Editors' Suggestion

Non-Hermitian Spectral Flows and Berry-Chern Monopoles

Lucien Jezequel and Pierre Delplace
Ens de Lyon, CNRS, Laboratoire de physique, F-69342 Lyon, France

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We propose a non-Hermitian generalization of the correspondence between the spectral flow and the topological charges of band crossing points (Berry-Chern monopoles). A class of non-Hermitian Hamiltonians that display a complex-valued spectral flow is built by deforming an Hermitian model while preserving its analytical index. We relate those spectral flows to a generalized Chern number that we show to be equal to that of the Hermitian case, provided a line gap exists. We demonstrate the homotopic invariance of both the non-Hermitian Chern number and the spectral flow index, making explicit their topological nature. In the absence of a line gap, our system still displays a spectral flow whose topology can be captured by exploiting an emergent pseudo-Hermitian symmetry.

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Non-Hermitian topology [1–3] is an emergent topic stimulated by the rise of topological physics in various quantum and classical physical systems [4–9] that display—naturally or on purpose—non-Hermitian effects [10–15]. Protected boundary states being certainly one of the most universal signatures in topological physics, the question of their existence in non-Hermitian systems has naturally led to numerous studies during the past few years. This central issue of the bulk-boundary correspondence is, however, quite involved in non-Hermitian systems. The reason being that the spectrum of the periodic system from which a topological index is usually computed, and that of the system with open boundary conditions, can be totally distinct [16,17]. In particular, eigenstates of non-Hermitian open systems were found to be localized near the boundary, in contrast with Bloch waves of Hermitian models. This so-called non-Hermitian skin effect [18] motivated the development of a non-Bloch bulk-boundary correspondence, where a generalized Brillouin zone is introduced to define the proper topological invariants in relation with the boundary states [18–23]. The calculation of this non-Hermitian Brillouin zone is, however, quite involved in itself, although recent analytical advances have been made to evaluate it in one dimension [24]. Another approach consists of considering a complete biorthogonal basis, made of left and right eigenstates of the non-Hermitian Hamiltonian in order to introduce a biorthogonal polarization that accounts for the appearance or disappearance of singular edge modes in non-Hermitian systems with open boundaries [25–29].

Instead of focusing on lattice problems and trying to adapt topological Bloch theory to the non-Hermitian realm, we address the issue of the non-Hermitian generalization of the correspondence between spectral flows and Berry-Chern monopoles, which are the topological charges associated to band crossing points [30–33]. In contrast with the bulk-boundary correspondence, the monopole-spectral flow correspondence does not involve open boundary conditions, but requires instead a variation in space of a physical quantity, such as a mass term or a vector potential. Moreover, when such a quantity varies linearly along a spatial direction—say x—all the "bulk" states are already localized around $x = 0$ in the Hermitian case.

Berry-Chern monopoles are abundant in physics and their associated spectral flows have different physical interpretations depending on the system at hand. For instance, they arise in the low energy description of smooth interfaces between topologically distinct two-dimensional (2D) topological insulators [31,34,35] and metals [36], they allow a suitable description of topological waves in various inhomogeneous continuous media [6,37–42], they account for the topological reorganization of quantum levels in molecules [43–46], and they also characterize the topology of 3D Weyl semimetals [47,48] and their generalizations [49,50].

Despite its ubiquity, the question of the monopole-spectral flow correspondence has been overlooked in the context of non-Hermitian physics, leaving us with the following key questions: Does the spectral flow survive non-Hermiticity? Is the correspondence between the spectral flow and the topological charge still valid, or does it break like the bulk-edge correspondence? In this Letter, we show how to construct a class of non-Hermitian models that preserve the spectral flow. We then show how to extend its topological description, and build different mappings to relate it with the Berry-Chern monopoles of Hermitian models.

To do so, let us consider the generic twofold band crossing Hamiltonian with a linear dispersion relation in the three directions λ, x, and p

$$H_0[\lambda; x, p] = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda & x - \mathrm{i}p \\ x + \mathrm{i}p & -\lambda \end{pmatrix}. \tag{1}$$

The variables x and p must be understood as two canonical conjugate classical (commuting) observables, while λ is a control parameter. The Hamiltonian (1) displays two bands of energy $E^{\pm} = \pm\sqrt{x^2 + p^2 + \lambda^2}$ that are separated by a gap except at the origin $\lambda = x = p = 0$ where they touch. It is well known that such a degeneracy point constitutes a source of Berry curvature, whose flux through a surface $\mathcal{S}$ enclosing the origin in (λ, x, p) space is a topological index called the first Chern number $\mathcal{C}_0$ [30,51,52], hence the name Berry-Chern monopole. This Chern number can be computed for each band as

$$\mathcal{C}_0^{\pm} = \frac{1}{2\pi i}\int_S \mathrm{tr}(P_0^{\pm} dP_0^{\pm} \wedge dP_0^{\pm}), \tag{2}$$

where $P_0^{\pm} = |\psi^{\pm}\rangle\langle\psi^{\pm}|$ is the spectral projector on the positive and negative band, with $|\psi^{\pm}\rangle$ the two eigenstates of H_0 and $dP_0^{\pm}$ is the one-form $\partial_\lambda P_0^{\pm} d\lambda + \partial_x P_0^{\pm} dx + \partial_p P_0^{\pm} dp$. In that case the Chern numbers of the bands are equal to $\mathcal{C}^{\pm} = \pm 1$. This nonzero value characterizes the impossibility to define a smooth gauge for the eigenstates of H_0 over the parameter space. In more formal words, it is a topological property of the $U(1)$-fiber bundle over the base space S^2. This bundle terminology simply means that at each point (λ, x, p) of the base space, the eigenstates $|\psi^{\pm}\rangle$ are defined up to a phase, owing to their normalization which is preserved due to Hermiticity of H_0. Extending this topological property to non-Hermitian Hamiltonians is therefore not obvious, precisely because this $U(1)$-fiber bundle structure is, in general, lost.

To circumvent this difficulty, we interpret $H_0[\lambda; x, p]$ as the *symbol* Hamiltonian, or semiclassical limit, of the *operator* Hamiltonian [31–33],

$$\mathcal{H}_0[\lambda] = \begin{pmatrix} \lambda & \hat{a}^\dagger \\ \hat{a} & -\lambda \end{pmatrix} = \hat{a}\sigma_- + \hat{a}^\dagger\sigma_+ + \lambda\sigma_z, \tag{3}$$

where $\sigma_\pm = \sigma_x \pm \mathrm{i}\sigma_y$ with σ_i $(i = x,\ y,\ z)$ the Pauli matrices, and $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators that satisfy $[\hat{a}, \hat{a}^\dagger] = 1$. Those operators are related to the (noncommuting) position and momentum operators that satisfy $[\hat{x}, \hat{p}] = \mathrm{i}$ $(\hbar = 1)$, as $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$, $\hat{a}^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}$. Up to the rescaling $\lambda/\sqrt{2} \to \lambda$, the symbol Hamiltonian (1) is recovered by simply replacing the operators $\hat{x}$ and $\hat{p}$ by their classical commuting counterpart x and p. Note that the same model is recovered for a Weyl fermion in a magnetic field where the magnetic momenta in the orthogonal plane become canonical conjugated observables (such as $\hat{x}$ and $\hat{p}$), while the longitudinal momentum plays the role of λ [31].

As shown in Fig. 1 (up-left), the spectrum of $\mathcal{H}_0$ is made of two branches $\pm$ given by the set

$$\mathcal{E}_{0,n}^{\pm} = \pm\sqrt{\lambda^2 + (n+1)} \quad n \in \mathbb{N}, \tag{4}$$

$$\mathcal{E}_{-1} = \lambda. \tag{5}$$

The additional mode of energy $\mathcal{E}_{-1} = \lambda$ constitutes the *spectral flow*, as it transits from the negative to the positive branch when λ varies from $-\infty$ to $+\infty$. In other words, the positive branch gains $\mathcal{N}^+ = 1$ mode while the negative branch gains $\mathcal{N}^- = -1$ mode. It is a remarkable result that those two numbers are precisely given by the Chern numbers (2) of the symbol Hamiltonian (1) as $\mathcal{N}^{\pm} = \mathcal{C}^{\pm}$. This relation is precisely the monopole-spectral flow correspondence. Importantly, for that model, the spectral flow is directly accounted for by the analytical index

$$\mathrm{ind}\mathcal{D} \equiv \dim\ \mathrm{Ker}\mathcal{D} - \dim\ \mathrm{Ker}\mathcal{D}^\dagger = \mathcal{N}^+ \tag{6}$$

with $\mathcal{D} = \hat{a}\sigma_-$ [31]. The relation between this analytical index and the Chern index of the symbol Hamiltonian is known as the index theorem [53,54].

Remarkably, the analytical index (6) is defined irrespective of the diagonal part of $\mathcal{H}_0$ and remains also unchanged when multiplying the off-diagonal elements $\mathcal{D}$ and $\mathcal{D}^\dagger$ by an arbitrary complex number. Those two crucial points allow us to deform $\mathcal{H}_0$ in two different non-Hermitian ways (i.e., diagonal and nondiagonal) that leave the analytical index invariant. We are thus led to introduce the non-Hermitian operator Hamiltonian

$$\mathcal{H}[\lambda] = \begin{pmatrix} \lambda e^{\mathrm{i}\varphi} & z_\alpha \hat{a}^\dagger \\ z_\beta \hat{a} & -\lambda e^{\mathrm{i}\varphi} \end{pmatrix} \tag{7}$$

with $z_{\alpha/\beta} \in \mathbb{C}$ and φ a phase, that has the same analytical index as $\mathcal{H}_0$. All the parameters introduced in (7) break Hermiticity, but they actually do not play the same role. Indeed, one can reduce the analysis of this Hamiltonian to that of the much simpler one

$$\mathcal{H}_\theta[\lambda] = \begin{pmatrix} \lambda & e^{\mathrm{i}\theta}\hat{a}^\dagger \\ e^{\mathrm{i}\theta}\hat{a} & -\lambda \end{pmatrix} \tag{8}$$

through the transformation

$$A^{-1}\mathcal{H}[\sqrt{r_\alpha r_\beta}\lambda]A = \sqrt{r_\alpha r_\beta} e^{\mathrm{i}\varphi}\mathcal{H}_\theta[\lambda], \tag{9}$$

where we have introduced $\theta \equiv (\varphi_\alpha + \varphi_\beta)/2 - \varphi$, with $z_{\alpha/\beta} = r_{\alpha/\beta} e^{\mathrm{i}\varphi_{\alpha/\beta}}$ and

$$A = \begin{pmatrix} z_\alpha^{1/2} & 0 \\ 0 & z_\beta^{1/2} \end{pmatrix}. \tag{10}$$

Note that $\mathcal{H}_\theta$ corresponds to the Hermitian Hamiltonian $\mathcal{H}_0$ introduced above when $\theta = 0$. The nonunitary transformation (9) means that if $|\psi(\lambda)\rangle$ is an eigenstate of $\mathcal{H}_\theta[\lambda]$ with

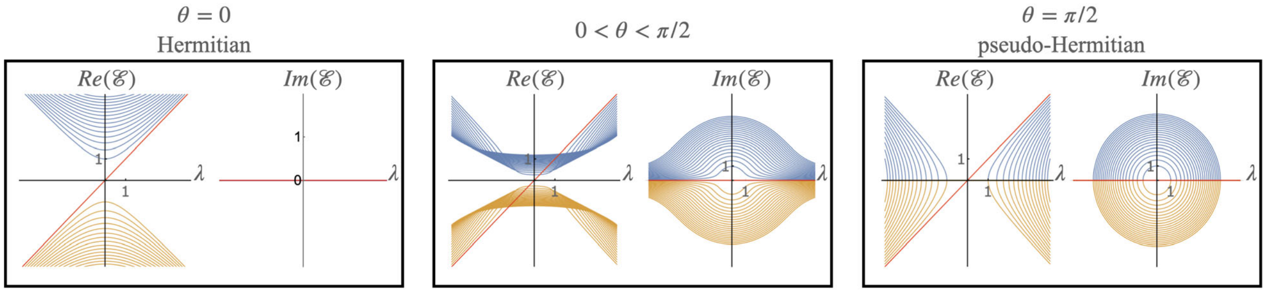

FIG. 1. Real and imaginary part of the eigenvalue spectrum $\mathcal{E}^{\pm}_{\theta,n}$ of $\mathcal{H}_\theta$ as a function of λ, up to $n = 20$, for $\theta = 0, 5\pi/12, \pi/2$. The spectral flow (in red) is unaffected by the non-Hermitian θ deformation.

the eigenvalue $\mathcal{E}_\theta(\lambda)$, then $A|\psi(\lambda/\sqrt{r_\alpha r_\beta})\rangle$ is an eigenstate of $\mathcal{H}[\lambda]$ with the eigenvalue $\sqrt{r_\alpha r_\beta}e^{i\varphi}\mathcal{E}_\theta(\lambda/\sqrt{r_\alpha r_\beta})$. So, up to a rescaling by a factor $\sqrt{r_\alpha r_\beta}$ and to a global rotation of angle φ of the spectrum in the complex plane, the study of $\mathcal{H}$ reduces to that of $\mathcal{H}_\theta$, whose spectrum reads (see Supplemental Material [55])

$$\mathcal{E}^{\pm}_{\theta,n} = \pm\sqrt{\lambda^2 + e^{2i\theta}(n+1)} \quad n \in \mathbb{N}, \tag{11}$$

$$\mathcal{E}_{-1} = \lambda. \tag{12}$$

The asymptotic behavior $\mathcal{E}^{\pm}_{\theta,n} \to \pm\mathrm{sgn}(\lambda)$ for each n and every θ when $|\lambda| \to \infty$ implies that $\mathcal{E}_{-1} \to \mathcal{E}^{\pm}_{\theta,n}$ when $\lambda \to \pm\infty$. This means that the operator Hamiltonian $\mathcal{H}_\theta$ (and therefore $\mathcal{H}$) displays a spectral flow with respect to the parameter λ. Examples are shown in Fig. 1 for different values of θ, and additional spectra of $\mathcal{H}$ are shown in Supplemental Material [55].

Figure 2 summarizes the role of the different parameters that break Hermiticity in the full model. A first important remark is that for $\varphi = 0$ modulo π (and thus in particular for $\mathcal{H}_\theta$), the spectral flow eigenvalue $\mathcal{E}_{-1} = \lambda$ remains purely real, unlike the rest of the spectrum which is in general complex valued. We furthermore find that the corresponding eigenmode remains localized around $x = 0$ (see Supplemental Material [55]). Therefore, the spectral flow mode of $\mathcal{H}_0$ remains invariant under the action of all the non-Hermitian terms introduced in (7) but the parameter φ which rotates its energy. Interpreting the spectral flow as a chiral mode at a smooth interface between two Chern insulators, this result contrasts that of [1] where the interface state's localization length is found to be affected by the non-Hermiticity at a sharp steplike domain wall.

If now, in addition to $\varphi = 0$, we also assume $\theta = 0$, then the full spectrum of $\mathcal{H}$ becomes real-valued, irrespective of the non-Hermitian asymmetry $r_\alpha/r_\beta \neq 1$. Real spectra are an unusual property of non-Hermitian matrices, and their existence in physical systems stimulated recent works [56,57]. Here, this property is explained by the transformation (9) that maps, in that case, $\mathcal{H}$ onto the Hermitian operator $\mathcal{H}_0$.

Finally, $\mathcal{H}_\theta$ displays a striking spectrum for $\theta = \pi/2$. There, the spectral flow does not bridge two branches separated by a spectral gap in energy, but relates instead two branches separated by a range in λ where the energy of the modes become purely imaginary and self-conjugated. As we detail below, this case, that requires a specific treatment, can be understood as a pseudo-Hermitian symmetry breaking phase.

For now, let us focus on the interpolation between the Hermitian case $\theta = 0$ and the pseudo-Hermitian one $\theta = \pi/2$. As θ increases, the real part of the spectrum of $\mathcal{H}_\theta$ "unfolds" around the spectral flow level $\mathcal{E}_{-1}$, while the imaginary part grows from zero but remains finite, leaving

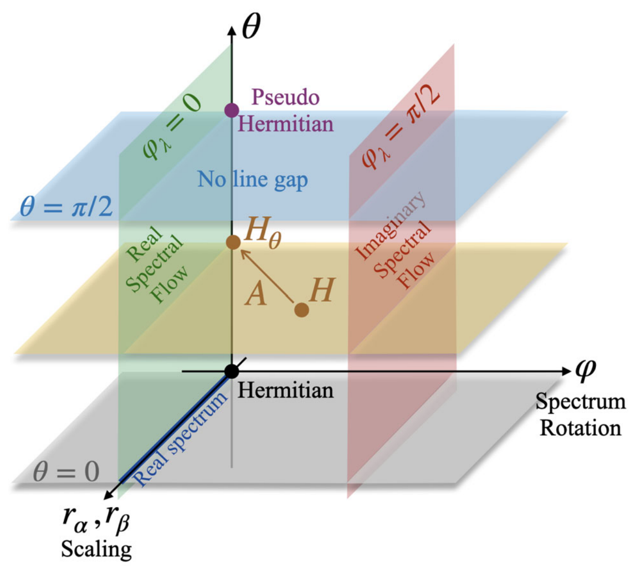

FIG. 2. Diagram of the different parameters that break Hermiticity. The origin, where the model becomes Hermitian, is taken as $(r_\alpha, r_\beta, \varphi, \theta) = (1, 1, 0, 0)$. The transformation A "projects" the non-Hermitian Hamiltonian H into H_θ. Under this transformation, the model becomes Hermitian when $\theta = 0$, and pseudo-Hermitian when $\theta = \pi/2$. Whenever $\theta = \pi/2$, there is no line gap in the spectrum of the symbol Hamiltonian but the operator Hamiltonian still displays a spectral flow. The spectral flow is in general complex valued, except for $\varphi = 0$ and $\varphi = \pi/2$ where it becomes purely real or purely imaginary, respectively.

the spectral flow intact (Fig. 1). This robustness can be interpreted topologically from the non-Hermitian symbol of $\mathcal{H}_\theta$ that reads

$$H_\theta[\lambda; x, p] = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda & e^{i\theta}(x - ip) \\ e^{i\theta}(x + ip) & -\lambda \end{pmatrix}. \quad (13)$$

Its spectrum $E^\pm = \pm\sqrt{\lambda^2 + e^{2i\theta}(x^2 + p^2)}$ preserves the degeneracy point at $\lambda = x = p = 0$. Owing to the lost of Hermiticity of H_θ, one needs to find a non-Hermitian generalization of the Chern number to capture its topological properties [1,19,29,58]. Here we propose the quantity

$$\mathcal{C}_\theta^\pm \equiv \frac{1}{2\pi i} \int_{(\lambda,x,p)\in S^2} \mathrm{tr}(P_\theta^\pm dP_\theta^\pm \wedge dP_\theta^\pm), \quad (14)$$

where the $P_\theta^\pm \equiv |\psi^\pm\rangle\langle\tilde{\psi}^\pm|$ are the nonorthogonal spectral projectors of the complex energy bands $\pm$, with $|\psi^\pm\rangle$ and $|\tilde{\psi}^\pm\rangle$ the right and left eigenstates of H_θ, respectively. These projectors are well defined as long as the two spectral bands $E^\pm$ are separated by a line in the complex plane (the so-called line gap [12]).

We then use the invariance by homotopy of $\mathcal{C}_\theta^\pm$ with respect to θ to show the equality between this generalized Chern number and the usual one defined in the Hermitian case, i.e., $\mathcal{C}_\theta^\pm = \mathcal{C}_0^\pm$, as long as the line gap is preserved (see Supplemental Material [55]). As the spectral flow index (6) is also invariant in the non-Hermitian regime (see Supplemental Material [55]), this extends the monopole-spectral flow correspondence to the non-Hermitian systems given by the operator Hamiltonian $\mathcal{H}_\theta$ and its symbol H_θ. This correspondence finally generalizes to the full non-Hermitian system defined by the operator Hamiltonian $\mathcal{H}$ given in (7) and its symbol H, since the transformation (9) preserves the Chern number (see Supplemental Material [55]).

The existence of a line gap is required for the projectors $P_\theta^\pm$ to be well defined. One can check that such a line gap indeed exists along the imaginary axis for H_θ (and is tilted by and angle φ in the full problem) except for $\theta = \pi/2$, as sketched in Fig. 2. In that case, the generalized Chern number (14) is ill defined, and we thus need to come with another strategy to capture the topology of $H_{\pi/2}$.

For that purpose, let us notice that $H_{\pi/2}$ (as well as $\mathcal{H}_{\pi/2}$) owns the pseudo-Hermitian symmetry $\sigma_z H_{\pi/2}[\lambda; x, p]\sigma_z = H_{\pi/2}^\dagger[\lambda; x, p]$. This symmetry implies that the eigenenergies are either real or appear as complex conjugate pairs. When a parameter is varied (e.g., λ here), eigenenergies can switch from the first case to the other in what is called a spontaneous pseudo-Hermitian symmetry breaking [12,29,59], as observed here. This pseudo-Hermiticity allows us to map $H_{\pi/2}$ onto a Hermitian Hamiltonian for which the topology is well defined (see details in the Supplemental Material [55]).

The idea behind this mapping is to notice that the iso-energy surfaces in parameter space for $\theta = \pi/2$ describe hyperboloids $E^2 = \lambda^2 - x^2 - p^2$, in contrast with the usual Hermitian case $\theta = 0$ where they describe spheres $E^2 = \lambda^2 + x^2 + p^2$. Actually, the pseudo-Hermitian case also yields a sphere, but a different one, since we have $\lambda^2 = E^2 + x^2 + p^2$. This hint suggests that one should change our point of view and consider our model as an eigenvalue problem in λ in parameter space (E, x, p). Thinking of λ as a momentum in a y direction, and of the energy E as the quantum number associated to ∂_t, such a transformation could be formally thought as a space-time $y \leftrightarrow t$ change of axes.

To do so, it is convenient to first perform a unitary transformation $UH_{\pi/2}U^\dagger = \tilde{H}_{\pi/2}$, that preserves the pseudo-Hermiticity, which, for the Hamiltonian operator yields

$$\tilde{\mathcal{H}}_{\pi/2}[\lambda] \equiv \begin{pmatrix} \lambda & \hat{a}^\dagger \\ -\hat{a} & -\lambda \end{pmatrix} \quad (15)$$

with $U = e^{i(\pi/4)\sigma_z}$. The eigenvalue problem in $\mathcal{E}$ for $\tilde{\mathcal{H}}_{\pi/2}$ is actually equivalent to the following eigenvalue problem in λ,

$$\begin{pmatrix} \mathcal{E} & \hat{a}^\dagger \\ \hat{a} & -\mathcal{E} \end{pmatrix} |\psi'\rangle = \lambda|\psi'\rangle, \quad (16)$$

with $|\psi'\rangle = -\sigma_z|\psi\rangle$. The equation (16) is nothing but $\mathcal{H}_0[\mathcal{E}]|\psi'\rangle = \lambda|\psi'\rangle$, meaning that the eigenvalue problem in $\mathcal{E}$ parametrized by λ at $\theta = \pi/2$, is the same as the Hermitian eigenvalue problem in λ, parametrized by $\mathcal{E}$. Thus, interchanging $\theta = \pi/2 \leftrightarrow \theta = 0$ amounts to swapping the roles of $\mathcal{E}$ and λ as parameters and eigenvalues, as it actually appears in Fig. 1 when both $\mathcal{E}$ and λ are real.

The realness of λ, being the eigenvalue of $\mathcal{H}_0[\mathcal{E}]$ in (16), is ensured by imposing the realness of $\mathcal{E}$, since $\mathcal{H}_0[\mathcal{E}]$ is Hermitian in that case. By doing so, we only keep the eigenmodes of $\tilde{\mathcal{H}}$ with a real energy and dismiss those in the spontaneously broken phase. As a consequence, the pseudo-Hermitian symmetry broken phase appearing in the $\mathrm{Re}\,\mathcal{E}$ spectrum parametrized by λ, is now interpreted as a disappearance of eigenvalue, namely, a gap in the spectrum in λ parametrized by $\mathrm{Re}\,\mathcal{E}$. Since the mapping (16) is also valid for the symbol Hamiltonian H_θ, we conclude that the spectral flow of $\mathcal{H}_{\pi/2}[\lambda]$ is captured by the Chern numbers of $H_0[\mathrm{Re}E, x, p]$ whose value is ± 1. Indeed, this Hamiltonian is formally equivalent to the previously discussed symbol Hamiltonian $H_0[\lambda, x, p]$ after the substitution $\mathrm{Re}E \leftrightarrow \lambda$, for which the Chern numbers take the values $\mathcal{C}_0 = \pm 1$. In other words, the spectral flow of $\mathcal{H}_{\pi/2}[\lambda]$ is thus determined by the Chern numbers of the Hermitian monopole.

Summary.— This Letter provides a non-Hermitian generalization of the correspondence between a spectral flow of an operator and the Chern numbers of the eigenstate bundle associated to the degeneracy point of the (local) symbol Hamiltonian. Our analysis has direct applications in the investigation of topological properties in various systems such as in photonics, fluids, and plasmas where non-Hermitian effects are abundant. Finally, the two-band model we have considered, seen as a spin $1/2$ model, can be generalized to higher spins and nonlinear dispersion relations in future works, as it is known that Hermitian spectral flows appear in such models [6,31,38,49,50].

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[1] Huitao Shen, Bo Zhen, and Liang Fu, Topological Band Theory for Non-Hermitian Hamiltonians, Phys. Rev. Lett. **120**, 146402 (2018).
[2] Ananya Ghatak and Tanmoy Das, New topological invariants in non-Hermitian systems, J. Phys. Condens. Matter **31**, 263001 (2019).
[3] Zongping Gong, Yuto Ashida, Kohei Kawabata, Kazuaki Takasan, Sho Higashikawa, and Masahito Ueda, Topological Phases Of Non-Hermitian Systems, Phys. Rev. X **8**, 031079 (2018).
[4] Ling Lu, John D. Joannopoulos, and Marin Soljačić, Topological photonics, Nat. Photonics **8**, 821 (2014).
[5] Lisa M. Nash, Dustin Kleckner, Alismari Read, Vincenzo Vitelli, Ari M. Turner, and William T. M. Irvine, Topological mechanics of gyroscopic metamaterials, Proc. Natl. Acad. Sci. U.S.A. **112**, 14495 (2015).
[6] Pierre Delplace, J. B. Marston, and Antoine Venaille, Topological origin of equatorial waves, Science **358**, 1075 (2017).
[7] Suraj Shankar, Anton Souslov, Mark J. Bowick, M. Cristina Marchetti, and Vincenzo Vitelli, Topological active matter, Nat. Rev. Phys. **4**, 380 (2022).
[8] Xiujuan Zhang, Meng Xiao, Ying Cheng, Ming-Hui Lu, and Johan Christensen, Topological sound, Commun. Phys. **1**, 97 (2018).
[9] Jeffrey B. Parker, Topological phase in plasma physics, J. Plasma Phys. **87**, 835870202 (2021).
[10] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Topologically protected bound states in photonic parity-time-symmetric crystals, Nat. Mater. **16**, 433 (2017).
[11] Shuo Liu, Shaojie Ma, Cheng Yang, Lei Zhang, Wenlong Gao, Yuan Jiang Xiang, Tie Jun Cui, and Shuang Zhang, Gain- and loss-induced topological insulating phase in a non-Hermitian electrical circuit, Phys. Rev. Appl. **13**, 014047 (2020).
[12] Yuto Ashida, Zongping Gong, and Masahito Ueda, Non-Hermitian physics, Adv. Phys. **69**, 249 (2020).
[13] Miguel A. Bandres, Steffen Wittek, Gal Harari, Midya Parto, Jinhan Ren, Mordechai Segev, Demetrios N. Christodoulides, and Mercedeh Khajavikhan, Topological insulator laser: Experiments, Science **359**, eaar4005 (2018).
[14] Daria Smirnova, Daniel Leykam, Yidong Chong, and Yuri Kivshar, Nonlinear topological photonics, Appl. Phys. Rev. **7**, 021306 (2020).
[15] Stefano Longhi, Non-Hermitian tight-binding network engineering, Phys. Rev. A **93**, 022102 (2016).
[16] Tony E. Lee, Anomalous Edge State in a Non-Hermitian Lattice, Phys. Rev. Lett. **116**, 133903 (2016).
[17] Ye Xiong, Why does bulk boundary correspondence fail in some non-Hermitian topological models, arXiv:1705.06039.
[18] Shunyu Yao and Zhong Wang, Edge States and Topological Invariants of Non-Hermitian Systems, Phys. Rev. Lett. **121**, 086803 (2018).
[19] Shunyu Yao, Fei Song, and Zhong Wang, Non-Hermitian Chern Bands, Phys. Rev. Lett. **121**, 136802 (2018).
[20] Wojciech Brzezicki and Timo Hyart, Hidden Chern number in one-dimensional non-Hermitian chiral-symmetric systems, Phys. Rev. B **100**, 161105(R) (2019),
[21] Tian-Shu Deng and Wei Yi, Non-Bloch topological invariants in a non-Hermitian domain wall system, Phys. Rev. B **100**, 035102 (2019).
[22] Dan S. Borgnia, Alex Jura Kruchkov, and Robert-Jan Slager, Non-Hermitian Boundary Modes and Topology, Phys. Rev. Lett. **124**, 056802 (2020).
[23] Gang-Feng Guo, Xi-Xi Bao, and Lei Tan, The analysis of bulk boundary correspondence under the singularity of the generalized brillouin zone in non-hermitian system, arXiv:2106.06384.
[24] Zhesen Yang, Kai Zhang, Chen Fang, and Jiangping Hu, Non-Hermitian Bulk-Boundary Correspondence and Auxiliary Generalized Brillouin Zone Theory, Phys. Rev. Lett. **125**, 226402 (2020).
[25] Flore K. Kunst, Elisabet Edvardsson, Jan Carl Budich, and Emil J. Bergholtz, Biorthogonal Bulk-Boundary Correspondence in Non-Hermitian Systems, Phys. Rev. Lett. **121**, 026808 (2018).
[26] Elisabet Edvardsson, Flore K. Kunst, and Emil J. Bergholtz, Non-hermitian extensions of higher-order topological phases and their biorthogonal bulk-boundary correspondence, Phys. Rev. B **99**, 081302(R) (2019),
[27] Matteo Brunelli, Clara C. Wanjura, and Andreas Nunnenkamp, Restoration of the non-Hermitian bulk-boundary correspondence via topological amplification, arXiv:2207.12427.
[28] Fei Song, Shunyu Yao, and Zhong Wang, Non-Hermitian Topological Invariants in Real Space, Phys. Rev. Lett. **123**, 246801 (2019).
[29] Zaur Z. Alisultanov and Edvin G. Idrisov, Towards the theory of types III and IV non-Hermitian Weyl fermions, arXiv:2110.13714.
[30] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2009).
[31] Pierre Delplace, Berry-Chern monopoles and spectral flows, SciPost Phys. Lect. Notes 39 (2022) 10.21468/SciPostPhysLectNotes.39.
[32] Frédéric Faure, Manifestation of the topological index formula in quantum waves and geophysical waves, arXiv:1901.10592.
[33] Antoine Venaille, Yohei Onuki, Nicolas Perez, and Armand Leclerc, From ray tracing to topological waves in continuous media, arXiv:2207.01479.

[34] Jean Bellisard, Change of the Chern number at band crossings, arXiv:cond-mat/9504030v1.

[35] Jean-Baptiste Touchais, Pascal Simon, and Andrej Mesaros, Robust propagating in-gap modes due to spin-orbit domain walls in graphene, Phys. Rev. B **106**, 035139 (2022).

[36] Lavi K. Upreti and Pierre Delplace, Topological chiral interface states beyond insulators, Phys. Rev. A **102**, 023520 (2020).

[37] Manolis Perrot, Pierre Delplace, and Antoine Venaille, Topological transition in stratified fluids, Nat. Phys. **15**, 781 (2019).

[38] M. Marciani and P. Delplace, Chiral Maxwell waves in continuous media from Berry monopoles, Phys. Rev. A **101**, 023827 (2020).

[39] A. Venaille and P. Delplace, Wave topology brought to the coast, Phys. Rev. Res. **3**, 043002 (2021).

[40] Ziyan Zhu, Christopher Li, and J. B. Marston, Topology of rotating stratified fluids with and without background shear flow, arXiv:2112.04691.

[41] Hong Qin and Yichen Fu, Topological Langmuir-cyclotron wave, arXiv:2205.02381.

[42] Nicolas Perez, Pierre Delplace, and Antoine Venaille, Unidirectional Modes Induced by Nontraditional Coriolis Force in Stratified Fluids, Phys. Rev. Lett. **128**, 184501 (2022).

[43] T. Iwai and B. Zhilinskii, Topological phase transitions in the vibration-rotation dynamics of an isolated molecule, Theor. Chem. Acc. **133**, 1501 (2014).

[44] T. Iwai and B. Zhilinskii, Band rearrangement through the 2d-dirac equation: Comparing the aps and the chiral bag boundary conditions, Indagat. Math **27**, 1081 (2016).

[45] F. Faure and B. Zhilinskii, Topological Chern Indices in Molecular Spectra, Phys. Rev. Lett. **85**, 960 (2000).

[46] F. Faure and B. Zhilinskii, Topological properties of the Born-Oppenheimer approximation and implications for the exact spectrum, Lett. Math. Phys. **55**, 219 (2001).

[47] A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, Phys. Rev. B **86**, 115133 (2012).

[48] A. A. Burkov, Chiral anomaly and transport in Weyl metals, J. Phys. Condens. Matter **27**, 113201 (2015).

[49] Barry Bradlyn, Jennifer Cano, Zhijun Wang, M. G. Vergniory, C. Felser, R. J, Cava, and B. Andrei Bernevig, Beyond Dirac and Weyl fermions: Unconventional quasi-particles in conventional crystals, Science **353**, (2016).

[50] Motohiko Ezawa, Chiral anomaly enhancement and photo-irradiation effects in multiband touching fermion systems, Phys. Rev. B **95**, 205201 (2017).

[51] J. E. Avron, L. Sadun, J. Segert, and B. Simon, Chern numbers, quaternions, and Berry's phases in Fermi systems, Commun. Math. Phys. **124**, 595 (1989).

[52] Xiangang Wan, Ari M. Turner, Ashvin Vishwanath, and Sergey Y. Savrasov, Topological semimetal and fermi-arc surface states in the electronic structure of pyrochlore iridates, Phys. Rev. B **83**, 205101 (2011).

[53] Michael Francis Atiyah and Isadore Manuel Singer, The index of elliptic operators: I, Ann. Math. **87**, 484 (1968).

[54] Mikio Nakahara, *Geometry, Topology and Physics* (Hilger, Bristol, 2018).

[55] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.066601 for a study of the spectra of $\mathcal{H}$ and its explicit topological caracterisation through nonhermitian invariants. We also generalise the mapping $E \leftrightarrow \lambda$ of the main text.

[56] Russell Yang, Jun Wei Tan, Tommy Tai, Jin Ming Koh, Linhu Li, Stefano Longhi, and Ching Hua Lee, Designing non-Hermitian real spectra through electrostatics, Sci. Bull. **67**, 1865 (2022).

[57] Kohei Kawabata and Masatoshi Sato, Real spectra in non-Hermitian topological insulators, Phys. Rev. Res. **2**, 033391 (2020).

[58] Yong Xu, Sheng-Tao Wang, and L.-M. Duan, Weyl Exceptional Rings in a Three-Dimensional Dissipative Cold Atomic Gas, Phys. Rev. Lett. **118**, 045701 (2017).

[59] Pierre Delplace, Tsuneya Yoshida, and Yasuhiro Hatsugai, Symmetry-Protected Multifold Exceptional Points and Their Topological Characterization, Phys. Rev. Lett. **127**, 186602 (2021).