2 - 151)$

$h_{43} = \gamma^2(4\gamma^8 - 59\gamma^4 + 35\gamma^2 + 60)$

$h_{44} = -525\gamma^7 + 1065\gamma^5 - 3883\gamma^3 + 1263\gamma$

$h_{45} = 175\gamma^7 - 150\gamma^6 - 355\gamma^5 + 210\gamma^4 + 185\gamma^3 - 66\gamma^2 - 37\gamma + 6$

$h_{46} = -175\gamma^7 + 355\gamma^5 - 185\gamma^3 + 37\gamma$

$h_{47} = \gamma(525\gamma^6 - 1065\gamma^4 - 2773\gamma^2 + 1041)$

$h_{48} = 96\gamma^{10} - 8464\gamma^8 + 54616\gamma^6 - 70104\gamma^4 + 9916\gamma^2 + 13895$

$h_{49} = 6144\gamma^{16} - 587336\gamma^{14} + 4034092\gamma^{12} - 417302\gamma^{10} - 5560073\gamma^8 - 142640\gamma^6 + 35710\gamma^4 - 8250\gamma^2 + 1575$

$h_{50} = -3747\gamma^6 + 3249\gamma^4 + 8535\gamma^2 + 1051$

$h_{51} = 24576\gamma^{18} + 213480\gamma^{16} - 1029342\gamma^{14} - 1978290\gamma^{12} + 3752006\gamma^{10} + 816595\gamma^8 - 55260\gamma^6 + 13690\gamma^4 - 3100\gamma^2 + 525$

$h_{52} = \gamma(16\gamma^6 + 204\gamma^4 - 496\gamma^2 - 869)$

$h_{53} = \gamma^2(8\gamma^4 - 6\gamma^2 - 9)$

$h_{54} = \gamma(2\gamma^2 - 3)(8\gamma^6 - 6\gamma^4 - 51\gamma^2 - 8)$

$h_{55} = -4321\gamma^6 + 3387\gamma^4 + 15261\gamma^2 + 2057$

$h_{56} = 2100\gamma^7 - 4996\gamma^6 + 1755\gamma^5 + 4332\gamma^4 - 6422\gamma^3 + 4212\gamma^2 - 1209\gamma + 36$

$h_{57} = -1249\gamma^6 + 1083\gamma^4 + 1053\gamma^2 + 9$

$h_{58} = -1823\gamma^6 + 1221\gamma^4 + 13155\gamma^2 + 2039$

$h_{59} = -24\gamma^6 + 18\gamma^4 + 111\gamma^2 + 16$

$h_{60} = \gamma(26\gamma^2 - 9)$

$$+\frac{(3h_{46}+2h_{57})\log(\frac{\gamma+1}{2})\mathrm{arccosh}(\gamma)}{4(\gamma^2-1)^2}+\frac{(h_{44}+2h_{55})\log(2)\mathrm{arccosh}(\gamma)}{4(\gamma^2-1)^2}+\frac{(h_{47}+2h_{58})\log(\gamma)\mathrm{arccosh}(\gamma)}{4(\gamma^2-1)^2}$$
$$+\mathrm{arccosh}(\gamma)\left(\frac{53760\gamma^9 h_{35}-14\gamma h_{51}+h_{40}}{3360\gamma^9(\gamma^2-1)^3}-\frac{32(15h_{36}-10h_{41}-3h_{52})}{15(\gamma^2-1)^{3/2}}\right)+\frac{(h_{56}-6h_{45})\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{4(\gamma^2-1)^2}$$
$$-\frac{(h_{56}-6h_{45})\mathrm{Li}_2(\frac{\gamma-1}{\gamma+1})}{16(\gamma^2-1)^2}-\frac{(3h_{46}+2h_{57})\mathrm{Li}_2(\sqrt{\gamma^2-1}-\gamma)}{2(\gamma^2-1)^2}-\frac{(h_{47}+2h_{58})\mathrm{Li}_2[-(\gamma-\sqrt{\gamma^2-1})^2]}{8(\gamma^2-1)^2}\Bigg]\Bigg\}. \quad (8)$$

Expanding in small velocities ($v_\infty^2=\gamma^2-1\ll 1$) we find,

$$\frac{b^4\Delta E^{4\mathrm{PM}}_{\mathrm{hyp}}}{G^4M^5\nu^2}=\frac{1568}{45v_\infty}+\left(\frac{18\,608}{525}-\frac{1136\nu}{45}\right)v_\infty+\frac{3136v_\infty^2}{45}+\left(\frac{764\nu^2}{45}-\frac{356\nu}{63}+\frac{220\,348}{11\,025}\right)v_\infty^3+\left(\frac{1216}{105}-\frac{2272\nu}{45}\right)v_\infty^4$$
$$+\left(-\frac{622\nu^3}{45}+\frac{3028\nu^2}{1575}-\frac{199\,538\nu}{33\,075}-\frac{151\,854}{13\,475}\right)v_\infty^5+\left(\frac{1528\nu^2}{45}-\frac{8056\nu}{1575}+\frac{117\,248}{1575}\right)v_\infty^6+\mathcal{O}(v_\infty^7). \quad (9)$$

A notable feature of the 4PM momentum is the emergence of a total recoil along the perpendicular direction. The full result is rather lengthy, however, performing a velocity expansion we find (with $\Delta_m\equiv(m_1-m_2)/M$)

$$\frac{b^4P^{4\mathrm{PM}}_{b,\mathrm{rad}}}{\pi\Delta_m G^4M^5\nu^2}=\frac{37}{30}+\frac{1661v_\infty^2}{560}+\frac{1491v_\infty^3}{400}+\frac{23\,563v_\infty^4}{10\,080}-\frac{26\,757v_\infty^5}{5600}+\frac{700\,793v_\infty^6}{506\,880}+\mathcal{O}(v_\infty^7). \quad (10)$$

Both expressions in (9) and (10) are consistent with the state of the art in the PN expansion [119–124]. On the other hand, in the opposite limit, as γ gets large,

$$\frac{b^4\Gamma\Delta E^{4\mathrm{PM}}_{\mathrm{hyp}}}{G^4M^5\nu^2}\to\frac{13696}{105}\gamma^3\nu\log(2\gamma), \quad (11)$$

which signals the presence of (logarithmic) mass singularities. We return to these limits in the conclusions.

GW energy flux.—The B2B map allows us to connect the radiated energy for the scattering process to its counterpart over a period of an ellipticlike orbit via $\Delta E_{\mathrm{ell}}(j)=\Delta E_{\mathrm{hyp}}(j)-\Delta E_{\mathrm{hyp}}(-j)$ [30], where $j\equiv(p_\infty b/GM^2\nu)$, with $p_\infty\equiv(M\nu/\Gamma)v_\infty$, is the (reduced) angular momentum. However, similarly to the periastron advance at $\mathcal{O}(G^3)$ [28,29], this expression vanishes at 4PM. To obtain radiative observables for generic orbits we derived instead the energy flux. Since nonlinear radiation-reaction terms do not contribute to the energy loss at this order we can resort to an adiabatic expansion. By writing the PM-expanded energy flux in an isotropic gauge as [30]

$$\frac{dE}{dt}=\frac{M}{r}\sum_n\mathcal{F}_E^{(n)}(\gamma)\left(\frac{GM}{r}\right)^{(n+3)}, \quad (12)$$

we find at 4PM order (see [30] for the 3PM term)

$$M\pi\xi\mathcal{F}_E^{(1)}=\frac{3\pi\Gamma^2\nu}{4(\gamma^2-1)^{3/2}}\Delta E^{(1)}_{j\mathrm{hyp}}-\frac{2\nu^3\Delta E^{(0)}_{j\mathrm{hyp}}}{(\gamma^2-1)^2\Gamma^6\xi^2}$$
$$\times[(\gamma-1)^3(10\gamma^3-10\gamma^2-9\gamma+5)\nu^2$$
$$+4(5\gamma^5-8\gamma^4+\gamma^3+4\gamma^2-3\gamma+1)\nu$$
$$+8\gamma^4-4\gamma^2-1], \quad (13)$$

with $\Delta E(j)=\sum_{n=0}^\infty(\Delta E_j^{(n)}/j^{n+3})$, $\xi\equiv(E_1E_2/E^2)$, $E_a\equiv\sqrt{p_\infty^2+m_a^2}$.

Conclusions.—We completed the knowledge of the total relativistic impulse in the scattering of nonspinning bodies at 4PM order, including linear, nonlinear, and hereditary radiation-reaction effects. We also derived the total radiated spacetime momentum at $\mathcal{O}(G^4)$ and extracted the GW energy flux, which can then be used to compute observables for generic (un)bound orbits incorporating an infinite series of velocity corrections. The most intricate part of the calculation involves a series of master integrals with retarded propagators, which we are able to compute to all orders in the velocity through the methodology of differential equations, without resorting to PN resummations. The boundary conditions in the near-static limit are obtained via the method of regions, thus making direct contact with derivations in the PN regime with potential and radiation modes. We find perfect agreement with various partial calculations in the literature. Explicit values can be found in the Supplemental Material and ancillary file.

There are, however, some key aspects of the structure of the impulse at 4PM order which deserve further study. First, concerning the high-energy limit, although nontrivial cancellations occur we find that the $\Delta^{(4)} p_1$ impulse (in particular the $c_{1b}^{(4)}$ component) does not transition smoothly, but rather it diverges when $m_1 \to 0$ while $\gamma m_1 m_2$ is held fixed. Since all the integrals in (6) vanish in dimensional regularization when the velocities obey the null condition ($u_a^2 = 0$), the divergent terms must arise due to the enforcement of timelike worldlines ($u_a^2 = 1$) in the massive theory from the onset. Moreover, similarly to what happens at $\mathcal{O}(G^3)$ [97], the total radiated energy at $\mathcal{O}(G^4)$ also diverges in the $\gamma \gg 1$ limit, this time featuring a factor of factor of $\log(\gamma)$ with respect to the 3PM case, see (11). We expect this behavior to be tamed in the nonperturbative solution (see, e.g., [127,128] and references therein).

Second, there is the issue of the mass scaling of the impulse [27], and violations thereof, e.g., [117,118]. As we mentioned, some of the radiative contributions affect only the total radiated momentum in the $\hat{b}$ direction, while conserving energy. Moreover, they are even under time reversal, see (10). To gain intuition about these terms, from the impulse and total recoil, we derived the relative deflection angle (see the Supplemental Material). After expanding in small velocities, we find perfect consistency with the (odd-in-velocity) PN values in [27]. Yet, starting at 5PN order, we encounter *conservativelike* (even-in-velocity) contributions at $\mathcal{O}(\nu^2)$, beyond the Feynman-only part [129]. In principle, depending on their origin, these terms could be incorporated into a *relative* Hamiltonian. We will discuss these issues in more detail elsewhere.

In summary, in addition to the natural connections to GW science, e.g., [99,100], the solution of the (classical) relativistic scattering problem at $\mathcal{O}(G^4)$ presented here demonstrates how the worldline EFT approach [15,18] combined with the methodology of differential equations and integration by regions—already successfully implemented both in the conservative [62–67] and dissipative [72–76] sectors—are very powerful tools to tackle the entire two-body dynamics in general relativity within the PM expansion. Complete results at higher PM orders, including spin and tidal effects, are underway.

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