Linear Response and Fluctuation-Dissipation Relations for Brownian Motion under Resetting

Igor M. Sokolov *
Institut für Physik, Humboldt-Universität zu Berlin, Newtonstraße 15, D-12489 Berlin, Germany and IRIS Adlershof, Humboldt-Universität zu Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany

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We consider fluctuation-dissipation relations (FDRs) for a Brownian motion under renewal resetting with arbitrary waiting time distribution between the resetting events. We show that if the distribution of waiting times of the resetting process possesses the second moment, the usual (generalized) FDR and the equivalent generalized Einstein's relation (GER) apply for the response function of the coordinate. If the second moment of waiting times diverges but the first one stays finite, the static susceptibility diverges, the usual FDR breaks down, but the GER still applies. In any of these situations, the fluctuation dissipation relations define the effective temperature of the system which is twice as high as the temperature of the medium in which the Brownian motion takes place.

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Random processes under resetting have found applications for the description of various phenomena and processes in physics, chemistry, biophysics and biochemistry, in biology (e.g., in movement ecology), and also in computer science, see Refs. [1–3] for a review. A beautiful introduction is also given in the beginning of Ref. [4]. The topic of resetting is therefore currently under extensive investigation.

A random process under resetting (a reset process) is a combination of two random processes: the continuous-time displacement process $x(t)$ between resetting events, and a resetting process $\{t_i\}$, a point process on a non-negative real time line. In the simplest situation considered here, at $t = t_i$ the coordinate $x(t)$ is set back to $x = x_0$. The time-dependent coordinate of a reset process stemming from the displacement process $x(t)$ will be denoted by $X(t)$. For non-Markovian displacement processes, the resetting of the coordinate may or may not be accompanied by erasing the internal memory of $x(t)$ [5–7]. In the first case one speaks about the complete resetting, in the second case the resetting is incomplete. Incomplete resetting is also possible for a Markovian displacement process with more than one relevant variable [i.e., for vector-valued $\mathbf{x}(t)$]: some coordinates are reset, while other ones stay unaffected by the resetting event.

Some systems pertinent to physics or chemistry allow for manipulation of the displacement process by application of an external force F (which is easy, e.g., in colloidal systems). Experimental realizations of the resetting scheme in such systems were discussed in [8–10], and therefore the response of such a process to external forcing is amenable to experimental investigation. In what follows, we consider the simplest case when the displacement process is a one-dimensional Brownian motion (which only allows for complete resetting). Brownian motion (BM) under Poissonian resetting was the very first example, Ref. [11], which provoked the whole wave of interest to reset processes. At difference with [11], we consider the resetting process to be a generic ordinary renewal process with a continuous waiting time density $\psi(t)$ which is not necessarily exponential. The process starts with a resetting event at t_0. Our example is indeed simple but far from trivial.

Brownian motion is a nonstationary Markov process with stationary increments. Resetting may transform this nonstationary process into a stationary reset one, possessing a nonequilibrium steady state (NESS). This state is far from thermodynamic equilibrium (see, e.g., the discussion in Ref. [12]), and violates the time-reversal symmetry being a key property of the equilibrium state. Since the time-reversal symmetry, or, equivalently, the detailed balance condition, is often considered as a cornerstone of linear response theory and fluctuation-dissipation relations (FDRs), a question arises, what kinds of FDRs, if any, are valid for reset processes.

The situation under resetting is close to many other cases for FDRs in a NESS (see Refs. [12–15] for discussions), but has some specific features due to the fact that there is no general way to define the Hamiltonian of the whole system including an appliance (or a physical person, or a demon) performing the resetting. The general resetting scheme also precludes using approaches based on Fokker-Planck equations ([16,17], see Ref. [14] for more discussion) which cannot be put down for such schemes. The pathway to the FDR (of a rather naive sort) not relying on these types of description is given in the Supplemental Material (SM) [18]. The discussion of our simple example therefore sheds light not only on some intrinsic properties of reset processes, but also on some properties of FDRs out of equilibrium, which, up to my best knowledge, were not reported so far.

Linear response.—A linear response of the variable V to an external force F means that its mean value is a linear functional of F:

$$\langle V(t)\rangle = \int_{t_0}^{t} dt' \chi^*(t,t',t_0)F(t'). \quad (1)$$

The variable V is centered in such a way that $\langle V\rangle$ vanishes in the absence of the force. The response is defined by a kernel $\chi^*(t,t',t_0)$ of a general form, and may bear the dependence on the preparation time t_0 even if the system itself does not show any physical aging. If the system's properties do not change with time, the response is stationary; the kernel (the response function) does not depend on t_0, and is only a function of $t-t'$,

$$\langle V(t)\rangle = \int_{t_0}^{t} dt' \chi(t-t')F(t'). \quad (2)$$

This is exactly the form of response shown by systems close to equilibrium, in which case one can shift $t_0 \to -\infty$. The static susceptibility Ξ of the system, describing its long-time response to a constant force, $\langle V\rangle = \Xi F$, is given by $\Xi = \int_0^\infty \chi(t)dt$, provided the integral converges. If it diverges, the response $\langle V\rangle$ to the constant force grows indefinitely in time. We will call the first kind of response "elastic," and the second one "fluid."

The response relations, Eq. (2), with $t_0 \to -\infty$ are well known, e.g., from the electrodynamics of media, and come essentially in two different types: as a relation between the thermodynamically conjugated coordinate and force, like the relation between the dielectric polarization **P** and the electric field **E** in a dielectric, or as the relation between the thermodynamic flux and the force, e.g., Ohm's law, i.e., the relation between the electric current **j** and the electric field **E** in a conductor. Thermodynamically, these relations have a different background, since in the first case a state of the system attained after long time under action of a constant field is a new equilibrium state, while in the second case it is a NESS. The corresponding FDRs are termed to be of the first and of the second kind, respectively [29]. In the classical setting, both kinds of FDRs can be considered within a unified framework [14].

If the form of the linear response, Eq. (2), is assumed or proven, the properties of response functions can be connected with the ones of spontaneous fluctuations in the absence of the perturbation. To do so one considers special forms of time dependence of the force, e.g., the force switched off at some time T_{off}, $F(t) = F_-\Theta(T_{\mathrm{off}} - t)$, or a force switched on at time T_{on}, $F(t) = F_+\Theta(t - T_{\mathrm{on}})$.

An example of the FDR of the first kind is the relation

$$\langle V(t)|F_-\rangle = \frac{F_-}{k_B T} C_{VV}(t), \quad (3)$$

which holds for $t > 0$. Here $\langle V(t)|F_-\rangle$ is the mean value of $V(t)$ for a system, which was at equilibrium under the action of the constant force F_- switched off at $T_{\mathrm{off}} = 0$, and the correlation function $C_{VV}(t) = \langle V(t')V(t'+t)\rangle$ is calculated at equilibrium in the absence of the force. This relation immediately follows from the Onsager's regression principle [30], see Ref. [14] for a detailed discussion.

Equation (3) implies that after switching off the force one has $\langle V(t)|F_-\rangle = F_- \int_{-\infty}^{0} \chi(t-t')dt' = F_- \int_t^\infty \chi(t'')dt''$ with $t'' = t - t'$. Denoting $\Xi(t) = \int_0^t \chi(t')dt'$ we may write Eq. (3) as $\Xi - \Xi(t) = (1/k_B T)C_{VV}(t)$ with $\Xi = \Xi(\infty)$. This allows for obtaining a relation between the response function and the autocorrelation function C_{VV}:

$$\chi(t) = -\frac{1}{k_B T}\frac{d}{dt} C_{VV}(t). \quad (4)$$

The assumption of finite static susceptibility is crucial since $\langle V(t)|F_-\rangle$ is otherwise undefined.

The FDR can be put in a different form, connecting the response with a mean squared displacement (MSD) of V from its initial value, $\langle \Delta V^2(t)\rangle \equiv \langle [V(t) - V(0)]^2\rangle$, in the absence of the force. Let us consider the response of V to the force switched on at $T_{\mathrm{on}} = 0$. Now, $\langle V(t)|F_+\rangle = F_+ \int_0^t \chi(t-t')dt'$. Using Eq. (4) we get $\langle V(t)|F_+\rangle = -F_+ \int_0^t (1/k_B T)(d/dt')C_{VV}(t')dt' = (F_+/k_B T)[C_{VV}(0) - C_{VV}(t)]$. Since the MSD during time t in a stationary process is $\langle \Delta V^2(t)\rangle = \langle [V(t)-V(0)]^2\rangle = \langle V^2(t)\rangle + \langle V^2(0)\rangle - 2\langle V(t)V(0)\rangle = 2C_{VV}(0) - 2C_{VV}(t)$, we get

$$\langle \Delta V(t)|F_+\rangle = \frac{F_+}{2k_B T}\langle \Delta V^2(t)|F = 0\rangle. \quad (5)$$

This relation will be called the generalized Einstein's relation (GER) in what follows. The expression for the response function via the fluctuations then reads $\chi(t) = (1/2k_B T)(d/dt)\langle \Delta V^2(t)|F = 0\rangle$.

The reason to call the relation Eq. (5) the generalized Einstein's relation is as follows. The Einstein's relation connects the mobility μ of a particle in a quiescent fluid with its diffusion coefficient D,

$$\mu = \frac{1}{k_B T} D, \quad (6)$$

and was the very first example of the fluctuation-dissipation relation (of the second kind). The mobility of the particle in a fluid medium describes the response of its mean velocity to the external force, $\langle v\rangle = \mu F$. The response of the particle's coordinate to the force F_+ switched on at $T_{\mathrm{on}} = 0$ is $\langle \Delta x(t)|F_+\rangle = \mu F_+ t$. The diffusion coefficient, on the other hand, defines the MSD of the particle from its initial position, $\langle \Delta x^2(t)\rangle = 2Dt$ (here in one dimension). From these expressions it follows that $\langle \Delta x(t)|F_+\rangle = (F_+/2k_B T)\langle \Delta x^2(t)|F = 0\rangle$, having the same form as our GER, Eq. (5).

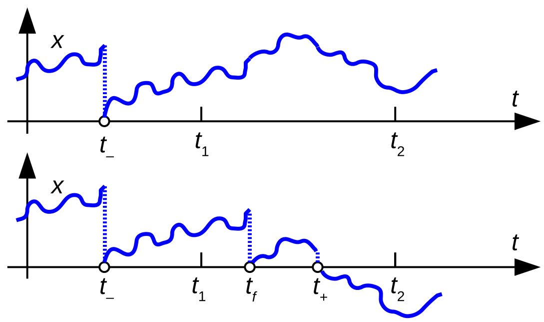

FIG. 1. Different realizations of the reset process. The two panels show the two mutually excluding situations: In the realization shown in the upper panel, no resetting event takes place between t_1 and t_2. In the realization shown below, the first resetting event after t_1 took place at time t_f and the last resetting event preceding t_2 at time t_+. The time t_- is the time of the last resetting before the beginning of observation.

Now let us turn to our process under resetting and calculate $\langle \Delta X(t)|F_+\rangle$ and $\langle \Delta X^2(t)\rangle$ for this process. Before doing so we need some preparative work.

Some conditional and joint probability densities for renewals.—We assume the resetting process to start at $t_0 = 0$, and consider two measurement times, t_1 and t_2, so that $0 < t_1 < t_2$. Let t_- be the last resetting time preceding t_1 (it might well coincide with t_0) and t_+ be the time of the last resetting preceding t_2 (this may well coincide with t_-), see Fig. 1, and let t_f be the time of the first renewal after t_1 (the forward recurrence time). We now calculate the probabilities and the probability density functions (PDFs) needed for our discussion following the pattern outlined in [5–7]. The explicit calculations and proofs are given in [18].

First we calculate $p_f(t_f|t_1)$, the PDF of the forward recurrence time conditioned on t_1. Let the last renewal before t_1 take place at t_-. Then $p_f(t_f|t_1) = \int_0^{t_1} k(t_-)\psi(t_f - t_-)dt_-$, where $k(t)$ is the time-dependent rate of renewals, so that $k(t)dt$ is a probability that a renewal event took place between t and $t + dt$. The rate of renewals (the intensity of the point process) at time t is $k(t) = \delta(t) + \psi(t) + [\psi * \psi](t) + \cdots$ (with star denoting the convolution), which follows from the fact that this renewal may be a zeroth, first, second one, etc., and these events are mutually excluding. This rate is time dependent unless the renewal process is a Poissonian one.

The probability that t_f is larger than t_2 [i.e., the probability that no resetting takes place between t_1 and t_2, $P_0(t_2, t_1)$] is then $P_0(t_1,t_2) = \int_{t_2}^{\infty} p_f(t_f|t_1)dt_f = \int_{t_2}^{\infty} dt_f \int_0^{t_1} dt_- k(t_-)\psi(t_f - t_-) = \int_0^{t_1} dt_- k(t_-) \int_{t_2}^{\infty} \psi(t_f - t_-)dt_f$, so that we get $P_0(t_2, t_1) = \int_0^{t_1} dt_- k(t_-)\Psi(t_2 - t_-)$ with $\Psi(t) = \int_t^{\infty} \psi(t')dt'$ being the survival probability in a resetting process.

Now we calculate the joint PDF of t_- and t_+ provided there was at least one renewal on the interval between t_1 and t_2. The overall situation is as follows: There was a renewal at t_-, then t_1 comes, then we have a renewal at t_f, then maybe further renewals [following at the rate $k(t - t_f)$] until the last renewal at t_+, and then no renewals until t_2, so that $p(t_-, t_+|t_1, t_2) = \Theta(t_1 - t_-)\times \int_{t_1}^{t_2} dt_f k(t_-)\psi(t_f - t_-)k(t_+ - t_f)\Psi(t_2 - t_+)\Theta(t_+ - t_f)$. The Heaviside Θ functions denote the facts that t_- *must* precede t_1 and t_+ *cannot* precede t_f. The last one indicates that the integral in t_f goes essentially up to t_+:

$$p(t_-, t_+|t_1, t_2) = \Theta(t_1 - t_-)k(t_-) \times \int_{t_1}^{t_+} dt_f \psi(t_f - t_-)k(t_+ - t_f)\Psi(t_2 - t_+). \tag{7}$$

This density is nonproper, since its integral over the two variables t_-, t_+ is $1 - P_0$. By taking the corresponding integrals of the joint density, Eq. (7), we obtain marginal probability densities $p_\pm(t_\pm|t_1, t_2)$ for t_+ and t_-, see Ref. [18] for details.

For the case of an equilibrated resetting process, which will be of importance in what follows, there exists a finite limiting value $k = \lim_{t\to\infty} k(t) = \langle\tau\rangle^{-1} > 0$, with $\langle\tau\rangle$ being the mean waiting time for the resetting events; the equations simplify, and for $t_1 \gg \langle\tau\rangle$ one obtains

$$\begin{aligned} P_0(t_1, t_2) &= k\int_0^{t_1} \Psi(t_2 - t')dt', \\ p_+(t_+|t_1, t_2) &= k\Psi(t_2 - t_+)\Theta(t_+ - t_1), \\ p_-(t_-|t_1, t_2) &= k\Theta(t_1 - t_-)[\Psi(t_1 - t_-) - \Psi(t_2 - t_-)]. \end{aligned} \tag{8}$$

Now we return to our reset process.

Linear response under resetting.—First we show that the reset process inherits the linearity of response from the displacement process. To see this it is enough to note that the mean position in the reset process is equal to the particle's mean displacement since the last renewal. Therefore, if the mean displacement $\langle x(t)\rangle$ is given by Eq. (2) with $V(t) = x(t)$ and the response function $\chi_d(t)$, the mean position at time t under the action of the time-dependent force for the reset process is

$$\langle X(t)\rangle = \int_{t_0}^{t} dt_- p_b(t_-|t) \int_{t_-}^{t} \chi_d(t - t')F(t')dt'. \tag{9}$$

Here $p_b(t_-|t)$ is the backward recurrence PDF (i.e., the PDF of the last renewal before time t) which is given by $p_b(t_-|t) = k(t_-)\Psi(t - t_-)$.

Thus, $\langle X(t)\rangle$ is a linear functional of the force. Interchanging the sequence of integrations in t' and t_-, one obtains $\langle X(t)\rangle = \int_{t_0}^{t} dt' F(t')\chi_d(t - t') \int_{t_0}^{t'} p_b(t_-|t)dt_-$ defining the response of a general type, Eq. (1), with $\chi^*(t, t', t_0) = \chi_d(t - t') \int_{t_0}^{t'} p_b(t_-|t)dt_-$. To check, whether this response gets time-homogeneous in the limit when the lag between the preparation and the observation times gets long enough [i.e., that the function $\chi^*(t, t', t_0)$ gets to be a function of $\Delta t = t - t'$ only], one has to discuss the properties of the integral $I(t, t', t_0) = \int_{t_0}^{t'} p_b(t_-|t) dt_-$ for fixed t_0 and for $t \to \infty$ with Δt fixed. As we show in [18], the response is time-homogeneous if the resetting process equilibrates. The two sufficient conditions for this are as follows: The resetting times are not the multiples of some given time (like for resetting at a constant pace), and the mean waiting time $\langle \tau \rangle$ is finite. In this case $k(t)$ stagnates, and Eqs. (8) apply. For the reset BM with $\chi_d(t) = \mu$ the response function is then given by

$$\chi(t) = \mu k \int_t^\infty \Psi(t') dt'.$$

To check whether response to the external force is elastic or fluid, one investigates the integrability of $\chi(t)$. It is integrable provided the second moment $\langle \tau^2 \rangle$ of the waiting time exists, since $\int_0^\infty dt \int_t^\infty \Psi(t') dt' = \frac{1}{2} \langle \tau^2 \rangle$.

As we show in [18], for the Brownian motion biased by a constant force F the stationary PDF of displacements exists under the same conditions under which the response is time-homogeneous. This PDF is given by

$$W(X|F) = k \int_0^\infty \frac{\Psi(t')}{\sqrt{4\pi D t'}} e^{-\frac{(X-\mu F t')^2}{4Dt'}} dt', \tag{10}$$

which is a special case of Eq. (2.41) of Ref. [1]. Calculating the moments $\langle X|F_- \rangle = \mu k F_- \int_0^\infty t' \Psi(t') dt' = \frac{1}{2} \mu k F_- \langle \tau^2 \rangle$ and $\langle X^2 \rangle = kD \langle \tau^2 \rangle$ we see that both diverge when the second moment of waiting time diverges, i.e., in the case of fluid response. This precludes applicability of the standard FDR for this case. However, if we consider a situation when the force F_+ is switched on at time $T_+ = t_1$ and the displacement of the particle $\Delta X(t_1, t_2) = X(t_2) - X(t_1)$ is measured at time $t_2 > t_1$, the first two moments of this displacement, $\langle \Delta X(t_1, t_2)|F_+ \rangle$ and $\langle \Delta X^2(t_1, t_2)|F = 0 \rangle$, stay finite as long as the mean waiting time $\langle \tau \rangle$ exists. These first two moments of displacement in the reset BM can be calculated explicitly as functions of t_1 and t_2 [18]:

$$\langle \Delta X|F_+ \rangle = \mu F_+ \left[\Delta t P_0(..) + \int_{t_1}^{t_2} dt_+ (t_2 - t_+) p_+(..) \right]$$

$$\begin{aligned} \langle \Delta X^2 \rangle_0 \equiv \langle \Delta X^2|F = 0 \rangle = 2D \Bigg[& \Delta t P_0(..) \\ & + \int_{t_1}^{t_2} dt_+ (t_2 - t_+) p_+(..) \\ & + \int_0^{t_1} dt_- (t_1 - t_-) p_-(..) \Bigg] \end{aligned} \tag{11}$$

with $\Delta t = t_2 - t_1$ and the expressions for $P_0(t_1, t_2)$ and $p_\pm(t_\pm|t_1, t_2)$ given by Eqs. (8). Moreover, under the same condition, limits of both $\langle \Delta X(t_1, t_2)|F_+ \rangle$ and $\langle \Delta X^2(t_1, t_2)|F = 0 \rangle$ for $t_1 \to \infty$ and $t_2 = t_1 + \Delta t$ stay finite and get to be functions of Δt only, see Ref. [18] for details:

$$\begin{aligned} \langle \Delta X(\Delta t)|F_+ \rangle &= \mu F_+ k \left[\Delta t \int_{\Delta t}^\infty \Psi(t') dt' + \int_0^{\Delta t} t' \Psi(t') dt' \right] \\ \langle \Delta X^2(\Delta t) \rangle_0 &= 4Dk \left[\Delta t \int_{\Delta t}^\infty \Psi(t'') dt'' + \int_0^{\Delta t} t'' \Psi(t'') dt'' \right]. \end{aligned} \tag{12}$$

Thus,

$$\langle \Delta X(\Delta t)|F_+ \rangle = \frac{\mu F_+}{4D} \langle \Delta X^2(\Delta t)|F = 0 \rangle \tag{13}$$

for any time lag Δt. If we assume that the Einstein's relation, Eq. (6), holds for the BM, Eq. (13) represents the GER, Eq. (5)

$$\langle \Delta X(\Delta t)|F_+ \rangle = \frac{F_+}{4k_B T} \langle \Delta X^2(\Delta t)|F = 0 \rangle. \tag{14}$$

This form of the GER implies that the effective temperature of the reset BM as defined via the FDR, see Ref. [31] for the discussion, is always twice the temperature of the medium in which the BM takes place. Equation (14) holds independently of the shape of $\psi(t)$ and on whether the response is elastic or fluid. This is due to a remarkable compensation effect: while in the fluid case for longer t_1 the variance $\langle X_1^2 \rangle$ grows, the probability of having a renewal between t_1 and $t_1 + \Delta t$ tends to zero due to aging, so that the scenario shown in the lower panel in Fig. 1 gets less and less probable [18].

We also note that in the case when the response is fluid, the GER *does not* correspond to a FDR of the second kind: Using Eq. (13) we get $\langle v(\Delta t)|F_+ \rangle = (d/d\Delta t) \langle \Delta X|F_+ \rangle = k\mu F_+ \int_{\Delta t}^\infty \Psi(t'') dt''$, and see that the velocity shows an unusual response to the force. For longer Δt, the velocity tends to zero, independently on whether the response is elastic or fluid.

To understand the GER, Eq. (14), one considers the variable Y conjugated to the force in our reset process. This is defined by [18]

$$Y(X) = \lim_{F \to 0} \frac{1}{W(X|0)} \frac{\partial}{\partial F} W(X|F). \tag{15}$$

Using Eq. (10) we obtain $Y = (\mu X / 2D) = (X / 2k_B T)$. The GER for this conjugated variable $\langle \Delta Y(\Delta t)|F_+ \rangle = (F_+/2) \langle \Delta Y^2(\Delta t)|F = 0 \rangle$ corresponds to Eq. (13) for the response variable X. Therefore, in the case of elastic response, Eq. (14) corresponds to a standard out-of-equilibrium FDR.

Our calculations, however, showed that Eq. (13) is still valid in the domain of fluid response, where the stationary PDF does not possess moments. This indicates that for nonequilibrium systems the GERs may have a larger domain of applicability than the usual FDRs. This issue definitely deserves further investigation.

Summary and outlook.—Let us summarize our findings. A Markovian process under complete renewal resetting inherits linear response from the displacement process. For the reset Brownian motion, the corresponding response fulfills the usual FDR for the case when the resetting process possesses a finite second moment for its waiting times. If the second moment diverges but the first moment still exists, the standard FDR is not applicable, but the generalized Einstein's relation still holds. The effective temperature of the reset Brownian motion, as defined via the fluctuation-dissipation relation, is universally twice as high as the temperature of the medium in which the motion takes place.

These results can be experimentally tested within the colloidal setup of [8] with small passive colloids performing Brownian motion, where the external forcing may correspond to a slow flow in suspending fluid. In the present Letter only the case of immediate resetting, i.e., fast return to the origin, is considered, and therefore the resetting procedure must be fast, or, alternatively, the return times must be censored from data. A two-dimensional variant of our model may be used for the discussion of reset motion of massive colloids (two variables, position and velocity), and two-dimensional variants of simple diffusion (akin to the experiment [8]) with complete or incomplete resetting. For measurements including return times, a more complicated, two stage model of the displacement process taking into account the motion on return with constant speed or under the action of the constant force, see, e.g., [32], must be used, for which the discussion of the FDRs is still missing.

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*Corresponding author.
igor.sokolov@physik.hu-berlin.de

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[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.067101 which contains the following: (a) Some conditional and joint probability densities for renewals mentioned in the main text. (b) The proof that a Markov process stays Markovian under complete resetting. (c) Conditions for stationarity of the single-point PDF and for the response. (d) The derivations of the expressions for the first and second moments of displacement in reset BM. (e) Derivation of the generalized FDR for Markov processes. It refers to additional literature, Refs. [19–28].
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