Advantages of Multicopy Nonlocality Distillation and Its Application to Minimizing Communication Complexity

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Nonlocal correlations are a central feature of quantum theory, and understanding why quantum theory has a limited amount of nonlocality is a fundamental problem. Since nonlocality also has technological applications, e.g., for device-independent cryptography, it is useful to understand it as a resource and, in particular, whether and how different types of nonlocality can be interconverted. Here we focus on nonlocality distillation which involves using several copies of a nonlocal resource to generate one with more nonlocality. We introduce several distillation schemes which distill an extended part of the set of nonlocal correlations including quantum correlations. Our schemes are based on a natural set of operational procedures known as wirings that can be applied regardless of the underlying theory. Some are sequential algorithms that repeatedly use a two-copy protocol, while others are genuine three-copy distillation protocols. In some regions we prove that genuine three-copy protocols are strictly better than two-copy protocols. By applying our new protocols we also increase the region in which nonlocal correlations are known to give rise to trivial communication complexity. This brings us closer to an understanding of the sets of nonlocal correlations that can be recovered from information-theoretic principles, which, in turn, enhances our understanding of what is special about quantum theory.

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Introduction.—A bound on the strength of correlations realizable between pairs of measurement inputs and outputs in any local theory was first shown by Bell [1,2]. This bound is exceeded in quantum theory and there are even stronger correlations theoretically possible without enabling signalling [3,4]. One way to better understand quantum theory is to consider it in light of possible alternative theories, which can be compared in terms of the correlations they can create, and the implications access to such correlations would have. For instance, it is known that theories that permit strong enough correlations have trivial communication complexity [5]. Furthermore, nonlocal correlations have found applications in cryptography, where they form a necessary resource for device-independent quantum key distribution [6–9] and randomness expansion [10–12], for example. Since nonlocal correlations serve as resources for information processing, it is natural to ask about their interconvertability. In this Letter, we look at nonlocality distillation [13], i.e., whether with access to several copies of some nonlocal resource we can generate stronger ones, which would have implications for the study of device-independent tasks in noisy regimes, for instance.

Nonlocality distillation is often analyzed in terms of *wirings* [13–19], which means interacting with systems by choosing inputs and receiving and processing outcomes from those systems. This has the advantage that, firstly, the distillation procedures apply to nonlocal quantum correlations no matter how complicated the system these have been obtained from and, secondly, these procedures are applicable beyond quantum theory. A general theory will prescribe various different ways to measure systems [in quantum theory, for instance, a measurement is described by a positive operator valued measure (POVM)]. Wirings form an operationally natural subclass that can be performed in any generalized probabilistic theory (GPT) [20] (including quantum theory).

Previous work on nonlocality distillation has focused on specific protocols for the distillation of two copies of a nonlocal resource (see, e.g., [13–15,17,19]). The case of more copies remains largely open, with only few specific results [16,18]. In part, this is because analyzing nonlocality distillation is challenging: distillation protocols act nonlinearly on the correlations and hence cannot be easily optimized. Furthermore, applying a successful two-copy protocol twice often decreases the nonlocality again (see, e.g., [14] for an exception). Hence, understanding two-copy protocols provides little insight into the n-copy case.

In this Letter, we describe a sequential adaptive algorithm that uses wirings to distill nonlocality. We use this algorithm to explore the distillable region within the set of nonlocal correlations, and the amount of distillation possible. We demonstrate new wirings that allow distillation of correlations that cannot be distilled with any two-copy wiring protocol.

Our results have implications for communication complexity. In this problem, Alice with input x and Bob with input y want to enable Alice to compute $f(x,y)\colon \{0,1\}^k \times \{0,1\}^m \to \{0,1\}$. We ask how much communication from Bob to Alice is required to do so. Communication complexity is said to be trivial if any such function (no matter how large k and m) can be computed using only one bit of communication. Shared maximally nonlocal resources are known to make communication complexity trivial in this sense [21]. A probabilistic notion of trivial communication complexity was introduced in [5] in which for any f we require the existence of $p > 1/2$ such that Alice can obtain the correct value of $f(x,y)$ with probability at least p for all x and y. In this Letter, when we talk about trivial communication complexity we mean it in this probabilistic sense. A larger set of shared states that render communication complexity trivial were found in Refs. [5,14]. Our results further enlarge this set, demonstrating advantages of wirings beyond two copies.

Nonlocality and wirings.—Correlations of inputs x, y and outputs a, b are described by conditional probability distributions $P(ab|xy)$, and we refer to these as a *box* or a *behavior*. In the context of nonlocality, we usually imagine these correlations as generated by two parties, Alice and Bob, who each choose an input (x and y, respectively) and obtain an output (a and b, respectively). The correlations they can generate according to any theory that is consistent with special relativity have to be *nonsignalling*, meaning

$$\sum_b P(ab|xy) = \sum_b P(ab|xy') \quad \forall\ a,x,y,y',$$

and the same holds with the roles of Alice and Bob (i.e., a, x and b, y) exchanged. A box is called *local* if it can be written

$$P(ab|xy) = \sum_\lambda P(a|x\lambda)P(b|y\lambda)P(\lambda) \quad \forall\ a,b,x,y.$$

In the language of Bell inequalities, there is a variable Λ that takes the value λ with probability $P(\lambda)$. Boxes that cannot be written in this form are *nonlocal*.

In the case of two binary inputs and outputs, i.e., $a,b,x,y \in \{0,1\}$, the set of all local boxes is the convex hull of 16 local deterministic (L) boxes $P_i^{\mathrm{L}}(ab|xy) = \delta_{a,\mu x\oplus\nu}\delta_{b,\sigma y\oplus\tau}$ for $\mu,\nu,\sigma,\tau \in \{0,1\}$, $i = 1+\tau+2\sigma+4\nu+8\mu$, and the set of all nonsignalling boxes is the convex hull of these local boxes and 8 extremal non local (NL) boxes [4,22] $P_i^{\mathrm{NL}}(ab|xy) = \frac{1}{2}\delta_{a\oplus b, xy\oplus\mu x\oplus\nu y\oplus\sigma}$ for $\mu,\nu,\sigma \in \{0,1\}$, $i = 1+\sigma+2\nu+4\mu$. Up to symmetry, the Clauser-Horne-Shimony-Holt (CHSH) inequality [23] is the only one that restricts the set of local boxes. Nonlocality can hence be quantified in terms of the CHSH value $\mathrm{CHSH}[P(ab|xy)] = E_{00}+E_{01}+E_{10}-E_{11}$, with $E_{xy} = P(a=b|xy) - P(a\neq b|xy)$.

Because we work in a black-box picture, the most general operation we consider for each party is a wiring. We describe here the deterministic wirings; the most general wirings are convex combinations of these. Consider a party with access to n boxes with inputs x_j and outputs a_j with $j = 1,\ldots,n$. They "wire" these together to form a new box with input x and output a. The most general deterministic wiring comprises choosing a box to make the first input to and then making a chosen input, then using the output of that box to choose the second box and the input to that second box and so on. We label the ith box chosen $j_i(x, a_{j_1},\ldots,a_{j_{i-1}})$ and its input $x_{j_i}(x, a_{j_1},\ldots,a_{j_{i-1}})$. The final outcome is chosen depending on the overall input and all previous outcomes $a(x, a_{j_1},\ldots,a_{j_n})$. Thus, if Alice and Bob each do wirings on shares of n boxes, they generate a new box $P(ab|xy)$.

Our main question is then, given several copies of a nonlocal box, are there wirings for Alice and for Bob such that the resulting box is more nonlocal than the original? In the case of two nonsignalling boxes each with binary inputs and outputs, the possible wirings have been fully characterized [24]. Nevertheless, even in this case, deciding whether these can result in more nonlocality for a specific box is computationally intensive: there are 82 deterministic wirings that each party can perform for each input [24], leading to a total of 82^4 possibilities (one of the 82 for each input of each party). To make the computation more tractable, we optimize the wirings of one party with a linear program, while iterating over 82^2 wirings for the other (see the Supplemental Material [25] for more detail). We use this linear programming technique to illustrate the regions in which distillation is possible for various two-dimensional cross sections (CSs) of the no-signalling polytope in Fig. 1. In this Letter, we consider three regions:

$$\begin{aligned}
\mathrm{CS\,I}&: \omega P_1^{\mathrm{NL}} + \frac{\eta}{2}(P_1^{\mathrm{L}} + P_6^{\mathrm{L}}) + (1-\omega-\eta)P^{\mathrm{O}} \\
\mathrm{CS\,II}&: \omega P_1^{\mathrm{NL}} + \eta P_1^{\mathrm{L}} + (1-\omega-\eta)P^{\mathrm{O}} \\
\mathrm{CS\,III}&: \omega P_1^{\mathrm{NL}} + \frac{\eta}{2}(P_1^{\mathrm{L}} + P_9^{\mathrm{L}}) + (1-\omega-\eta)P^{\mathrm{O}}, \qquad (1)
\end{aligned}$$

where the origin (O) box $P^{\mathrm{O}} = 3/4P_1^{\mathrm{NL}} + 1/4P_2^{\mathrm{NL}}$ is local and η, $\omega \geq 0$ with $\eta + \omega \leq 1$.

We analyzed the distillability within these cross sections. Among the optimal protocols we recovered several that were previously known [15,30]. The protocols of [15] (called $\mathrm{ABL}^{+}1$, $\mathrm{ABL}^{+}2$) are sufficient to characterize the two-copy distillability in CS II (see Fig. 1), and CS III is two-copy nondistillable. The observation that $\mathrm{ABL}^{+}2$ achieves no distillation in CS I shows that optimal protocols depend on the cross section.

The above analysis is generally not useful for analyzing whether repeated distillation of a box can lead to a certain

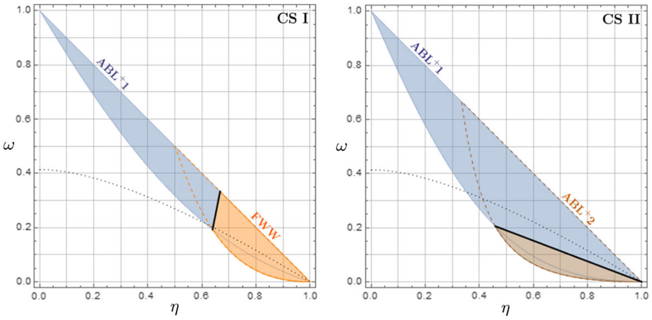

FIG. 1. Protocols sufficient to characterize the two-copy distillability (both the distillable region and the strongest amplification) for two CSs [cf. Eq. (1)]. The optimal two-copy protocols for CS II are the two protocols from [15] ($\mathrm{ABL}^{+}1$, $\mathrm{ABL}^{+}2$), while for CS I the protocol of [13] (called FWW here) is optimal in some cases. The shading indicates where the corresponding protocol is optimal, with the boundary indicated by the black line (see the Supplemental Material [25] for details of the protocols). The dotted curve indicates the boundary of the set of correlations realizable in quantum theory (computed using the conditions in [3,29]).

CHSH value. Applying a wiring that works for two boxes to two copies of the generated box often does not give a further increase in nonlocality, in which case a switch of wirings is needed to distill further. While there are boxes that cannot be distilled at all with wirings (e.g., isotropic boxes [31]), the maximum CHSH value that can be distilled using multiple copies of a specific resource box is unknown. This means that we do not know how resourceful (multiple copies of) most nonlocal boxes are for information processing. For instance, shared boxes render communication complexity trivial if their initial CHSH value is greater than $\mathrm{CHSH}[P(ab|xy)] = 4\sqrt{2/3}$ [5]. The complete set of boxes that render communication complexity trivial is unknown, although an additional region was found with the protocol of [14].

Sequential algorithms for nonlocality distillation and reduction of communication complexity.—While a repeated application of a successful two-copy protocol often does not increase the nonlocality further, there are various ways to combine different two-copy wirings (see the Supplemental Material [25]). Here, we focus on the specific structure illustrated in Fig. 2. Our serial algorithm consists in optimizing the wiring to be applied in every step, which is done in terms of a hybrid procedure of iterating over wirings and linear programming (see the Supplemental Material [25] for a detailed description of the algorithm). Applying our serial algorithm, we are able to extend the region of nonlocal boxes known to trivialize communication complexity, see Fig. 3.

Our algorithm furthermore provides us with a way to systematically derive new nonlocality distillation protocols for multicopy nonlocality distillation. When performing two steps of the serial algorithm, we find the three-copy protocol below to be successful.

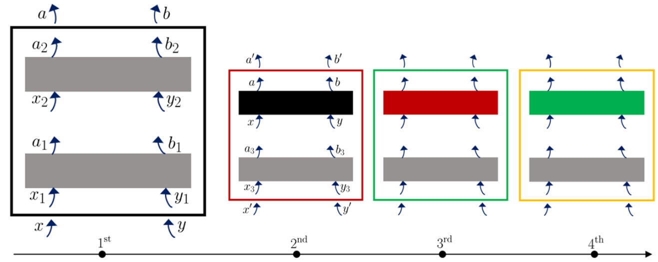

FIG. 2. A serial architecture for combining nonlocal resources (gray) in a sequential manner. The first step on the left depicts the usual two-copy distillation scheme. Each subsequent iteration uses another copy of the original box and the previously generated one. Our sequential algorithm optimizes the protocol at each round. See the Supplemental Material [25] for details.

In the first step, a box is created from two copies of a box P with inputs (outputs) labeled x_1, y_1 (a_1, b_1) and x_2, y_2 (a_2, b_2), respectively (first step in Fig. 2). Then this is wired to another copy of P, $P(a_3 b_3|x_3 y_3)$, using the functions [32]

$$\begin{aligned} &x_1 = x = x', \quad x_2 = x \oplus \bar{a}_1, \quad a = a_1 \oplus a_2, \quad x_3 = x\bar{a} \\ &y_1 = y = y', \quad y_2 = y b_1, \quad b = b_1 \oplus b_2, \quad y_3 = y \oplus b, \\ &a' = a \oplus a_3, \quad b' = b \oplus b_3, \end{aligned} \tag{2}$$

where $\oplus$ is the logical XOR and $\bar{z} = z \oplus 1$. This new protocol distills in CS II a strict superset of nonlocal boxes

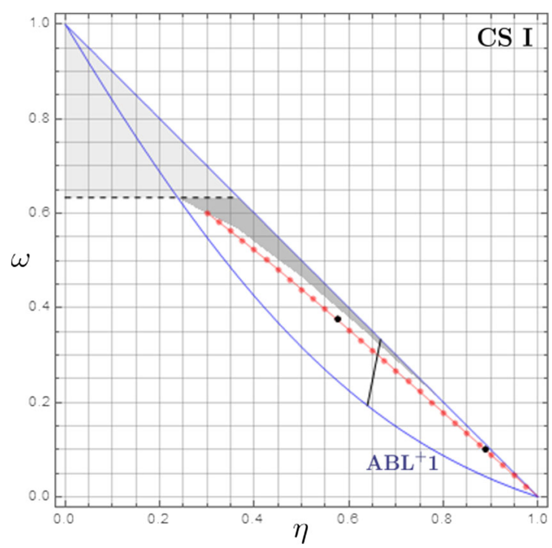

FIG. 3. Region of trivial communication complexity in CS I. The light-gray part was identified in [5]. The dark-gray region includes boxes that trivialize communication complexity through (up to four) iterations of $\mathrm{ABL}^{+}1$. The red points (and everything on their right) collapse communication complexity using our serial algorithm. The black solid chord is that of Fig. 1 (left) and indicates a change in protocol for the red points—see the Supplemental Material [25] for details, including an analysis of the black points in the figure.

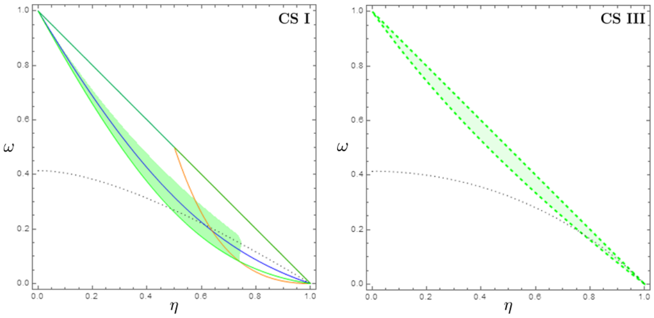

FIG. 4. Region of distillation by means of the three-copy wiring of Eq. (3) bounded by the green lines. The blue and orange lines show the region of optimal two-copy distillation in CS I, as in Fig. 1 (left). The green shaded area in CS I depicts where our protocol leads to higher CHSH values than all previously known protocols (i.e., two-copy and three-copy FWW, $\mathrm{ABL}^{+}1$, HR). In CS III no two-copy nonlocality distillation is possible and the ability to distill is unlocked only when given access to at least three copies of a nonlocal box where use of a genuine three-copy protocol is imperative. The dotted curve indicates the boundary of the set of quantum-realizable correlations.

compared to the previously known three-copy distillation protocol of [16] (in contrast to CS I where the protocol of [16] is superior). For completeness we introduce the protocol from [16] in the Supplemental Material [25] and we refer to it as HR. The region in which the new protocol distills in CS II is also shown in the Supplemental Material [25].

Genuine three-copy distillation protocols.—When considering three-copy distillation, the variety of possible protocols is vastly increased. In this case we can derive new protocols that outperform the previous ones in terms of the boxes for which they offer distillation. For this, we introduce a *genuine three-copy distillation protocol*, which is one that cannot be reduced to any concatenation of two-copy protocols, i.e., is not of the form of Fig. 2. Consider the following wiring, where $\vee$ denotes the logical OR operation:

$$\begin{aligned} &x_1 = x_2 = \bar{x}, \quad x_3 = \bar{x}a_1 \vee \bar{x}a_2, \quad a = a_1a_3 \vee a_2a_3 \vee \bar{a}_1\bar{a}_2\bar{a}_3, \\ &y_1 = y_2 = y, \quad y_3 = yb_1 \vee yb_2 \vee \bar{y}\bar{b}_1\bar{b}_2, \\ &b = \bar{y}b_1b_3 \vee \bar{y}b_2b_3 \vee yb_1\bar{b}_3 \vee yb_2\bar{b}_3 \vee \bar{y}\bar{b}_1\bar{b}_2\bar{b}_3 \vee y\bar{b}_1\bar{b}_2b_3. \quad (3) \end{aligned}$$

We find larger regions of distillable boxes as compared to the two-copy case, see Fig. 4. In CS III no two-copy distillation is possible, while with three copies it is. Furthermore, the increase in the region of boxes that allow for distillation is considerably larger than that of HR (which is nearly indistinguishable from $\mathrm{ABL}^{+}1$, see also Fig. III in the Supplemental Material [25]).

Additionally we find three-copy protocols that increase the region where communication complexity is trivial. In particular

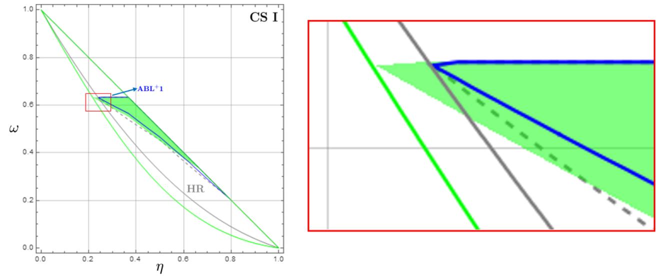

FIG. 5. Regions of trivial communication complexity with various protocols. The green region is from repeated use of our genuine three-copy protocol of Eq. (4), the blue bounded region is from repeated use of $\mathrm{ABL}^{+}1$, and the dashed gray bounded region is from repeated use of HR. In the magnified view (right) we see a small region where our new three-copy protocol outperforms HR and any possible two-copy protocol.

$$\begin{aligned} &x_1 = x_2 = x, \quad x_3 = xa_2 \vee x\bar{a}_1 \vee \bar{x}\bar{a}_2a_1, \\ &a = a_3a_2 \vee a_3\bar{a}_1 \vee \bar{a}_3\bar{a}_2a_1, \quad y_1 = y_2 = y, \quad y_3 = yb_2 \vee y\bar{b}_1, \\ &b = b_3b_2 \vee b_3\bar{b}_1 \vee \bar{b}_3\bar{b}_2b_1. \quad (4) \end{aligned}$$

We illustrate the use of this protocol for trivializing communication complexity in Fig. 5. In addition, we find that in CS I, starting from any point with $\omega > 0$ on the line $\omega = 1 - \eta$ we can distill arbitrarily close to a PR box by repeatedly iterating this protocol (see Section IV of the Supplemental Material [25]). We observe, that as compared to using two-copy protocols (even sequentially), three-copy protocols provide further advantages.

Additionally, all the protocols introduced here, i.e., those of Eqs. (2)–(4) work in a full dimensional subset of the space of no-signalling correlations. This space is 8 dimensional for bipartite non-signalling boxes with binary inputs and outputs. The form of our distillation protocols (and many others in the literature) implies that the difference between the initial and final CHSH value is a polynomial in the parameters of the initial box $P(ab|xy)$ and hence continuous in these parameters. Thus, for any distillable point not on the boundary of the polytope, there exists an eight-dimensional ball around it that is also distillable.

Conclusions.—We have found a genuine three-copy protocol that distills nonlocality for boxes in which distillation with two copies is impossible and shown that there are three-copy protocols that outperform *all* two-copy protocols (and sequential applications thereof). For the latter we employed an optimization technique for two-copy wiring protocols. Although this optimization furthers our understanding, it remains limited to cases with small numbers of inputs and outputs and there remains much more to discover about nonlocality distillation.

Whether the principle of nontrivial communication complexity [5] defines a closed set of correlations [33] that allows for a simple characterization and lies well between quantum and nonsignalling sets is an open question of interest for the foundations of quantum theory. Indeed, finding a sensible generalized probabilistic theory that leads to a set of correlations between the nonsignalling and quantum set with a simple geometric description has been a conundrum. The present work suggests that a better understanding of multicopy nonlocality distillation may give us insights into such a set, namely, that of a GPT whose only restriction is imposed by the principle of nontrivial communication complexity. This would further advance the recent research program of experimentally ruling out generalized probabilistic theories due to the correlations they produce in networks [34,35].

Some of our distillation protocols work within the set of quantum correlations (see Fig. 4). [See also [36] for recent work aiming to distill quantum correlations.] Being wirings, they are much simpler to perform than entanglement distillation protocols [37]. It would be interesting to explore applications of these for information processing.

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[1] J. S. Bell, On the Einstein-Podolsky-Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[2] J. S. Bell, The theory of local beables, in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987), pp. 52–62.
[3] B. Cirel'son, Quantum generalizations of Bell's inequality, Lett. Math. Phys. **4**, 93 (1980).
[4] S. Popescu and D. Rohrlich, Quantum nonlocality as an axiom, Found. Phys. **24**, 379 (1994).
[5] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, Limit on Nonlocality in any World in Which Communication Complexity is not Trivial, Phys. Rev. Lett. **96**, 250401 (2006).
[6] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).
[7] D. Mayers and A. Yao, Quantum cryptography with imperfect apparatus, in *Proceedings of the 39th Annual Symposium on Foundations of Computer Science (FOCS-98)* (IEEE Computer Society, Los Alamitos, CA, USA, 1998), pp. 503–509.
[8] J. Barrett, L. Hardy, and A. Kent, No Signalling and Quantum Key Distribution, Phys. Rev. Lett. **95**, 010503 (2005).
[9] S. Pironio, A. Acin, N. Brunner, N. Gisin, S. Massar, and V. Scarani, Device-independent quantum key distribution secure against collective attacks, New J. Phys. **11**, 045021 (2009).
[10] R. Colbeck, Quantum and relativistic protocols for secure multi-party computation, Ph.D. thesis, University of Cambridge, 2007, also available as arXiv:0911.3814.
[11] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).
[12] R. Colbeck and A. Kent, Private randomness expansion with untrusted devices, J. Phys. A **44**, 095305 (2011).
[13] M. Forster, S. Winkler, and S. Wolf, Distilling Nonlocality, Phys. Rev. Lett. **102**, 120401 (2009).
[14] N. Brunner and P. Skrzypczyk, Nonlocality Distillation and Postquantum Theories with Trivial Communication Complexity, Phys. Rev. Lett. **102**, 160403 (2009).
[15] J. Allcock, N. Brunner, N. Linden, S. Popescu, P. Skrzypczyk, and T. Vértesi, Closed sets of nonlocal correlations, Phys. Rev. A **80**, 062107 (2009).
[16] P. Høyer and J. Rashid, Optimal protocols for nonlocality distillation, Phys. Rev. A **82**, 042118 (2010).
[17] L.-Y. Hsu and K.-S. Wu, Multipartite nonlocality distillation, Phys. Rev. A **82**, 052102 (2010).
[18] X.-J. Ye, D.-L. Deng, and J.-L. Chen, Nonlocal distillation based on multisetting bell inequality, Phys. Rev. A **86**, 062103 (2012).
[19] G.-Z. Pan, C. Li, M. Yang, G. Zhang, and Z.-L. Cao, Nonlocality distillation for high-dimensional correlated boxes, Quantum Inf. Process. **14**, 1321 (2015).
[20] J. Barrett, Information processing in generalized probabilistic theories, Phys. Rev. A **75**, 032304 (2007).
[21] W. van Dam, Implausible consequences of superstrong nonlocality, arXiv:quant-ph/0501159.
[22] B. S. Tsirelson, Some results and problems on quantum Bell-type inequalities, Hadronic J. Suppl. **8**, 329 (1993).
[23] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[24] A. J. Short and J. Barrett, Strong nonlocality: A trade-off between states and measurements, New J. Phys. **12**, 033034 (2010).
[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.100201 for further details on the optimizations and comparisons between different nonlocality schemes. This also includes Refs. [26–28].
[26] A. J. Short, S. Popescu, and N. Gisin, Entanglement swapping for generalized nonlocal correlations, Phys. Rev. A **73**, 012101 (2006).
[27] S. G. A. Brito, M. G. M. Moreno, A. Rai, and R. Chaves, Nonlocality distillation and quantum voids, Phys. Rev. A **100**, 012102 (2019).
[28] G. Eftaxias, Theory-independent topics towards quantum mechanics: ψ-ontology and nonlocality distillation, Ph.D. thesis, Quantum Engineering Centre for Doctoral Training, University of Bristol, United Kingdom, 2022.
[29] L. Masanes, Necessary and sufficient condition for quantum-generated correlations, arXiv:quant-ph/0309137.
[30] N. Brunner, D. Cavalcanti, A. Salles, and P. Skrzypczyk, Bound Nonlocality and Activation, Phys. Rev. Lett. **106**, 020402 (2011).

[31] S. Beigi and A. Gohari, Monotone measures for non-local correlations, IEEE Trans. Inf. Theory **61**, 5185 (2015).

[32] See Fig. 2 for the meaning of x, y, a, b and x', y', a', b'.

[33] B. Lang, T. Vértesi, and M. Navascués, Closed sets of correlations: Answers from the zoo, J. Phys. A **47**, 424029 (2014).

[34] M. Weilenmann and R. Colbeck, Self-Testing of Physical Theories, or, is Quantum Theory Optimal with Respect to Some Information-Processing Task?, Phys. Rev. Lett. **125**, 060406 (2020).

[35] M. Weilenmann and R. Colbeck, Toward correlation self-testing of quantum theory in the adaptive Clauser-Horne-Shimony-Holt game, Phys. Rev. A **102**, 022203 (2020).

[36] S. G. Naik, G. L. Sidhardh, S. Sen, A. Roy, A. Rai, and M. Banik, Distilling nonlocality in quantum correlations, arXiv:2208.13976.

[37] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).