Oberlack *et al.* Reply: Comment [1] proposes the approximation of the scaling laws developed in [2] based on symmetry analyses for the instantaneous moments H by neglecting moments for the fluctuations R. In this case, not only the key scaling is lost, but very large deviations from the direct numerical simulation (DNS) data are obtained for high moments. Comment [3] proposes to compare the R moments, which result directly from the above scaling laws of the H moments, with the DNS data. This approach is meaningless because uncertainty for R moments $n > 2$ grow exponentially due to finite DNS datasets.

All complete theories of turbulence, i.e., the Lundgren probability density functions hierarchy [4], the Hopf functional equation [5], and the characteristic function equations by Monin [6] are based on the full instantaneous velocities with the distinct advantage that all these equations are linear. This immediately implies classical and, in a certain sense, generic symmetries of linear equations. However, they go beyond those of the Navier-Stokes equations and represent statistical properties whose deep physical meaning was worked out in detail by Oberlack and co-workers in the following papers [2,7–9].

Both [1] and [3] suggest approximating the H moment scaling laws in [2] using the mean velocity $\bar{U}_1$ and R moments, where, following [7], we call the moments based on the instantaneous velocities H moments and those based on the fluctuations R moments. Indeed, the nth power of $\bar{U}_1$ is of the same order of magnitude as the nth H moment. However, both authors do not recognize that the absolute magnitude of the numerical values makes no statement about the validity of a scaling law. The indicator function is the most sensitive parameter for assessing logarithmic or algebraic scaling laws. For this, we compare the log-region scaling law (16) in [2] for the moments $n \geq 2$ using the power-law indicator function (18). In Fig. 1, we compare the indicator function, which with the symmetry-based scaling law naturally leads to constant exponents $\omega(n-1)$ and which matches DNS data in the log region very well. However, the nth power of the mean velocity in Fig. 1 apparently is a poor approximation. This becomes especially clear for large moments, where the value deviates significantly from the theoretical Γ_n and is not constant.

The mentioned approximation of the H moments breaks down if the mean velocity vanishes. However, the symmetry theory is completely independent of whether the moments contain the mean velocity. This can be seen by considering the moments of U_2 and U_3 in the channel center because almost identical deficit scaling laws apply to them as for the U_1 velocity, i.e.,

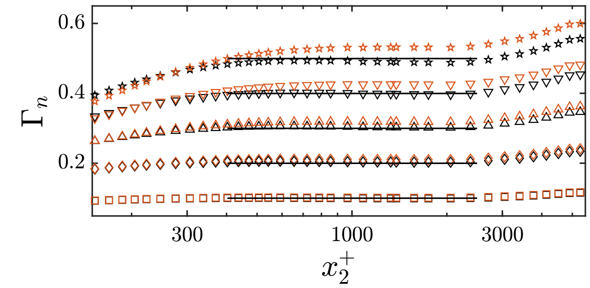

FIG. 1. Comparison of H moments of U_1 using the indicator function for $n = 2, \ldots, 6$ from DNS [10] (black points), the symmetry-based theory, i.e., $\overline{U_1^n}^+ = C_n(y^+)^{\omega(n-1)} - B_n$ in [2] (black lines), and an approximation via the power of the mean velocity $\bar{U}_1^n$ (orange points) as suggested in [3].

$$\frac{\overline{U_i^n} - \overline{U_i^n}^{(0)}}{u_\tau^n} = C'_{i,n}\left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2} \quad \text{with}$$
$$i = 2, 3 \quad \text{and} \quad C'_{i,n} = \alpha'_i e^{\beta'_i n}. \tag{1}$$

Above, $\overline{U_i^n}^{(0)}$ defines the nth moment on the centerline.

The data for this were already available to us since the end of the DNS runs in [10]. However, we deliberately did not publish them because the statistical convergence is less good than for the U_1 moments. The above laws (1) with exactly the same numbers for the exponents as in [2] are compared with DNS data in Fig. 2. The agreement speaks for itself, though a slight waviness due to insufficient convergence is visible.

Symmetry theory has evidently made a matching prediction for the scaling of U_2 and U_3 moments, which is founded inherently on the statistical symmetries. Both authors of [1] and [3] apparently misinterpret the concept of symmetries and statistical symmetries, which also leads to the fundamentally incorrect statement in [3] that "the scaling laws in [2] are not solutions to the statistical Navier-Stokes equations."

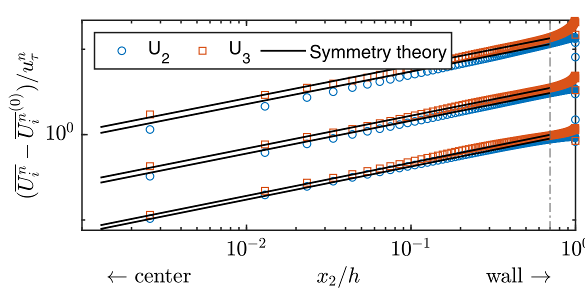

FIG. 2. Deficit scaling plot of the moments of U_2 and U_3 in the channel center for the H moment orders $n = 2, 4, 6$ according to formula (1). For a better visibility, the curves are each shifted by 10^{n-2}. The lines refer to the scaling laws (1), where $\sigma_1 = 1.95$ and $\sigma_2 = 1.94$ taken from [2] for the U_1 moments were used unchanged. Squares and circles refer to the DNS data taken from the datasets published in [10].

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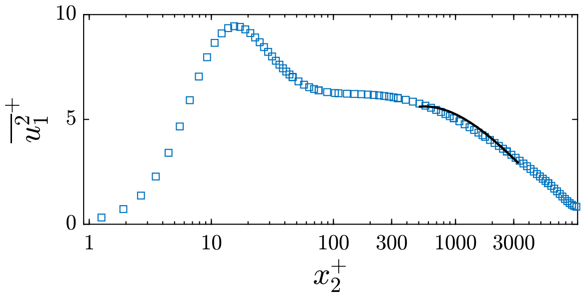

FIG. 3. $\overline{u_1^2}^+$: Squares represent the DNS data, and the line refers to the symmetry-based scaling law taken from [2], i.e., $\overline{u_1^2}^+ = C_2(x_2^+)^\omega - B_2 - [(1/\kappa)\ln(x_2^+) + B]^2$, where also the original values of the parameters were taken thereof.

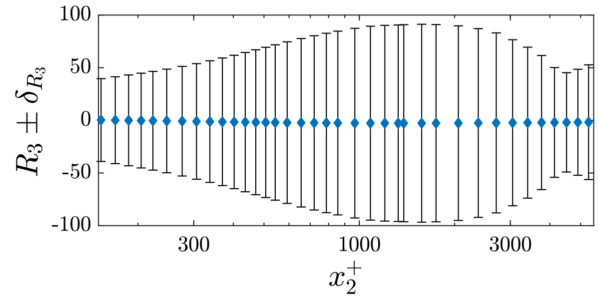

FIG. 4. The third moment of U_1 fluctuations, i.e., $R_3 = \overline{u_1^3}$, calculated from (3) based on the DNS data and the uncertainty estimate represented by error bars δ_{R_3} defined in (5).

Finally, we examine the proposal propagated, in particular, in [3] to analyze the R moments of fluctuations. For the second moment of the fluctuations R_2, symmetry theory and DNS agree reasonably well, as seen in Fig. 3. For R_2, the uncertainty of the DNS data, defined in (5), is still very small and below 2%, so there is sufficient reliability for this quantity.

However, in the following, we verify indubitably that this makes absolutely no sense for R moments with $n > 2$, as their uncertainties grow exponentially. To see this, consider the uncertainties of $\bar{U}_1$, the H and the R moments,

$$\bar{U}_1 \to \bar{U}_1 + \delta_{\bar{U}_1}, \qquad H_n \to H_n + \delta_{H_n}, \qquad R_n \to R_n + \delta_{R_n}. \tag{2}$$

The δ defines the uncertainty caused by a finite DNS dataset (see, e.g., [11]), and we have used a simplified notation for the H and R moments.

The H moments are the base values because only these can be calculated directly from the DNS data. The error bars for the H moments have been calculated according to [11]. The R moments uncertainty must be calculated from the H moments uncertainties in a second step. For this, we start with the relation between the moments, i.e.,

$$R_n = \sum_{k=0}^{n} (-1)^{n+1} \frac{n!}{k!(n-k)!} H_{n-k} \bar{U}_1^k = H_n + \cdots + (-1)^n (n-1) \bar{U}_1^n. \tag{3}$$

Substituting (2) into the formula of Gaussian uncertainty propagation, which, for this case, reads

$$\delta_{R_n} = \left|\frac{\partial R_n}{\partial H_n}\right| \delta_{H_n} + \left|\frac{\partial R_n}{\partial H_{n-1}}\right| \delta_{H_{n-1}} + \cdots + \left|\frac{\partial R_n}{\partial \bar{U}_1}\right| \delta_{\bar{U}_1}, \tag{4}$$

we obtain uncertainty measures for the R moments, i.e.,

$$\delta_{R_n} = \delta_{H_n} + \cdots + n(n-1)\bar{U}_1^{n-1}\delta_{\bar{U}_1}. \tag{5}$$

The leading-order uncertainty terms of the nth order R moments scale with the mean velocity to the power of $n-1$, i.e.,

$$\delta_{R_n} \sim \bar{U}_1^{n-1}\delta_{\bar{U}_1}. \tag{6}$$

Already for $n = 3$, we can see in Fig. 4 that the uncertainties are so large that a valid statement about R_3 is meaningless, and for higher n this is, of course, even more true. Hence, moments of order $n > 2$ derived from the fluctuations exhibit extreme uncertainties, and, obviously, moments from instantaneous velocities are preferable.

Finally, we argue that the high complexity of the equations of moments for the fluctuations impressively shows that these equations are ineffective as a basis. This is especially true for moments $n > 2$ of the fluctuations, as has become very clear above. Therefore, it is natural to employ the equations based on the instantaneous velocities, i.e., Lundgren [4], Hopf [5], and Monin [6] equations for any analysis and symmetry analysis to derive scaling laws thereof.

In summary, Comments [1] and [3] provide little that is valid and especially the suggestion of comparing DNS data with R moments for $n > 2$ is meaningless.

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