Optimal Strategies of Quantum Metrology with a Strict Hierarchy

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One of the main quests in quantum metrology is to attain the ultimate precision limit with given resources, where the resources are not only of the number of queries, but more importantly of the allowed strategies. With the same number of queries, the restrictions on the strategies constrain the achievable precision. In this Letter, we establish a systematic framework to identify the ultimate precision limit of different families of strategies, including the parallel, the sequential, and the indefinite-causal-order strategies, and provide an efficient algorithm that determines an optimal strategy within the family of strategies under consideration. With our framework, we show there exists a strict hierarchy of the precision limits for different families of strategies.

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Introduction.—Quantum metrology [1,2] features a series of promising applications in the near future [3]. In the prototypical setting of quantum metrology, the goal is to estimate an unknown parameter carried by a quantum channel, given N queries to it. A pivotal task is to design a *strategy* that utilizes these N queries to generate a quantum state with as much information about the unknown parameter as possible. This often involves, for example, preparing a suitable input probe state [4–6] and applying intermediate quantum control [7–10] as well as quantum error correction [11–14].

In reality, the implementation of strategies is subject to physical restrictions. In particular, within the noisy and intermediate-scale quantum (NISQ) era [15], we have to adjust the strategy to accommodate the limitations on the system. For example, for systems with short coherence time it might be favorable to adopt the parallel strategy [Fig. 1(a)], where multiple queries of the unknown channel are applied simultaneously on a multipartite entangled state [4]. When the system has longer coherence time and can be better controlled, one could choose to query the channel sequentially [Fig. 1(b)], which may potentially enhance the precision. In addition to the parallel and sequential strategies, it was recently discovered that the quantum SWITCH [16], a primitive where the order of making queries to the unknown channel is in a quantum superposition [Fig. 1(c)], can be employed to generate new strategies of quantum metrology [17–19] that may even break the Heisenberg limit [19]. Moreover, indefinite causal structures beyond the quantum SWITCH [16,20,21] [Figs. 1(d) and 1(e)] have recently been shown to further boost the performance of certain information processing tasks [22,23]. The ultimate performance of these strategies in quantum metrology, however, remains unknown. This is mainly due to the lack of a systematic method that optimizes the probe state, the control and other degrees of freedom in a strategy in a unified fashion which leads to the ultimate precision limit.

In this Letter, we develop a semidefinite programming (SDP) method of evaluating the optimal precision of single-parameter quantum metrology for finite N (which we call *the nonasymptotic regime*) over a family of admissible strategies. With this method, we show a strict hierarchy (see Fig. 2) of the optimal performances under different families of strategies, which include the parallel, the sequential, and the indefinite-causal-order [16,20,21] ones (see Fig. 1).

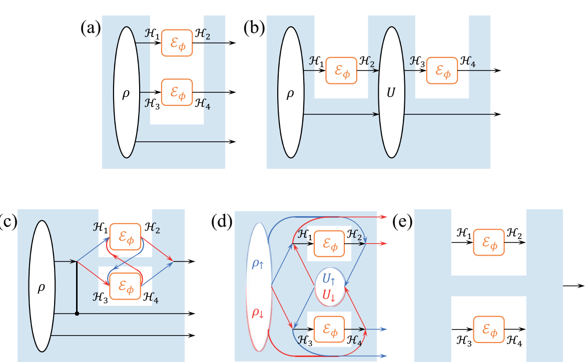

FIG. 1. Prototypical strategies of quantum metrology (for the $N = 2$ case). $\mathcal{E}_\phi$ is a quantum channel carrying an unknown parameter ϕ, and the blue shaded area represents a *strategy.* (a) A parallel strategy. (b) A sequential strategy, where U is a control operation. (c) A quantum SWITCH strategy. The blue and red lines, respectively, correspond to two different execution orders entangled with a control qubit. (d) A causal superposition strategy. Two sequential strategies, plotted in blue and in red, respectively, are entangled with a control qubit (not shown in the figure) and the output will be measured with the control qubit collectively. (e) A general indefinite-causal-order strategy.

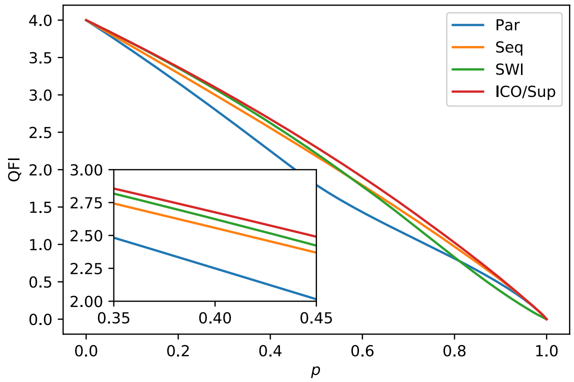

FIG. 2. Hierarchy of QFI using parallel, sequential, and indefinite-causal-order strategies. We take $N = 2$ and $\phi = 1.0$, fix $t = 1.0$ and vary the decay parameter p. The gaps can be seen more clearly by zooming in on the interval [0.35, 0.45] of the value of p. $J^{(\mathsf{Sup})} = J^{(\mathsf{ICO})}$ with an error tolerance of no more than 10^{-8} in this case, but the gap between $J^{(\mathsf{Sup})}$ and $J^{(\mathsf{ICO})}$ could exist for larger N or for other types of noise, which can be observed by randomly sampling noise channels.

In conjunction, we design an algorithm to obtain an optimal strategy achieving the highest precision. For the strategy set that admits a symmetric structure, we develop a method of reducing the complexity of our algorithms by an exponential factor.

Quantum Fisher information.—The uncertainty $\delta\hat{\phi}$ of estimating an unknown parameter ϕ encoded in a quantum state ρ_ϕ, for any unbiased estimator $\hat{\phi}$, can be determined via the quantum Cramér-Rao bound (QCRB) as $\delta\hat{\phi} \geq 1/\sqrt{\nu J_Q(\rho_\phi)}$ [24–26], where $J_Q(\rho_\phi)$ is the quantum Fisher information (QFI) of the state ρ_ϕ and ν is the number of repeated measurements [27]. For single-parameter estimation, the QCRB is achievable, and the QFI thus quantifies the amount of information that can be extracted from the quantum state. One way to compute the QFI is [28,29]

$$J_Q(\rho_\phi) = 4 \min_{\mathrm{Tr}_A(|\Psi_\phi\rangle\langle\Psi_\phi|)=\rho_\phi} \langle\dot{\Psi}_\phi|\dot{\Psi}_\phi\rangle, \tag{1}$$

where $|\Psi_\phi\rangle$ is the purification of ρ_ϕ with an ancillary space $\mathcal{H}_A$, Tr_A denotes the partial trace over $\mathcal{H}_A$, and $\dot{\Psi} \coloneqq \partial\Psi/\partial\phi$. When the parameter is carried by a quantum channel $\mathcal{E}_\phi$, i.e., a completely positive trace-preserving (CPTP) map, the channel QFI can be defined as the maximal QFI of output states using the optimal input assisted by arbitrary ancillae [28,30–32]: $J_Q^{(\mathrm{chan})}(\mathcal{E}_\phi) = \max_{\rho_{\mathrm{in}}\in\mathcal{S}(\mathcal{H}_S\otimes\mathcal{H}_A)} J_Q[(\mathcal{E}_\phi \otimes \mathcal{I}_A)(\rho_{\mathrm{in}})]$, where $\mathcal{S}(\mathcal{H})$ denotes the space of density operators on the Hilbert space $\mathcal{H}$, $\mathcal{H}_{S/A}$ denotes the Hilbert space of the system or ancillae, and $\mathcal{I}_A$ is the identity on $\mathcal{H}_A$.

We denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators on the finite-dimensional Hilbert space $\mathcal{H}$, and $\mathcal{L}[\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{H}_2)]$ denotes the set of linear maps from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{H}_2)$. By the Choi-Jamiołkowski (CJ) isomorphism, a parametrized quantum channel $\mathcal{E}_\phi \in \mathcal{L}[\mathcal{L}(\mathcal{H}_{2i-1}), \mathcal{L}(\mathcal{H}_{2i})]$ (for $1 \leq i \leq N$) can be represented by a positive semidefinite operator (called the CJ operator) $E_\phi = \mathsf{Choi}(\mathcal{E}_\phi) = \mathcal{E}_\phi \otimes \mathcal{I}(|I\rangle\rangle\langle\langle I|)$, where $|I\rangle\rangle = \sum_i |i\rangle|i\rangle$. The CJ operator of N identical quantum channels is $N_\phi = E_\phi^{\otimes N} \in \mathcal{L}(\otimes_{i=1}^{2N} \mathcal{H}_i)$.

Strategy set in quantum metrology.—A *strategy* is an arrangement of physical processes (the blue shaded area in Fig. 1) which, when concatenated with given queries to $\mathcal{E}_\phi$, generates an output quantum state carrying the information about ϕ. A strategy can be described by a CJ operator on $\mathcal{L}(\mathcal{H}_F \otimes_{i=1}^{2N} \mathcal{H}_i)$, where $\mathcal{H}_F$ denotes the output Hilbert space of the concatenation, referred to as the *global future* space. The concatenation of two processes is characterized by the link product [33,34] of two corresponding CJ operators $A \in \mathcal{L}(\otimes_{a\in\mathcal{A}} \mathcal{H}_a)$ and $B \in \mathcal{L}(\otimes_{b\in\mathcal{B}} \mathcal{H}_b)$ as

$$A * B \coloneqq \mathrm{Tr}_{\mathcal{A}\cap\mathcal{B}}[(\mathbb{1}_{\mathcal{B}\setminus\mathcal{A}} \otimes A^{T_{\mathcal{A}\cap\mathcal{B}}})(B \otimes \mathbb{1}_{\mathcal{A}\setminus\mathcal{B}})], \tag{2}$$

where T_i denotes the partial transpose on $\mathcal{H}_i$, and $\mathcal{H}_{\mathcal{A}/\mathcal{B}}$ denotes $\otimes_{i\in\mathcal{A}/\mathcal{B}} \mathcal{H}_i$. The output state lies in the global future F, which should not affect any state in the past.

Following the above formalism, given a sufficiently large ancillary Hilbert space, a *strategy set* determined by the relevant causal constraints is described by a subset P of

$$\mathsf{Strat} \coloneqq \{P \in \mathcal{L}(\mathcal{H}_F \otimes_{i=1}^{2N} \mathcal{H}_i) | P \geq 0, \mathsf{rank}(P) = 1\}. \tag{3}$$

Here, without loss of generality [35], we have restricted P to pure processes (rank-1 operators) due to the monotonicity of QFI [36]. Our goal is to identify the ultimate precision limit of parameter estimation characterized by the QFI within such constraints:

Definition 1: The QFI of N quantum channels $\mathcal{E}_\phi$ [37] given a strategy set P is

$$J^{(\mathsf{P})}(N_\phi) \coloneqq \max_{P\in\mathsf{P}} J_Q(P * N_\phi), \tag{4}$$

where $J_Q(\rho)$ is the QFI of the state ρ, and N_ϕ is the CJ operator of N channels.

In general we can write the ensemble decomposition [28] of the CJ operator N_ϕ as $N_\phi = \sum_{i=1}^r |N_{\phi,i}\rangle\langle N_{\phi,i}| = \mathbf{N}_\phi \mathbf{N}_\phi^\dagger$, where $\mathbf{N}_\phi \coloneqq (|N_{\phi,1}\rangle, \ldots, |N_{\phi,r}\rangle)$ and $r \coloneqq \max_\phi \mathsf{rank}(N_\phi)$. We also define $\dot{\tilde{\mathbf{N}}}_\phi \coloneqq \dot{\mathbf{N}}_\phi - \mathrm{i}\mathbf{N}_\phi h$ for $h \in \mathbb{H}_r$, where $\mathbb{H}_r$ denotes the set of $r \times r$ Hermitian matrices, and the performance operator [38]

$$\Omega_\phi(h) := 4(\dot{\tilde{\mathbf{N}}}_\phi \dot{\tilde{\mathbf{N}}}_\phi^\dagger)^T. \quad (5)$$

With these notions, we can show (see Ref. [39], which is analogous to the approach in [38]) that the QFI admits the form

$$J^{(\mathsf{P})}(N_\phi) = \max_{\tilde{P}\in\tilde{\mathsf{P}}} \min_{h\in\mathbb{H}_r} \mathrm{Tr}[\tilde{P}\Omega_\phi(h)], \quad (6)$$

with

$$\tilde{\mathsf{P}} := \{\tilde{P} = \mathrm{Tr}_F P | P \in \mathsf{P}\}. \quad (7)$$

To evaluate the QFI, we first exchange $\max_{\tilde{P}}$ and $\min_h$ without changing the optimal QFI, assured by the minimax theorem [58,59], since the objective function is concave on $\tilde{P}$ and convex on h [60]. Hence, the problem is cast into

$$J^{(\mathsf{P})}(N_\phi) = \min_h \max_{\tilde{P}\in\tilde{\mathsf{P}}} \mathrm{Tr}[\tilde{P}\Omega_\phi(h)]. \quad (8)$$

Then we fix h and formulate the dual problem of maximization over the set $\tilde{\mathsf{P}}$. Finally we further optimize the value of h. To simplify the calculation of QFI we require that

$$\tilde{\mathsf{P}} = \mathsf{Conv}\left\{\bigcup_{i=1}^{K} \{S^i \geq 0 | S^i \in \mathsf{S}^i\}\right\}, \quad (9)$$

where $\mathsf{Conv}\{\cdot\}$ denotes the convex hull, and each S^i for $i = 1, \ldots, K$ is an affine space of Hermitian operators. Adopting the above-mentioned method, we get

Theorem 1.—Given an arbitrary strategy set P such that $\tilde{\mathsf{P}}$ given by Eq. (7) satisfies the condition Eq. (9), the QFI of N quantum channels $\mathcal{E}_\phi$ can be expressed as the following optimization problem:

$$J^{(\mathsf{P})}(N_\phi) = \min_{\lambda, Q^i, h} \lambda,$$
$$\text{s.t. } \lambda Q^i \geq \Omega_\phi(h), \quad Q^i \in \bar{\mathsf{S}}^i, \quad i = 1, \ldots, K, \quad (10)$$

where $\bar{\mathsf{S}}^i := \{Q \text{ is Hermitian} | \mathrm{Tr}(QS) = 1, S \in \mathsf{S}^i\}$ is the dual affine space of S^i.

The proof can be found in [39]. We remark that similar optimization ideas have been applied to other tasks, such as quantum Bayesian estimation [61], quantum network optimization [62], non-Markovian quantum metrology [38], and quantum channel discrimination [23]. The minimization problem in Theorem 1 can be further written in the form of SDP and solved efficiently, with detailed numerically solvable forms given in [39], where the constraints in Eq. (10) can be further simplified in some cases.

Optimal strategies.—By itself, the QFI does not reveal how to implement the optimal strategy achieving the highest precision. Here, in addition to Theorem 1, we design an algorithm that yields a strategy attaining the optimal QFI for any strategy set satisfying Eq. (9). The method, which generalizes the method of finding an optimal probe state for a single channel [63,64], is summarized as Algorithm 1 (see Ref. [39] for its derivation).

Algorithm 1. Find an optimal strategy in the set P.

(i) Given N_ϕ the CJ operator of N channels, solve for an optimal value $h = h^{(\mathrm{opt})}$ in Eq. (10) of Theorem 1 via SDP.

(ii) Fixing $h = h^{(\mathrm{opt})}$, solve for an optimal value $\tilde{P}^{(\mathrm{opt})}$ of $\tilde{P} \in \tilde{\mathsf{P}}$ in Eq. (6) via SDP such that

$$\mathrm{Re}\{\mathrm{Tr}\{\tilde{P}^{(\mathrm{opt})}[-\mathrm{i}\mathbf{N}_\phi \mathscr{H}(\dot{\mathbf{N}}_\phi - \mathrm{i}\mathbf{N}_\phi h^{(\mathrm{opt})})^\dagger]^T\}\} = 0$$
$$\text{for all } \mathscr{H} \in \mathbb{H}_r, \quad (11)$$

where $\mathbf{N}_\phi := (|N_{\phi,1}\rangle, \ldots, |N_{\phi,r}\rangle)$. An optimal strategy $P^{(\mathrm{opt})} \in \mathsf{P}$ can be taken as a purification of $\tilde{P}^{(\mathrm{opt})}$.

By Algorithm 1 we obtain the CJ operator of a strategy that attains the optimal QFI. For strategies following definite causal order, there exists an operational method of mapping the CJ operator of the strategy to a probe state and a sequence of in-between control operations with minimal memory space [65]. For causal order superposition strategies (see the strategy set Sup), we show that they can always be implemented by controlling the order of operations in a circuit with a quantum SWITCH [39]. In this way, we obtain a systematic method to identify optimal sequential and causal superposition strategies, one of the key problems in quantum metrology.

Strategy sets.—We consider the evaluation of QFI for five different families of strategies. In all the following definitions the subscript i of an operator denotes the Hilbert space $\mathcal{H}_i$ it acts on.

The family of parallel strategies [see Fig. 1(a)] is the first and one of the most successful examples of quantum-enhanced metrology, featuring the usage of entanglement to achieve precision beyond the classical limit [66]. By making parallel use of N quantum channels together with ancillae, we can regard these N channels as one single channel from $\mathcal{L}(\otimes_{i=1}^N \mathcal{H}_{2i-1})$ to $\mathcal{L}(\otimes_{i=1}^N \mathcal{H}_{2i})$. A parallel strategy set Par is defined as the collection of $P \in \mathsf{Strat}$ such that [34]

$$\mathrm{Tr}_F P = \mathbb{1}_{2,4,\ldots,2N} \otimes P^{(1)}, \quad \mathrm{Tr} P^{(1)} = 1. \quad (12)$$

Note that the optimal QFI of parallel strategies can also be evaluated using the method in [28,30].

A more general protocol is to allow for sequential use of N channels assisted by ancillae, where only the output of the former channel can affect the input of the latter channel, and any control gates can be inserted between channels [see Fig. 1(b)]. A sequential strategy set Seq is defined as the collection of $P \in \mathsf{Strat}$ such that [34]

$$\mathrm{Tr}_F P = \mathbb{1}_{2N} \otimes P^{(N)}, \qquad \mathrm{Tr} P^{(1)} = 1,$$
$$\mathrm{Tr}_{2k-1} P^{(k)} = \mathbb{1}_{2k-2} \otimes P^{(k-1)}, \qquad k = 2, \ldots, N. \tag{13}$$

Unlike the case of parallel strategies, there is no existing way of evaluating the exact QFI using sequential strategies.

We also consider families of strategies involving indefinite causal order. The first one, denoted by SWI, takes advantage of the (generalized) quantum SWITCH [67,68], where the execution order of N channels is entangled with the state of an $N!$-dimensional control system [see Fig. 1(c)]. See Ref. [39] for the formal definition.

More generally, we consider the quantum superposition of multiple sequential orders, each with a unique order of querying the N channels [see Fig. 1(d)]. This can be implemented by entangling $N!$ definite causal orders with a quantum control system [69]. If $N = 2$ and the control system is traced out, this notion is equivalent to causal separability [20,21]. A causal superposition strategy set Sup is defined as the collection of $P \in$ Strat such that

$$\mathrm{Tr}_F P = \sum_{\pi} q^{\pi} P^{\pi}, \qquad \sum_{\pi \in S_N} q^{\pi} = 1,$$
$$P^{\pi} \in \mathsf{Seq}^{\pi}, \qquad q^{\pi} \geq 0, \qquad \pi \in S_N, \tag{14}$$

where each permutation π is an element of the symmetric group S_N of degree N, and each Seq^{π} denotes a sequential strategy set whose execution order of N channels is $\mathcal{E}_{\phi}^{\pi(1)} \to \mathcal{E}_{\phi}^{\pi(2)} \to \cdots \to \mathcal{E}_{\phi}^{\pi(N)}$, having denoted by $\mathcal{E}_{\phi}^{k}$ the channel from $\mathcal{L}(\mathcal{H}_{2k-1})$ to $\mathcal{L}(\mathcal{H}_{2k})$. Note that SWI is a subset of Sup, where the intermediate control is trivial. There are other strategies, such as quantum circuits with quantum controlled casual order (QC-QCs) and probabilistic QC-QCs [69,70], which we will not discuss here.

Finally, we introduce the family of general indefinite-causal-order strategy [see Fig. 1(e)], which is the most general strategy set considered in this Letter. Here the only requirement is that the concatenation of the strategy P with N arbitrary channels results in a legitimate quantum state. The causal relations in this case [21] are a bit cumbersome, but for our purpose what matters is the dual affine space (see Theorem 1), which is simply the space of no-signaling channels [16,62]. A general indefinite-causal-order strategy set ICO is defined as the collection of $P \in$ Strat such that

$$\rho_F = P * \left(\otimes_{j=1}^{N} E^j\right), \quad \rho_F \geq 0, \quad \mathrm{Tr}\rho_F = 1, \tag{15}$$

for any $E^j \in \mathcal{L}(\mathcal{H}_{2j-1} \otimes \mathcal{H}_{2j} \otimes \mathcal{H}_{A_j})$ that denotes the CJ operator of an arbitrary quantum channel with an arbitrary ancillary space $\mathcal{H}_{A_j}$.

We note that, unlike the previous strategies that can always be physically realized, the physical realization of the general ICO is untraceable [69,70]. The optimal value obtained with general ICO nevertheless serves as a useful tool that can gauge the performances of different strategies. For example, as we will show, in some cases the optimal QFI $J^{(\mathsf{Sup})}$ and $J^{(\mathsf{ICO})}$ are equal or nearly equal. This then shows that the physically realizable strategy obtained from the set Sup is already optimal or nearly optimal among all possible strategies, which we will not be able to tell without $J^{(\mathsf{ICO})}$.

Symmetry reduced programs for optimal metrology.—The complexity of the original optimization problems in Theorem 1 and Algorithm 1 can be reduced by exploiting the permutation symmetry. In [39], we prove that we can choose a permutation-invariant matrix h for Theorem 1 and solve for a permutation-invariant optimal strategy [71] by Algorithm 1 based on this choice, if any permutation $\pi \in S_N$ bijectively maps each affine space S^i [in Eq. (9)] to some affine space S^j. That is, for any $\pi \in S_N$ and any i, there exists a j such that the mapping $S \mapsto G_{\pi} S G_{\pi}^{\dagger}$ on S^i is a bijective function from S^i to S^j, where G_{π} is a unitary representation of π. Furthermore, if each space S^i itself is permutation invariant, we can restrict each $Q^i \in \bar{\mathsf{S}}^i$ to be permutation invariant, further reducing the complexity of optimization. For both optimization problems we can apply the technique of group-invariant SDP to reduce the size as there exists an isomorphism which preserves positive semidefiniteness, from the permutation-invariant subspace to the space of block-diagonal matrices [72] (Theorem 9.1). Table I compares the number of variables involved in QFI evaluation with and without exploiting the symmetry (see Ref. [39] for its derivation as well as the complexity of Algorithm 1, where by group-invariant SDP we also numerically evaluate the growth of QFI $J^{(\mathsf{ICO})}$ up to $N = 5$).

Hierarchy of strategies.—By substituting the definitions of different strategy sets into Theorem 1, we obtain the *exact* values of the optimal QFI. We find that a strict hierarchy of QFI exists quite prevalently. For demonstration purposes, here we show only the result for the amplitude damping channel for $N = 2$ and supplement our findings with bountiful numerical results in [39]. In this case, the process encoding ϕ is a z rotation $U_z(\phi) = e^{-\mathrm{i}\phi t \sigma_z/2}$, where t is the evolution time, followed by an amplitude damping channel described by two Kraus operators:

TABLE I. Complexity of QFI evaluation for each family of strategies (with respect to N). The asymptotic numbers of variables in optimization are compared between the original (Ori.) and group-invariant (Inv.) SDP. We denote $d \coloneqq \mathrm{dim}(\mathcal{H}_1)\,\mathrm{dim}(\mathcal{H}_2)$ and $s \coloneqq \max_{\phi} \mathrm{rank}(E_{\phi}) \leq d$.

SDP	Par	Seq	SWI	Sup	ICO
Ori.	$O(s^N)$	$O(d^N)$	$O(s^N)$	$O(N!d^N)$	$O(d^N)$
Inv.	$O(N^{d^2-1})$	$O(d^N)$	$O(N^{s^2-1})$	$O(d^N)$	$O(N^{d^2-1})$

$K_1^{(\mathrm{AD})} = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$ and $K_2^{(\mathrm{AD})} = \sqrt{p}|0\rangle\langle 1|$, with the decay parameter p.

In Fig. 2 we plot the QFI versus p for the amplitude damping noise with all 5 strategy sets for $N = 2$. A strict hierarchy of Par, Seq and ICO holds if p is neither 1 nor 0, i.e., $J^{(\mathsf{Par})} < J^{(\mathsf{Seq})} < J^{(\mathsf{ICO})}$. This is in contrast to the asymptotic regime of $N \to \infty$, where the relative difference between $J^{(\mathsf{Seq})}$ and $J^{(\mathsf{Par})}$ vanishes for this channel [64]. Besides, in this case general ICO cannot strictly outperform Sup, implying that causally superposing two sequential strategies is sufficient to achieve the general optimality in this particular scenario. The gap between $J^{(\mathsf{Sup})}$ and $J^{(\mathsf{ICO})}$, however, could be observed for the same channel with larger N or for other channels when $N = 2$ [39]. In fact, by randomly sampling noise channels from CPTP channel ensembles, we find that for 984 of 1000 random channels, a strict hierarchy $J^{(\mathsf{Par})} < J^{(\mathsf{Seq})} < J^{(\mathsf{Sup})} < J^{(\mathsf{ICO})}$ holds for $N = 2$, implying that there exist more powerful strategies than causal superposition strategies in these cases. We note that a strict hierarchy of strategies has been found for channel discrimination in [23], but much less is known in quantum metrology until our Letter.

Our method can also test the tightness of existing QFI bounds in the nonasymptotic regime, which has seldom been done until this Letter. Here we take the commonly used, asymptotically tight [64] upper bound for parallel strategies [see Ref. [28] (Theorem 4) or [30] [Eq. (16)]]. For $p = 0.5$, our result shows that the exact parallel QFI $J^{(\mathsf{Par})} = 1.795$ is 32.7% lower than the asymptotically tight parallel upper bound 2.667, and even the exact sequential QFI $J^{(\mathsf{Seq})} = 2.179$ is 18.3% lower than this parallel upper bound [73]. Similar phenomena are observed in other noise models and for different N (see Ref. [39]).

With Algorithm 1 we can also construct strategies to achieve the optimal QFI. Remarkably, we find that a simple strategy of applying a quantum SWITCH using a control qubit $|+\rangle_C \coloneqq (|0\rangle_C + |1\rangle_C)/\sqrt{2}$ (without any additional control operations on the probe) beats any sequential strategies (which can involve complex control) in certain cases (e.g., $p < 0.5$). To our best knowledge, this is the only instance of noisy quantum metrology so far, where the advantage of indefinite causal orders is established rigorously. In [39] we also present two explicit examples of implementing optimal sequential and causal superposition strategies, obtained by first applying Algorithm 1 and converting the CJ operators into quantum circuit consisting of single-qubit rotations and CNOT gates (as well as a quantum SWITCH for the case of Sup). For optimal causal superposition strategies, the permutation symmetry allows us to only control the execution order of channels while fixing state preparation and intermediate control [$\rho_\uparrow = \rho_\downarrow$ and $U_\uparrow = U_\downarrow$ in Fig. 1(d)], which can be implemented by a $(2N-1)$-quantum SWITCH of N channels $\mathcal{E}_\phi$ and $N-1$ intermediate operations.

Our result serves as a versatile tool for the demonstration of optimal quantum metrology and the design of optimal quantum sensors, especially in the context of control optimization [74,75] and indefinite causal orders [16–23].

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