Hidden Symmetries in Acoustic Wave Systems

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Latent symmetries are hidden symmetries which become manifest by performing a reduction of a given discrete system into an effective lower-dimensional one. We show how latent symmetries can be leveraged for continuous wave setups in the form of acoustic networks. These are systematically designed to possess latent-symmetry induced pointwise amplitude parity between selected waveguide junctions for all low frequency eigenmodes. We develop a modular principle to interconnect latently symmetric networks to feature multiple latently symmetric junction pairs. By connecting such networks to a mirror symmetric subsystem, we design asymmetric setups featuring eigenmodes with domain-wise parity. Bridging the gap between discrete and continuous models, our work takes a pivotal step towards exploiting hidden geometrical symmetries in realistic wave setups.

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Introduction.—Symmetries dictate the appearance of fundamental physical laws and allow us to make detailed predictions without solving the underlying equations of motion [1,2]. Selection rules for atoms and molecules [3,4], the emergence of Bloch states and band structures in crystals [5], and the explanation of spectral degeneracies [4] are all examples for the importance of symmetries.

Recently the concept of latent symmetry has been introduced, that is, a symmetry not of an original Hamiltonian, but of an equivalent dimensionally reduced effective Hamiltonian [6]. Importantly, the presence of a latent symmetry leaves its fingerprints in the original eigenvectors, thereby showing, e.g., a parity symmetry of certain eigenvector components only. This concept has proven fruitful in many different areas such as the analysis of complex networks [6–8], the explanation of a class of accidental degeneracies [9], and can be used to design lattices with flat bands [10], a topic of major current interest [11,12]. Besides this, a certain subclass of latent symmetries can be closely linked to the graph-theoretical concept of cospectrality [13–15], which is of importance in the context of (almost) perfect state transfer [16–20].

The theory of latent symmetries has so far been developed and applied only to discrete systems. In this Letter, we take the conceptual step of extending and applying the concept of latent symmetry to a continuous system. We systematically design networks with pointwise amplitude parity between selected waveguide junctions for all low frequency eigenmodes. Our construction principle yields asymmetric setups which possess eigenmodes with domain-wise parity.

Setup.—We investigate the eigenmodes of acoustic networks described by the 3D-Helmholtz equation

$$\Delta p + k^2 p = 0 \tag{1}$$

with Neumann hard boundary (wall) conditions on the rigid surfaces of waveguides or cavities, and with $p(\mathbf{r})$ denoting the acoustic pressure field. For simplicity, we consider the case where all structures possess the same thickness d, so that Eq. (1) can be separated into a 2D (x-y plane) and a 1D (z-axis) problem.

We begin with networks formed by interconnecting waveguides of equal length L. If such a network is spatially mirror symmetric, its eigenmodes have odd or even parity under the reflection, that is, $p(\mathcal{R}(\mathbf{r})) = \pm p(\mathbf{r})$ for all points $\mathbf{r}$, with $\mathcal{R}(\mathbf{r})$ denoting the reflection operation. In contrast to that, we design asymmetric networks that feature *pointwise parity* in their low-frequency eigenmodes, that is, $p(\mathbf{r}_n) = \pm p(\mathbf{r}_m)$ only for specific locations $\mathbf{r}_n$, $\mathbf{r}_m$. To reach this goal, we design the network to feature a latent symmetry by tuning the waveguide widths. In a next step, we show that those networks can be easily augmented by mirror-symmetric cavity subsystems such that (i) the coupled system features no geometrical symmetry while (ii) the low-frequency eigenmodes have *domain-wise parity*, that is, definite parity everywhere in the cavities.

Latent symmetries in eigenvalue problems.—Let us start by sketching the theory of latent symmetries [6]. It is based on the ordinary eigenvalue problem

$$H\mathbf{Y} = \lambda\mathbf{Y}, \tag{2}$$

with H denoting the Hamiltonian represented by a Hermitian matrix. To define latent symmetries, we first partition the underlying setup into two subsystems, S and its complement $\overline{S}$, and write Eq. (2) in block form as

$$\begin{pmatrix} H_{SS} & H_{S\overline{S}} \\ H_{\overline{S}S} & H_{\overline{S}\overline{S}} \end{pmatrix} \begin{pmatrix} Y_S \\ Y_{\overline{S}} \end{pmatrix} = \lambda \begin{pmatrix} Y_S \\ Y_{\overline{S}} \end{pmatrix}. \tag{3}$$

Assuming for simplicity that $\lambda \mathbb{1} - H_{\overline{S}\overline{S}}$ is invertible for any eigenvalue of H, we can formally solve the second equation for $Y_{\overline{S}}$ and insert it into the first. This gives us the nonlinear eigenvalue problem

$$\widetilde{H}_S(\lambda) Y_S = \lambda Y_S \tag{4}$$

with the effective Hamiltonian $\widetilde{H}_S(\lambda) = H_{SS} + H_{S\overline{S}}(\lambda \mathbb{1} - H_{\overline{S}\overline{S}})^{-1} H_{\overline{S}S}$ [6,9,13,21]. $\widetilde{H}_S(\lambda)$ is known as the "isospectral reduction" of H, since in general its (nonlinear) eigenvalues are equal to that of the original Hamiltonian H.

The effective Hamiltonian may or may not have symmetries. If it does, that is, if $[\widetilde{H}_S(\lambda), M] = 0$ for all λ, where the normal matrix M describes the symmetry operation, then the original Hamiltonian H is said to be *latently symmetric* in S. A latent symmetry has a profound impact on the eigenvectors $\mathbf{Y}$ of H: First, it follows from Eq. (4) that the restriction Y_S of $\mathbf{Y}$ on S must be an eigenvector of $\widetilde{H}_S(\lambda)$. Now, since $\widetilde{H}_S(\lambda)$ is M symmetric, it follows that (assuming no degeneracies) Y_S must follow this symmetry as well. In other words, $\mathbf{Y}$ must be locally M symmetric on S. Thus, by tuning the system to feature a latent symmetry with a specific matrix M, it is possible to tailor *local properties* of the eigenvectors. To demonstrate the principle of latent symmetry induced design of local properties, we will here focus on the special case of a latent mirror symmetry (LMS) described by $M \equiv \Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since the eigenvalues of Σ are $\sigma = \pm 1$, this means that one can choose the eigenvectors of H to have definite parity on $S := \{u, v\}$. That is, they feature *pointwise parity* [22].

Latent symmetry in acoustic waveguides.—In order to construct latently symmetric networks of acoustic waveguides, we focus on narrow waveguides, $w_n \ll L$, $d \ll L$, such that the propagation of low-frequency waves in the individual waveguides with width w_n and identical thickness d effectively becomes one-dimensional (see Sec. I of the Supplemental Material [23] for more details). In this regime, which we will consider throughout this work, the eigenmode amplitudes at the endpoints of waveguides can be described by the generalized eigenvalue problem (gEVP) [30–33]

$$A\mathbf{X} = \lambda B \mathbf{X}, \tag{5}$$

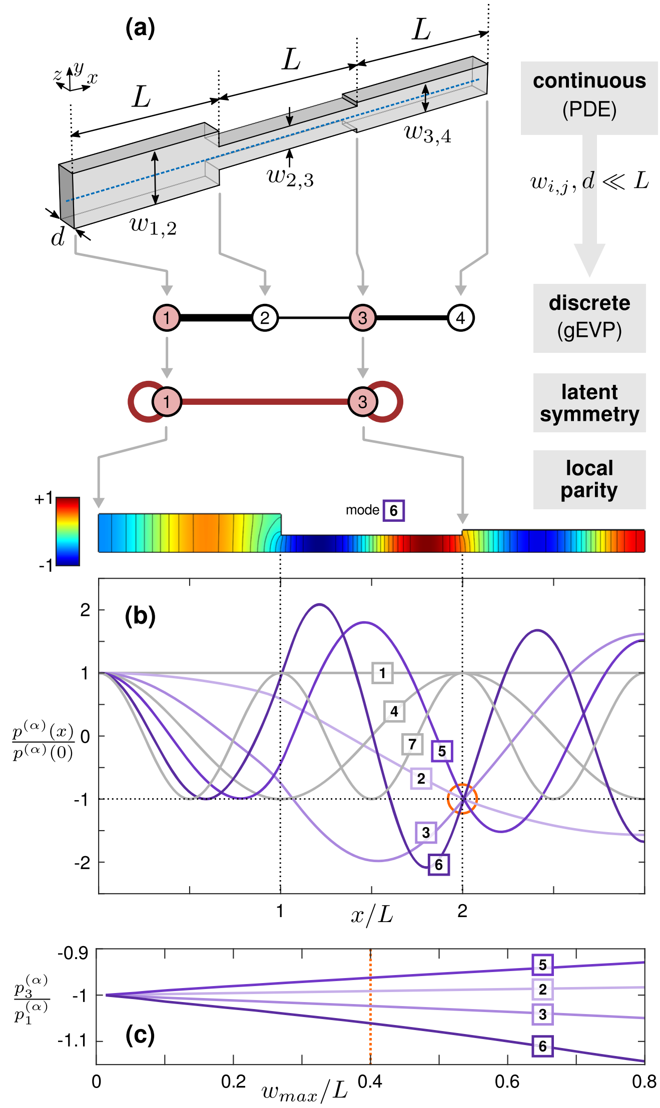

FIG. 1. (a) A setup consisting of three waveguides (top) and its mapping to a discrete model (bottom; see text for details). For $w_{1,2} = w_{3,4} + w_{2,3}$, the setup features a latent mirror symmetry with $S = \{1, 3\}$, resulting in pointwise parity of the eigenfunctions. (b) The amplitude ratio $p^{(\alpha)}(x)/p^{(\alpha)}(0)$ [evaluated along the blue dashed line depicted in (a)] for the first $\alpha = 1, \ldots, 7$ eigenmodes of the continuous Eq. (1). (c) The behavior of $p_3^{(\alpha)}/p_1^{(\alpha)}$ for the nonplane wave modes $\alpha = \{2, 3, 5, 6\}$ for varying $w_{\max}/L$ with $w_{\max} = \max(w_{1,2}, w_{2,3}, w_{3,4})$. For the modes depicted in (b), $w_{\max}/L = 0.4$.

with $\lambda = \cos(kL)$, $A_{n,m} = w_{n,m}$, and B diagonal with $B_{n,n} = \sum_m w_{n,m}$. Here, $w_{n,m}$ denotes the width of the waveguide between the end points n and m, with $w_{n,m} = 0$ if there is no waveguide. The eigenvector $\mathbf{X} = (p_1, \ldots, p_N)^T$ corresponds to the acoustic pressure on the end points of the waveguides. For our first example, the three-waveguide setup of Fig. 1(a), the discrete problem is four dimensional, and we have

$$A = \begin{pmatrix} 0 & w_{1,2} & 0 & 0 \\ w_{1,2} & 0 & w_{2,3} & 0 \\ 0 & w_{2,3} & 0 & w_{3,4} \\ 0 & 0 & w_{3,4} & 0 \end{pmatrix}, \quad (6)$$

$$B = \begin{pmatrix} w_{1,2} & 0 & 0 & 0 \\ 0 & w_{1,2}+w_{2,3} & 0 & 0 \\ 0 & 0 & w_{2,3}+w_{3,4} & 0 \\ 0 & 0 & 0 & w_{3,4} \end{pmatrix}. \quad (7)$$

Before we continue, we note that the gEVP in the form of Eq. (5) with A, B real-symmetric and B positive definite is widespread; it occurs in electronic structure models in a nonorthogonal basis in quantum chemistry [34], spring-mass systems, molecular or mechanical vibrations [3,35], and it also appears naturally in numerical finite element treatments of wave equations [36]. Now, since the matrix B is positive definite in all these cases, we can convert Eq. (5) to the ordinary symmetric eigenvalue problem Eq. (2) with $\mathbf{Y} = \sqrt{B}\mathbf{X}$ and the real-symmetric "Hamiltonian" $H = \sqrt{B}^{-1}A\sqrt{B}^{-1}$. This convenient transformation allows us to extend the concept of latent symmetries from ordinary [Eq. (2)] to generalized eigenvalue problems [Eq. (5)].

Let us now analyze the case where the Hamiltonian H corresponding to a gEVP features a LMS for $S \coloneqq \{u, v\}$. Assuming for simplicity that H features no degeneracies (see Sec. II of the Supplemental Material [23] for details), this latent symmetry induces point-wise parity on u, v onto the eigenvectors $\mathbf{Y}$ of H. Depending on the structure of B, the eigenvectors $\mathbf{X} = \sqrt{B}^{-1}\mathbf{Y}$ of Eq. (5) then may or may not feature pointwise parity. In the special case of acoustic waveguides, however, a LMS of H automatically induces pointwise parity, that is, $X_u = \pm X_v$ for any eigenvector $\mathbf{X}$ (see Secs. IC and II in the Supplemental Material [23]).

We now apply the concept of latent symmetries to a concrete setup. In Fig. 1, we show a particularly simple waveguide network which, for $w_{1,2} = w_{3,4} + w_{2,3}$, features a LMS for the two junctions $S = \{1, 3\}$. Thus, the eigenmodes $\mathbf{X}$ of the gEVP Eq. (5) have pointwise parity on 1,3; the corresponding Hamiltonian has no degeneracy. As a consequence, the acoustic pressure at the end points 1 and 3 [see Fig. 1(a)] has pointwise parity for low-frequency eigenmodes and narrow waveguides.

Two aspects are noteworthy. First, while the Hamiltonian H describing Fig. 1(a) is only four dimensional, the pointwise parity induced by its LMS has an impact on more than four eigenmodes of the underlying continuous Eq. (1). Indeed, and as we show in Sec. I of the Supplemental Material [23], as long as the low-frequency limit (monomode approximation) is valid, every eigenmode p of Eq. (1) features pointwise parity; this is demonstrated in Fig. 1(b). Second, we stress that our theoretical considerations of a latently mirror symmetric waveguide network (LMSWN) are based on approximating Eq. (1) by a *discrete* gEVP. Thus, one would expect that our results are valid only in the limiting case of very narrow waveguides. The pointwise parity of eigenmodes, however, is robust and it approximately persists even when departing from the limiting case $w_{\max}/L \to 0$, up to roughly $w_{\max}/L \simeq 0.2$. This is shown in Fig. 1(c). There, we scale all widths by an identical factor and analyze the deviation from -1 of the ratio $p_1^{(\alpha)}/p_3^{(\alpha)}$ for the eigenmodes $\alpha = \{2, 3, 5, 6\}$ in dependence of this scaling factor. For modes 1,4,7, we note that the pointwise parity is perfect, $p_1^{(\alpha)}/p_3^{(\alpha)} = 1$, because these modes—for which kL is a multiple of π—are exact solutions to the PDE of Eq. (1). Irrespective of the individual waveguide widths, these modes do always exist, and as can easily be shown, they exactly fulfill $p_1^{(\alpha)}/p_3^{(\alpha)} = 1$ even far away from the limit $w_{n,m} \ll L$.

Network design.—Having demonstrated a first instance of a LMSWN, let us now address the general construction of such networks. This task is equivalent to finding a network geometry with suitable widths $w_{i,j}$ and two sites $S = \{u, v\}$ for which $\widetilde{H}_S(\lambda)$ commutes, for all λ, with $\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Expanding $\widetilde{H}_S(\lambda)$ into a power series in λ shows that this commutation is equivalent to [9,13]

$$(H^k)_{u,u} = (H^k)_{v,v} \quad \forall \ 1 \le k \le N-1, \quad (8)$$

with N denoting the dimension of H. Interestingly, an analysis of these $N-1$ conditions on the diagonal elements of the matrix powers of H allows us to derive generic rules that a LMSWN has to fulfill (see Supplemental Material [23], Sec. I C). For example, the sum of widths of waveguides adjacent to u and v must be identical; that is, $B_{u,u} = B_{v,v}$.

The difficulty of finding a latently mirror symmetric configuration clearly depends on the network size and topology. In general, for a given H of size N, the choice of S would be done in principle by trying all $N(N-1)/2$ possible pairs $\{u, v\}$ and testing whether they obey Eq. (8). For small acoustic networks, as the one in Fig. 1, a suitable combination of widths and S can be even found analytically yielding the simple condition $w_{1,2} = w_{3,4} + w_{2,3}$; but for larger networks the complexity may become too high. Fortunately, there is an alternative means for this problem: As we now demonstrate, smaller LMSWNs can be combined to form *arbitrarily large* networks: Given two waveguide networks with latent mirror symmetries for $S_n = \{u_n, v_n\}$, $n = 1, 2$, we connect u_1 to u_2 by a narrow waveguide of arbitrary (though small) width w, and v_1 to v_2 by another waveguide with identical width w. As shown in Sec. III of the Supplemental Material [23], the resulting larger network is then guaranteed to feature latent mirror symmetries for both S_1 and S_2. Figure 2(a) demonstrates

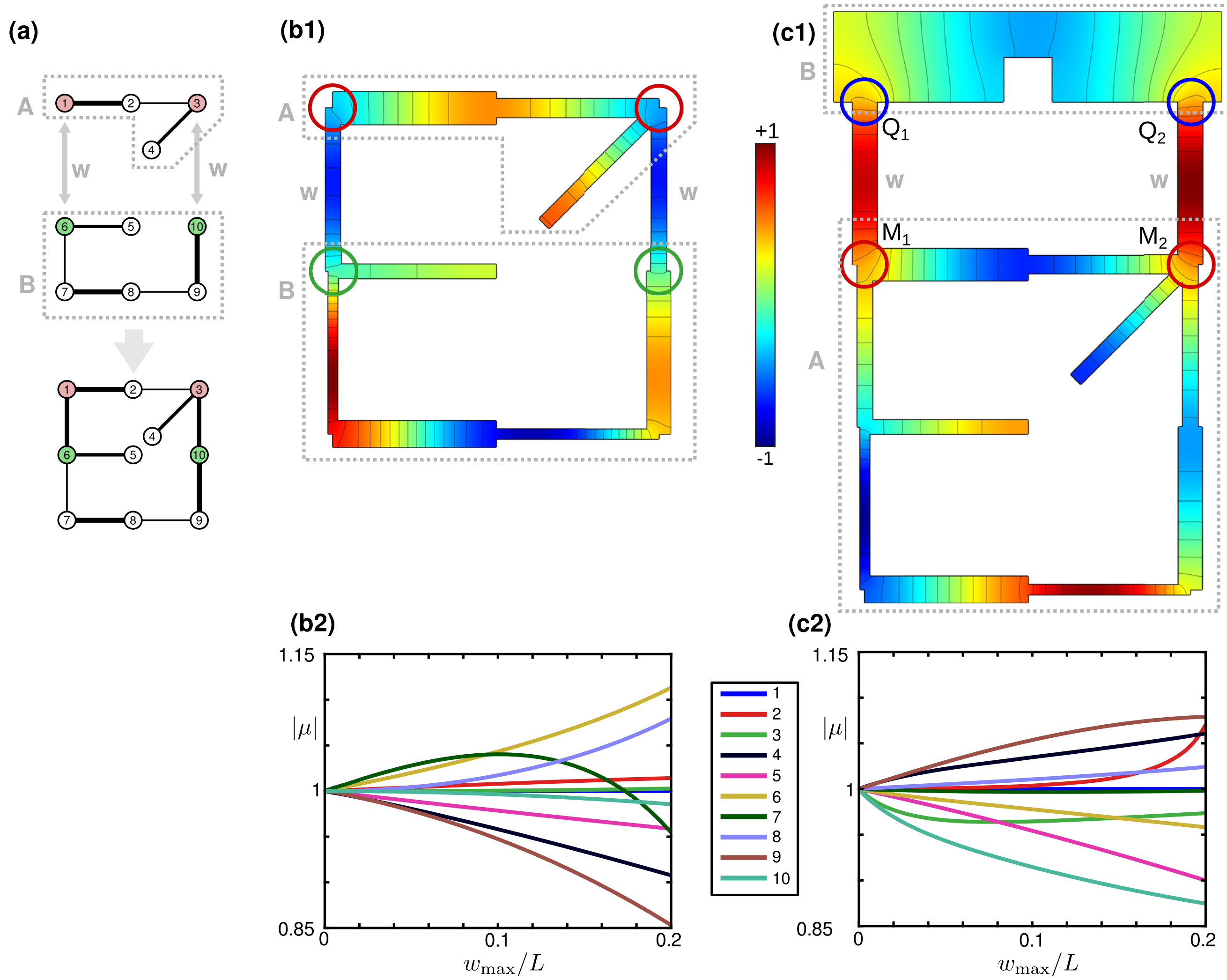

FIG. 2. (a) Connecting two networks A and B with latent mirror symmetries (in A for $\{1,3\}$, in B for $\{6,10\}$) through the corresponding latent symmetry points (see text for details). For the latent symmetry in B, the widths need to be chosen as $w_{5,6} = w_{9,10} - w_{6,7}$ and $w_{8,9} = \frac{w_{6,7}w_{7,8}w_{9,10}}{w_{5,6}w_{7,8}+w_{6,7}w_{9,10}}$. (b1) shows the ninth eigenmode of the network constructed by combining A and B in the above manner. The depicted eigenmode features pointwise parity on the junction points $\{1,3\}$ and $\{6,10\}$ (center points of the upper/lower circles, respectively). (c1) shows the tenth eigenmode of the system that is obtained by augmenting (b) with a cavity on top. This mode features domain-wise parity in the coupled-cavity subsystem. (b2) and (c2) show the absolute values of the amplitude ratio μ (see text) on the center of the red circles in the corresponding setup. For both (b1) and (c1), we have $w_{\max}/L = 0.2$.

this principle. Here, A denotes the first network, with $S_1 = \{1,3\}$, which is in fact the three-waveguide network we already encountered in Fig. 1. B denotes a second network of five waveguides which—by finding suitable widths and $S = \{u, v\}$ fulfilling Eq. (8)—has been designed to feature a LMS for $S_2 = \{6,10\}$. Figure 2(b) shows an eigenmode of the resulting setup featuring pointwise parity for all low-frequency eigenmodes *both* on S_1 and S_2, as predicted. The above principle can be repeated by analogously connecting a third network with $S_3 = \{u_3, v_3\}$ to either u_1, v_1 or u_2, v_2. A fourth network can then be connected to either of the three S_i, and so on, ultimately arriving at a modular construction principle.

Domain-wise parity.—Instead of coupling two latently symmetric networks, as done in Figs. 2(a) and 2(b), we could just as well couple a subsystem B with a *conventional* global geometrical symmetry to a latently symmetric network A. Interestingly, as we now demonstrate in Fig. 2(c), this can even be done when B is no longer a network of thin waveguides but a *spatially extended* setup. In that figure, B is an extended, mirror-symmetric cavity, while A corresponds to the setup from Fig. 2(b).

To understand the outcome of this procedure, let us investigate the composite system of the waveguide network (ending at points $M_{1,2}$) and the two waveguides w which end at the two points $Q_{1,2}$. Because of latent symmetry, all eigenmodes of this setup have parity, both on $M_{1,2}$ and on $Q_{1,2}$. When connecting this composite setup to the extended cavity, the latter "sees" only a two-port setup with an impedance relation $\mathbf{p} = Z\mathbf{p}'$, where the two-dimensional vectors $\mathbf{p}, \mathbf{p}'$ denote the pressure and normal derivative, respectively, at the two points Q_1, Q_2. Now, as shown in the Supplemental Material [23], in the low-frequency approximation we have $Z_{11}(k) = Z_{22}(k)$ *for all* k. As a result, the eigenmodes of the entire interconnected geometry have definite parity in the complete subsystem B. What is unexpected about this example is that the eigenmodes display this parity even though the geometry of the overall network is not symmetric. The *domain-wise* parity observed in Fig. 2(c) is an interesting extension of the other case examples shown in this work, whose eigenmodes featured only pointwise parity.

Similarly to our first setup of Fig. 1, the observed parity is robust and it remains approximately valid even for the case of waveguides that are not so thin. This is demonstrated in Figs. 2(b2) and 2(c2), where we show the absolute value $|\mu|$ of the pressure ratio $\mu = p_u^{(\alpha)}/p_v^{(\alpha)}$ for the first 10 eigenmodes, with u, v denoting the center points of the two red circles in Figs. 2(b1) and 2(c1).

Concluding remarks.—Geometrical symmetries form the basis of regularities and order in wave patterns. We have demonstrated that pointwise or even domain-wise parities can be systematically introduced in correspondingly asymmetric acoustic networks in their low-frequency eigenmodes. The origins of this behavior are hidden or latent symmetries which can be revealed by an effective Hamiltonian approach. This constitutes the basis for the design of networks with multiple latent symmetries. By putting symmetric or antisymmetric point sources where the eigenmodes feature latent-symmetry induced pointwise parity, one may control the symmetry properties of the wave field. Within this perspective, when opening up the waveguide network one can imagine to control, e.g., the reflection coefficients of the corresponding multiport scattering setup in the low-frequency regime [37]. In a more far-reaching perspective latent symmetries might be generalized to PT symmetric or topological waves.

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