# Estimating Quantum Hamiltonians via Joint Measurements of Noisy Noncommuting Observables

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Estimation of expectation values of incompatible observables is an essential practical task in quantum computing, especially for approximating energies of chemical and other many-body quantum systems. In this Letter, we introduce a method for this purpose based on performing a single joint measurement that can be implemented locally and whose marginals yield noisy (unsharp) versions of the target set of noncommuting Pauli observables. We derive bounds on the number of experimental repetitions required to estimate energies up to a certain precision. We compare this strategy to the classical shadow formalism and show that our method yields the same performance as the locally biased classical shadow protocol. We also highlight some general connections between the two approaches by showing that classical shadows can be used to construct joint measurements and vice versa. Finally, we adapt the joint measurement strategy to minimise the sample complexity when the implementation of measurements is assumed noisy. This can provide significant efficiency improvements compared to known generalizations of classical shadows to noisy scenarios.

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*Introduction.*—Measurement incompatibility, one of the defining nonclassical features of quantum theory, limits an observer's ability to measure certain physical properties of a system simultaneously. While typically viewed as a quantum resource [1,2], essential in applications such as nonlocality [3], quantum steering [4,5], and state discrimination [6,7], incompatibility is a major issue for variational quantum algorithms [8–11], which constitute one of the leading candidates for attaining quantum speedups in near-term quantum computers. These algorithms require the estimation of expectation values of a quantum many-body Hamiltonian (encoding, for example, a molecular system relevant for quantum chemistry) on states occurring in the course of a classical-quantum optimization loop. The Hamiltonian of interest, which, in general, cannot be measured in its eigenbasis, is described by a linear combination of Pauli operators (tensor products of single-qubit Pauli matrices). Estimating the expectation values of all relevant Pauli operators, and subsequently the Hamiltonian, involves measuring large collections of incompatible observables.

To overcome this computational burden, many strategies have been introduced [12–19], with a typical method involving some grouping of the observables into compatible sets, e.g., [20–27]. Another approach is a technique called *classical shadows* [28], based on a practical implementation of ideas from shadow tomography [29]. This framework involves a randomized measurement strategy, implemented with random unitary circuits, that builds a classical approximation of the unknown state to efficiently estimate linear and nonlinear functions of the state [28]. The protocol readily applies to the problem of estimating multiple expectation values of incompatible observables (as well as many other applications [30–35]) and leads to rigorous bounds on the sample complexity (i.e., the number of state preparations) to achieve accurate estimations.

In this Letter, we present a new approach, conceptually distinct from previous methods for estimating multiple expectation values, using ideas from the theory of measurement incompatibility [36–39]. While joint measurability and commutativity are equivalent notions for projective (von Neumann) measurements, general measurements, i.e., positive operator-valued measures (POVMs), do not necessarily require commutativity to be measured jointly. Importantly, a set of incompatible observables can be measured simultaneously *if* a sufficient amount of noise is added to the measurements. In our scheme, we use the randomization of local projective measurements to simultaneously perform noisy (unsharp) versions of Pauli measurements. This strategy is easily implementable on a quantum computer and can be extended to a locally biased $n$-qubit joint measurement which, after efficient classical postprocessing, reproduces the outcome statistics of any sufficiently noisy Pauli measurement (even though no physical noise resides in the system). The outcomes of the joint measurement are then used to efficiently construct unbiased estimators of the expectation values of the original noiseless observables, or some linear combination, such as a Hamiltonian (see Fig. 1).

 

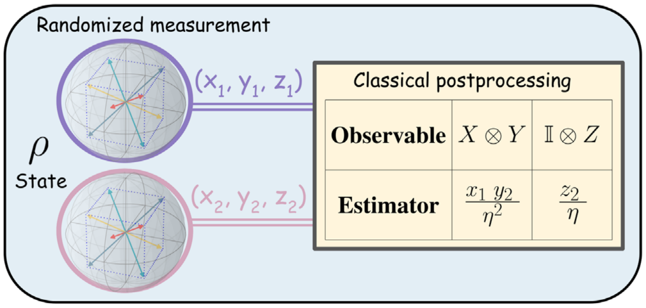


FIG. 1. Illustration of the protocol for simultaneously estimating expectation values of Pauli observables. A joint measurement of unsharp Pauli observables (with uniform noise $\eta$) of the form (2) is implemented on each qubit. The estimators of expectation values are constructed by taking a product of outcomes and dividing by the noise.

Our analysis offers a new and conceptually simpler perspective on understanding randomized protocols for estimating multiple noncommuting observables. We derive bounds on the sample complexity of the protocol, which we show to be identical to those derived for locally biased classical shadows [40], a generalized version of the original local classical shadow protocol [28]. We then formulate some basic connections between joint measurability and classical shadows, showing that joint measurements can be used to construct classical shadows and, conversely, the shadow protocol defines a joint measurement. Finally, we study the effects of physical noise on our protocol, adapting the joint measurement strategy to minimize the sample complexity when the implementation of physical measurements (used to simulate parent POVMs of unsharp Pauli observables) is affected by readout noise from the quantum computing device [41–43]. We find that this approach can provide significant performance improvements compared to known generalizations of classical shadows to noisy scenarios [44,45].

*Preliminaries.*—Let $\mathcal{H}$ be $d$-dimensional Hilbert space. A measurement is described by a POVM $\mathsf{M}$, with a finite outcome set $\Omega$, and consists of positive semidefinite matrices (effects) $\mathsf{M}(s) \geq 0$, for which $\sum_{s\in\Omega} \mathsf{M}(s) = \mathbb{1}$, where $\mathbb{1}$ is the identity of $\mathcal{H}$. For a quantum state $\rho$, the outcome probability distribution of $\mathsf{M}$ is given by $p(s|\rho) = \mathrm{tr}[\mathsf{M}(s)\rho]$. A finite collection of measurements $\mathsf{M}_1, \ldots, \mathsf{M}_m$ is *jointly measurable* (compatible) whenever their statistics can be reproduced by classical postprocessing of the statistics from a single POVM. In particular, there exists a measurement $\mathsf{G}$ with outcome set $\Omega_\mathsf{G}$ such that the effects $\mathsf{M}_j(s_j)$ can be obtained from $\mathsf{G}$ via a stochastic transformation:

$$\mathsf{M}_j(s_j) = \sum_{\lambda\in\Omega_\mathsf{G}} D(s_j|j,\lambda)\mathsf{G}(\lambda), \tag{1}$$

where $0 \leq D(s_j|j,\lambda) \leq 1$ and $\sum_{s_j} D(s_j|j,\lambda) = 1$, for all $j = 1, \ldots, m$. Equivalently, a set of observables is said to be jointly measurable if there exists a single POVM whose marginals yield all effects of the individual observables; otherwise, they are said to be incompatible [36,46]. A simple proof of the equivalence is provided in Appendix A in Supplemental Material [47] for completeness.

As an example, consider the qubit Pauli observables $X \coloneqq \sigma_x$, $Y \coloneqq \sigma_y$, and $Z \coloneqq \sigma_z$ and their noisy unsharp versions $\mathsf{M}_X^{\eta^x}(x) = \frac{1}{2}(\mathbb{1} + x\eta^x X)$, $\mathsf{M}_Y^{\eta^y}(y) = \frac{1}{2}(\mathbb{1} + y\eta^y Y)$, and $\mathsf{M}_Z^{\eta^z}(z) = \frac{1}{2}(\mathbb{1} + z\eta^z Z)$, with outcomes $x, y, z \in \{\pm 1\}$ and $0 \leq \eta^x, \eta^y, \eta^z \leq 1$. It is well known (cf. [39,48,49]) that the triple is jointly measurable if and only if $(\eta^x)^2 + (\eta^y)^2 + (\eta^z)^2 \leq 1$. The corresponding joint measurement, whose marginals are the unsharp Pauli observables, is

$$\mathsf{G}(x, y, z) = \frac{1}{8}(\mathbb{1} + x\eta^x X + y\eta^y Y + z\eta^z Z). \tag{2}$$

Given an arbitrary set of $m$ observables $O_1, \ldots, O_m$ and an unknown $n$-qubit quantum state $\rho$, our aim will be to provide estimators $\hat{O}_j$ to the expectation values $\mathrm{tr}[O_j\rho]$, up to a certain precision. In particular, we would like to know the number of copies of $\rho$ (i.e., the sample complexity $N$) such that, for all $j = 1, \ldots, m$, $|\mathrm{tr}[O_j\rho] - \hat{O}_j| < \epsilon$, with probability at least $1 - \delta$.

For simple single-shot estimators, such as those for local dichotomic observables, Hoeffding's inequality provides an effective way to bound the sample complexity of the estimation protocol. In other cases, such as a Hamiltonian, the median-of-means approach gives a simple classical postprocessing strategy which can reduce the effect of estimation errors [28]. This method (explained further in Appendix B [47]) depends on the variance of the estimator $\hat{O}_j$ and leads to the following bound on the sample complexity: $N = O([\log(m/\delta)/\epsilon^2]\max_{1\leq j\leq m}\mathrm{Var}[\hat{O}_j])$.

*Estimating expectations via joint measurements.*—The observables we wish to measure simultaneously are the set of $n$-qubit Pauli strings $P = \otimes_{i=1}^n P_i$ (with $\mathbb{P}_n$ denoting the set), where $P_i \in \{\mathbb{1}, X, Y, Z\}$. The joint-measurability strategy to estimate the expectation values $\mathrm{tr}[P\rho]$, for all $P \in \mathbb{P}_n$, can be described succinctly as follows. First, we perform a locally biased joint measurement to implement an observable of the form (2) on each qubit system. To obtain the outcome of the unsharp version of $P$, note that each local measurement provides an outcome tuple $(x_i, y_i, z_i)$; hence, we take the product of local outcomes $p_i$ (equal to either $x_i$, $y_i$, or $z_i$ corresponding to $P_i$). We obtain an unbiased estimator of $\mathrm{tr}[P\rho]$ by dividing $\prod_i p_i$ by the product of local noises (see also Fig. 1).

Formally, we define the *locally biased joint measurement* $\mathsf{F}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \coloneqq \otimes_{i=1}^n \mathsf{G}_i(\mathbf{x}_i)$, where each $\mathsf{G}_i$ [with outcomes $\mathbf{x}_i = (x_i, y_i, z_i)$] is of the form given in Eq. (2) and the noise parameters $\eta_i^x$, $\eta_i^y$, and $\eta_i^z$ are biased (and independent) for each qubit. The measurement $\mathsf{F}$ is then a joint measurement for the noisy Pauli measurements

$$\mathsf{M}_P^\eta(s_P) = \frac{1}{2}(\mathbb{1} + s_P\eta_P P), \tag{3}$$

with $s_P \in \{\pm 1\}$. The noise coefficient $\eta_P := \prod_{i\in\mathrm{supp}(P)} \eta_i^{\nu_i(P)}$ is a product of local noises dependent on the individual Pauli operators $P_i$, where $\nu_i(P) = x,\ y,\ z$ if $P_i = X,\ Y,\ Z$, respectively, and $\mathrm{supp}(P) = \{i | P_i \neq \mathbb{1}\}$. For example, if $P = X \otimes \mathbb{1} \otimes Z$, then $\eta_P = \eta_1^x \eta_3^z$. The classical postprocessing is defined as

$$D(s_P|P, \mathbf{x}) = \begin{cases} 1 & \text{if } s_P = \mu(P), \\ 0 & \text{if } s_P = -\mu(P), \end{cases} \tag{4}$$

where $\mu: P \mapsto \prod_{i\in\mathrm{supp}(P)} \mu_i(P)$ is the product of the relevant local outcomes, with $\mu_i(P) = x_i, y_i, z_i$ if $P_i = X,\ Y,\ Z$, respectively.

We estimate $\mathrm{tr}[P\rho]$ by sampling from the outputs $s_P$ of the corresponding unsharp measurement $\mathsf{M}_P^\eta$ [from Eq. (3)]. Importantly, for any input state $\rho$, the expectation value of $s_P$ equals $\eta_P \mathrm{tr}[P\rho]$, and, therefore, it is natural to set an unbiased estimator as $\hat{P} = (1/\eta_P)s_P$. The variance of $\hat{P}$ can be easily upper bounded by $\mathrm{Var}[\hat{P}] \leq \eta_P^{-2}$, which for the uniform case gives $\mathrm{Var}[\hat{P}] \leq 3^{w(P)}$, with $w(P) = |\mathrm{supp}(P)|$.

Importantly, the joint measurement of Eq. (2) can be implemented via classical randomization of qubit projective measurements and is, therefore, projective simulable [50]. For example, the uniform case can be simulated by a uniform mixture of four qubit projective measurements onto opposite vertices of a cube inscribed in the Bloch sphere (see Appendix C [47] for details) and is easily realized on a quantum computer.

*Joint measurements of Hamiltonians.*—We now apply the locally biased joint measurement $\mathsf{F}$ to estimate the expectation values $\mathrm{tr}[H\rho]$ of a Hamiltonian $H = \sum_{P\in\mathbb{P}_n} \lambda_P P$, written as a linear combination of Pauli strings, with $\lambda_P \in \mathbb{R}$. From the outcomes of our joint measurement on a $n$-qubit quantum state $\rho$, our single-shot estimator of $\mathrm{tr}[H\rho]$ is given by

$$\hat{H} = \sum_{P\in\mathbb{P}_n} \eta_P^{-1} \lambda_P s_P, \tag{5}$$

where the outcome $s_P \in \{\pm 1\}$ corresponds to the effect of the POVM defined in Eq. (3). Clearly, the estimator's expectation satisfies $\mathbb{E}[\hat{H}] = \sum_P \eta_P^{-1}\lambda_P \mathbb{E}[s_P] = \mathrm{tr}[H\rho]$, where the expectation value $\mathbb{E}$ is over the outcome statistics of the POVM $\mathsf{F}$ on state $\rho$.

To bound the sample complexity of estimating the expectation value of $H$ via our joint measurement, we obtain the following result.

*Proposition 1.*—The variance of the estimator $\hat{H}$ [defined in Eq. (5)] for the Hamiltonian $H$ is given by

$$\mathrm{Var}[\hat{H}] = \sum_{P,Q\in\mathbb{P}_n} \frac{\eta_{PQ} f(P,Q)}{\eta_P \eta_Q} \lambda_P \lambda_Q \mathrm{tr}[PQ\rho] - (\mathrm{tr}[H\rho])^2, \tag{6}$$

where $f(P,Q) = \prod_{i=1}^n f_i(P,Q)$ and

$$f_i(P,Q) = \begin{cases} 1 & \text{if } P_i = \mathbb{1} \text{ or } Q_i = \mathbb{1} \text{ or } P_i = Q_i, \\ 0 & \text{otherwise.} \end{cases}$$

The noise coefficient is defined as $\eta_{PQ} = \prod_{i\in\mathrm{supp}(PQ)} \eta_i^{\nu_i(PQ)}$, where $\nu_i(PQ)$ ignores the phase of $PQ$ and acts as defined earlier. The proof presented in Appendix D [47] involves calculating $\mathbb{E}[\hat{P}\hat{Q}] = (1/\eta_P\eta_Q)\mathbb{E}[s_P s_Q]$, which can be evaluated from the statistics of the joint measurement $\mathsf{M}_{P,Q}^\eta(s_P, s_Q) = \sum_{\mathbf{x}} D(s_P|P,\mathbf{x}) D(s_Q|Q,\mathbf{x}) \mathsf{F}(\mathbf{x})$. For example, if $P = X \otimes \mathbb{1}$ and $Q = Y \otimes Y$ such that $PQ = XY \otimes Y$, we have, in the uniform noise case, $\mathsf{M}_{P,Q}^\eta(s_P, s_Q) = \frac{1}{4}(\mathbb{1} + s_P\eta P + s_Q\eta^2 Q)$, and a simple calculation yields $\mathbb{E}[s_P s_Q] = 0$. On the other hand, if $P = X \otimes \mathbb{1}$ and $Q = X \otimes X$, then $\mathsf{M}_{P,Q}^\eta(s_P, s_Q) = \frac{1}{4}(\mathbb{1} + s_P\eta P + s_Q\eta^2 Q + s_P s_Q \eta PQ)$, and it follows that $\mathbb{E}[s_P s_Q] = \eta \mathrm{tr}[PQ\rho]$.

*Connections to classical shadows.*—A single round of a classical shadow protocol consists of three stages [28]. First, a quantum state is transformed via a unitary transformation $\rho \mapsto U\rho U^\dagger$, where $U$ is chosen randomly from an ensemble $\mathcal{U}$. This is followed by a measurement in the computational basis $\{|e\rangle : e \in \{0,1\}^n\}$. Finally, the unitary $U^\dagger$ is applied to the postmeasurement state $|\hat{e}\rangle$, i.e., $|\hat{e}\rangle\langle\hat{e}| \mapsto U^\dagger|\hat{e}\rangle\langle\hat{e}|U$. In expectation (over the unitary ensemble and measurement outcomes), this randomized measurement procedure can be described by a quantum channel $\mathcal{M}: \rho \mapsto \mathbb{E}_{U\sim\mathcal{U}} \sum_{e\in\{0,1\}^k} p(e) U^\dagger |e\rangle\langle e| U$, where $p(e) = \langle e|U\rho U^\dagger|e\rangle$. Assuming $\mathcal{M}$ is invertible, we construct a *classical shadow*, i.e., a set of estimators $\hat{\rho}^{(\ell)} = \mathcal{M}^{-1}(U^{\dagger,(\ell)}|\hat{e}^{(\ell)}\rangle\langle\hat{e}^{(\ell)}|U^{(\ell)})$ such that $\mathbb{E}[\hat{\rho}] = \mathcal{M}^{-1}[\mathcal{M}(\rho)] = \rho$, where $\ell$ labels different experimental realizations. The above formalism can be applied to predict many properties of the quantum state. For example, for any collection of observables $O_1, \ldots, O_m$, the function $\hat{O}_j^{sh} = \mathrm{tr}[O_j\hat{\rho}]$ is an unbiased estimator of $\mathrm{tr}[O_j\rho]$. The sample complexity can be found by bounding the variance of $\hat{O}_j^{sh}$ with the *shadow norm* [28], i.e., $\mathrm{Var}[\hat{O}_j^{sh}] \leq \|O_j\|_{\mathrm{sh}}^2$, as defined in Appendix E [47].

Two cases considered in Ref. [28] construct classical shadows via random local qubit or global $n$-qubit Clifford measurements and rely on the 3-design property of the Clifford group to compute the shadow norm. In the former case, the measurement procedure is equivalent to performing random Pauli measurements on each qubit, and the classical shadow has the form

$$\hat{\rho} = \bigotimes_{i=1}^{n} (3U_i^\dagger |\hat{e}_i\rangle\langle\hat{e}_i| U_i - \mathbb{1}). \tag{7}$$

For an arbitrary Pauli string $P \in \mathbb{P}_n$, the shadow norm is given by $\|P\|_{\text{sh}}^2 = 3^{w(P)}$.

A modified *locally biased* classical shadow approach for the product Clifford ensemble is described in Ref. [40] and is applied to estimate expectation values of Hamiltonians $H = \sum_P \lambda_P P$. Rather than implementing uniformly random Pauli measurements on each qubit, each Pauli $P_i \in \{X, Y, Z\}$ is randomly selected according to a probability distribution $\beta_i(P_i)$. Surprisingly, while the estimator $\hat{H}^{sh}$ of this protocol differs from $\hat{H}$ in Eq. (5), we observe the following relation between the two.

*Observation 1.*—The variance of the locally biased classical shadow estimator in Ref. [40] is equivalent to the variance of the joint-measurability estimator in Proposition 1.

This is shown explicitly in Appendix E [47] and requires $(\eta_i^x)^2$, $(\eta_i^y)^2$, and $(\eta_i^z)^2$ to be set as the probabilities $\beta_i(P_i)$ of sampling $X$, $Y$, and $Z$, respectively, for each qubit system. It follows that both approaches have the same sample complexity bounds. For a given Hamiltonian, strategies are provided in Refs. [40,51] to minimize the variance by optimizing over the probabilities $\beta_i(P_i)$. These techniques can also be applied directly to optimize the joint measurement.

We now highlight some further connections when joint measurability is viewed in the framework of classical shadows and vice versa.

*Observation 2.*—From the outcomes of the joint measurement $\mathsf{F}$, we can construct a locally biased classical shadow. In the unbiased setting, this has a similar form (and the same performance) as the shadow of Eq. (7).

For each Pauli string $P \in \mathbb{P}_n$, we construct the product $\mu(P) \coloneqq \prod_{i\in\text{supp}(P)} \mu_i(P)$ from the outcomes of $\mathsf{F}$. A single-shot classical approximation of the quantum state $\rho$ is given by

$$\hat{\rho}^{\text{JM}} = \frac{1}{2^n} \sum_{P\in\mathbb{P}_n} \eta_P^{-1} \mu(P) P = \bigotimes_{i=1}^{n} \frac{1}{2} (\mathbb{1} + \mathbf{e}_i \cdot \boldsymbol{\sigma}), \tag{8}$$

where $\mathbf{e}_i = [(x_i/\eta_i^x), (y_i/\eta_i^y), (z_i/\eta_i^z)]$ and $\|\mathbf{e}_i\|^2 = (\eta_i^x)^{-2} + (\eta_i^y)^{-2} + (\eta_i^z)^{-2}$. It follows (see Appendix F [47]) that, for the $i$th qubit, we have $\hat{\rho}_i^{\text{JM}} = \|\mathbf{e}_i\| \rho_{\tilde{\mathbf{e}}_i} + \frac{\mathbb{1}}{2}(1 - \|\mathbf{e}_i\|)$, where $\tilde{\mathbf{e}}_i = \mathbf{e}_i / \|\mathbf{e}_i\|$ and $\rho_{\tilde{\mathbf{e}}_i} = \frac{1}{2}(\mathbb{1} + \tilde{\mathbf{e}}_i \cdot \boldsymbol{\sigma})$. If we consider uniform noise such that $\|\mathbf{e}_i\| = (\sqrt{3}/\eta)$ and take $\eta = (1/\sqrt{3})$, the expression simplifies to $\hat{\rho}_i^{\text{JM}} = 3\rho_{\tilde{\mathbf{e}}_i} - \mathbb{1}$, which has a similar form to the classical shadow (7) but $\rho_{\tilde{\mathbf{e}}_i}$ is no longer a Pauli eigenstate. Note also that both shadows give the same variance for the estimators.

*Observation 3.*—Any classical shadow defines a joint measurement and provides a sufficient condition for the compatibility of an arbitrary set of measurements.

Suppose a classical shadow on $\mathcal{H}$ is constructed from a (global) unitary ensemble $\mathcal{U}$ which constitutes a 2-design. The randomized measurement procedure can be described by a single POVM: $\mathsf{G}(x, U) = (1/|\mathcal{U}|) U^\dagger |x\rangle\langle x| U$, where $U \in \mathcal{U}$ and $x \in \{0,1\}^n$. As a consequence of the 2-design property, each classical snapshot is given by $\hat{\rho}_{x,U} = (d+1) U^\dagger |x\rangle\langle x| U - \mathbb{1}$ [28]. While $\hat{\rho}_{x,U}$ is not necessarily positive semidefinite, it has $\text{tr}[\hat{\rho}_{x,U}] = 1$ and satisfies $\mathbb{E}[\hat{\rho}_{x,U}] = \rho$. For a POVM $\mathsf{M}_j$, we can compute $q(s|j, x, U) = \text{tr}[\mathsf{M}_j(s) \hat{\rho}_{x,U}]$, which, in expectation, yields the outcome statistics of the measurement $\mathsf{M}_j$.

To determine which observables can be measured jointly from classical shadows, we require that $q(s|j, x, U)$ is a classical postprocessing function, i.e., $\text{tr}[\mathsf{M}_j^\eta(s) \hat{\rho}_{x,U}] \geq 0$, where $\mathsf{M}_j^\eta(s) = \eta \mathsf{M}_j(s) + (1-\eta)\{\text{tr}[\mathsf{M}_j(s)]/d\}\mathbb{1}$. For the set of all observables, this holds if and only if $\eta \leq 1/(d+1)$ (see Appendix F [47]). Thus, we can simulate the measurements $\mathsf{M}_j^\eta$ for $\eta \leq 1/(d+1)$, from the classical postprocessing $q(s|j, x, U)$. Improved bounds can be achieved if $\text{tr}[\mathsf{M}_j(s) U^\dagger |x\rangle\langle x| U] > 0$ for all $j$. While the general condition does not give the exact joint-measurability region for an arbitrary collection of observables [52], the classical shadow $\hat{\rho}_i^{\text{JM}} = 3\rho_{\tilde{\mathbf{e}}_i} - \mathbb{1}$ constructed from the joint measurement of Pauli observables yields the precise incompatibility robustness threshold. Note that classical shadows can also be constructed from informationally complete measurements [53], for which the above analysis would also apply.

*Noisy joint measurements.*—The unbiased classical shadow protocol has been adapted to incorporate noise by applying a fixed quantum channel $\Lambda$ to the state after a unitary transformation $U \in \mathcal{U}$, i.e., $U\rho U^\dagger \mapsto \Lambda(U\rho U^\dagger)$, followed by the usual measurement in the computational basis [44,45]. When the unitary ensemble $\mathcal{U}$ is the product Clifford ensemble and $\Lambda$ describes uncorrelated product noise, the corresponding shadow norm can be calculated explicitly [44].

This noise model can be equivalently seen as a noisy measurement [described by the dual (unital) *CP* map $\Lambda^*$ acting on the standard measurement] performed on a perfectly prepared state. Given that the scheme already assumes implementation of perfect product Clifford gates, it is natural to allow implementation of arbitrary noiseless single-qubit unitaries prior to the noisy measurement. We note that if the noise is uncorrelated, one can design unitaries that (on each qubit) transform the noise channel to stochastic (classical) noise (see Appendix G [47]). It is worth noting that stochastic readout noise is indeed one of the main sources of errors in modern quantum devices (see, e.g., [41–43]).

TABLE I. Relative upper bounds on the variance of the estimators for popular quantum chemistry Hamiltonians (with different encodings) in the presence of readout noise, using an optimized joint-measurability strategy (see Appendix I in Supplemental Material [47] for details on the optimization heuristics). The quantities for each molecule and encoding are normalized by the value of the variance upper bound for the classical shadows strategy of the given molecule and encoding (see Appendix J [47] for explicit results).

| Encoding or molecule | $H_2$ | LiH | $BeH_2$ | $H_2O$ |
|---|---|---|---|---|
| Jordan-Wigner | 1.00 | 0.06 | 0.04 | 0.1 |
| Bravyi-Kitaev | 0.13 | 0.78 | 0.55 | 0.61 |
| Parity | 0.38 | 0.02 | 0.009 | 0.02 |

Crucially, our joint-measurability scheme can easily incorporate uncorrelated readout noise. To this end, we find strategies of implementing qubit POVMs via classical randomization of noisy qubit projective measurements and postprocessing. This class of measurements is then used to construct parent POVMs for unsharp versions of Pauli observables. Specifically, we modify the semidefinite program (SDP) derived in Ref. [50], which characterizes qubit projective simulability to include readout noise (see Appendix H [47]), and then incorporate the constraint into the standard joint-measurability SDP [38].

We use the above observations to compare the performance of classical shadows and joint-measurability schemes when applied to the energy estimation of popular quantum chemistry Hamiltonians in the presence of readout noise. We note that our methods allow for easily incorporating noise into *biased* strategies, where $\eta_i^{x,y,z}$ are weighted according to the frequency of certain Pauli operators in the Hamiltonian, whereas only *unbiased* noisy classical shadows have been developed so far [44,45]. Thus, we can optimize the measurement strategy for each Hamiltonian (using heuristics developed in Appendix I [47]).

The results of benchmarks for molecules and encodings [9,40,54] are shown in Table I. For each strategy, we calculate the upper bound on the variance [see Eq. (6) for the joint-measurability strategy or corresponding expressions for noisy classical shadows in Ref. [45]]. As a noise model, we use the noise data from (the most noisy) subsystems of IBM's Washington 126-qubit quantum device. Interestingly, even in the presence of noise, the unbiased joint-measurability strategy gives exactly the same performance as classical shadows. In contrast, optimized strategies allow us to obtain a reduction of the variance upper bound by as much as a factor of ≈100.

*Concluding remarks.*—In this Letter, we have applied a simple joint-measurability strategy, implemented locally, to simultaneously estimate expectation values of collections of incompatible observables. We apply this technique to estimate energies of quantum Hamiltonians and derive bounds on the sample complexity of the protocol. The application of joint measurability to quantum computing problems, as well as the connections to classical shadows, opens new research questions to explore. In particular, are there deeper fundamental connections between shadows and joint measurements? Can we gain further insight into the efficiency of computational tasks from the limits of joint measurability, or, conversely, can we construct optimal joint measurements from the performance limits of classical shadows? In general, this work motivates further studies of measurement incompatibility, especially the characterization of optimal joint-measurement schemes which are projective (or noisy projective) simulable.

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*Note added.*—Recently, we became aware of independent results by Nguyen *et al.* [55], which offer a complementary perspective on connecting measurement theory with classical shadows.

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