Rapid Thermalization of Spin Chain Commuting Hamiltonians

Ivan Bardet,[1,*] Ángela Capel,[2,3,4,†] Li Gao,[2,3,5,‡] Angelo Lucia,[6,7,§]
David Pérez-García,[6,7,∥] and Cambyse Rouzé[3,¶]
[1]Inria Paris, 75012 Paris, France
[2]Department of Mathematics, Technische Universität München, 85748 Garching, Germany
[3]Munich Center for Quantum Science and Technology (MCQST), 80799 München, Germany
[4]Fachbereich Mathematik, Universität Tübingen, 72076 Tübingen, Germany
[5]Department of Mathematics, University of Houston, Houston, Texas 77204, USA
[6]Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain
[7]Instituto de Ciencias Matemáticas, 28049 Madrid, Spain

(Received 12 January 2022; revised 20 November 2022; accepted 22 December 2022; published 9 February 2023)

We prove that spin chains weakly coupled to a large heat bath thermalize rapidly at any temperature for finite-range, translation-invariant commuting Hamiltonians, reaching equilibrium in a time which scales logarithmically with the system size. This generalizes to the quantum regime a seminal result of Holley and Stroock from 1989 for classical spin chains and represents an exponential improvement over previous bounds based on the nonclosure of the spectral gap. We discuss the implications in the context of dissipative phase transitions and in the study of symmetry protected topological phases.

DOI: 10.1103/PhysRevLett.130.060401

Introduction.—Understanding how thermal noise affects quantum systems is a major open problem in emerging quantum technologies. A key question is, how long does it take for a system to thermalize (i.e., to converge to its thermal Gibbs state)? Or more specifically, what is the dependency of the thermalization time, also known as *mixing or decoherence time*, on the temperature and the system size?

In particular, it is important to identify those scenarios in which the mixing time scales only logarithmically with the system size—such property is usually called *rapid mixing*. From a negative point of view, in this regime quantum properties that hold in the ground state but not in the thermal state are suppressed too fast for them to be of any reasonable use. On the positive side, thermal states with such short mixing time can be constructed very efficiently with a quantum device that simulates the effect of the corresponding thermal bath. Let us note that constructing thermal Gibbs states is one of the main expected applications of a quantum computer, both as an important self-standing problem [1], and also as a stepping stone in optimization problems, via simulated annealing type algorithms [2–6]. On top of that, rapidly mixing systems have very desirable properties, such as stability with respect to extensive perturbations in the noise operator [7,8].

Despite the importance of the question, very few mathematically rigorous results are known in this direction. The reason is the lack of mathematical techniques to tackle the problem. Indeed, the analogous results for classical systems already required sophisticated mathematical tools, in particular, the notion of log-Sobolev constant for the noise infinitesimal generator, whereas estimates for the spectral gap of the generator are not enough to guarantee such rapid mixing. Starting with the pioneer work of Glauber for the particular case of classical 1D Ising model in 1963 [9], Holley and Stroock [10] (building upon a previous result of Holley [11]) managed to prove the rapid mixing property for all 1D classical models at any temperature. This was done by showing that the log-Sobolev constant decreases at most logarithmically with the system size. Later, Zegarlinski [12] improved their result showing that the log-Sobolev constant was indeed bounded.

In the quantum regime all results have focused on systems with commuting interactions. Note that this does not imply at all that the system is classical. Indeed, such systems include all types of nonchiral topological phases of matter. However, most known results deal only with the spectral gap of the generator, which can only guarantee a mixing time that grows polynomially (and not logarithmically) with the system size [13]. For instance, Alicki *et al.* [14] proved that the spectral gap has a uniform bound independent of the system size for the quantum ferromagnetic 1D Ising model and for Kitaev's toric code in 2D at all temperatures. This result was extended later for all Abelian [15] and non-Abelian [16] Kitaev's quantum double models in 2D, as well as for all 1D models with commuting interactions [13].

In Ref. [17], a log-Sobolev inequality was also introduced in the quantum regime. In particular, a bounded (or logarithmically growing) associated constant is known to imply rapid mixing [18,19]. Since then, several works have appeared trying to estimate such a constant for different noise models in many body quantum systems with commuting interactions [18–21]. Despite considerable effort, the state of the art is that estimates have been obtained either for rather artificial noise operators [22,23] or only for sufficiently high temperature [21].

In this Letter, we prove that at any temperature, 1D quantum systems with commuting interactions are rapidly mixing. That is, they thermalize in a time that scales only logarithmically with the system size. Our approach is to bound the log-Sobolev constant of the associated Davies generator, which is the standard choice for the action of a thermal bath in the weak coupling limit.

This result yields interesting consequences in the context of phase transitions. It is well known that the phases of a given system in thermal equilibrium can be classified according to its physical properties. Moreover, changes of the system allow for the transformation of one phase to another, sometimes abruptly, which results in the appearance of a phase transition. Such phase transitions can also occur in systems that are away from their thermal equilibrium. In this case, due to dissipation, the environment drives the system to the aforementioned equilibrium, which is represented by a state and depends on the system and the environment parameters. As such parameters change, the properties of the system might also change suddenly, yielding a so-called dissipative phase transition [24–28], sometimes also referred to as noise-driven quantum phase transitions [29] or simply quantum phase transitions driven by dissipation [30].

In many cases, such dissipative phase transitions are associated with an abrupt change in the scaling of the thermalization time [28,29,31,32]. Indeed, if the transition is driven by temperature, one expects a slowdown in the convergence to the thermal Gibbs state as one crosses the critical temperature from above, in line with the well-known behavior of the classical 2D Ising model, where the mixing time scaling grows from logarithmic to exponential when crossing the critical temperature [33,34]. Our main result shows that this type of slowdown never happens for 1D quantum systems with commuting interactions, since they all rapidly mix at any temperature.

The result also has implications in the context of symmetry protected topological (SPT) phases [35–37]. There has been a quite intensive study of SPT phases in open quantum systems [38–48] and there is yet no consensus on what is the fate of SPT in the presence of temperature (see, e.g., [40,43] for negative results and [42,49] for positive ones). The 1D cluster state [50]—which plays a key role in the paradigm of measurement based quantum computation [51,52]—has a commuting Hamiltonian and it is a nontrivial SPT phase under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry [53]. Hence, our result applies to it and gives the first example of a nontrivial interacting SPT phase with a provable decoherence time growing only logarithmically with the system size for thermal noise at every nonzero temperature, where in addition all relevant interactions in the problem can be asked to preserve the symmetry, at least in a weak sense. The result has the extra benefit of being stable to extensive perturbations, a general property of quasilocal dissipative evolutions with logarithmic decoherence time [7].

Our result does not apply, however, in the presence of a strong symmetry [54,55], a key condition identified in [47] to preserve SPT in open quantum systems, which emphasizes even more the totally different behavior between weak and strong symmetries in the context of SPT phases in nonzero temperature regimes.

The proof of our main result has two main steps. One step works in arbitrary dimension and gives a way to upgrade a bound on the spectral gap of the Davies generator to a bound of the log-Sobolev constant for commuting Hamiltonians. The proof requires among other things the theory of operator spaces, that has been already proven very useful in answering different questions within quantum information theory [56]. The other step is to show that 1D systems fulfill the hypothesis for such an upgrade to hold.

We expect the first step to be of independent interest, since it opens the possibility to upgrade to the log-Sobolev regime the recent result [16] showing that the Davies generator of quantum double models in 2D have a bounded gap.

Mixing times for Davies maps.—We now briefly recall the construction due to Davies [57], which under the assumption of a weak-coupling limit with a thermal bath at inverse temperature β, gives a description of the evolution of the system as a Markovian master equation. The joint Hamiltonian of the system and the environment can be decomposed as $H = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \lambda H_I$, where H_S is the Hamiltonian of the system, H_E the one of the bath, and H_I is the coupling term between the two of them, with the coupling constant $\lambda \geq 0$. We can decompose H_I as $H_I = \sum_\alpha S^\alpha \otimes B^\alpha$, where S^α, B^α are Hermitian. Renormalizing by the free evolution and sending $\lambda \to 0$ while keeping $\tau = \lambda^2 t$ constant, the reduced evolution of the system is given by $\rho(\tau) = \exp(\tau\mathcal{L})[\rho(0)]$ [57]. Here, $\mathcal{L}$ is a Lindbladian whose Lindblad operators, which we denote by $S^\alpha(\omega)$, satisfy $e^{itH_S} S^\alpha e^{-itH_S} = \sum_\omega S^\alpha(\omega) e^{-i\omega t}$, where the sum is over the Bohr frequencies ω of the system Hamiltonian H_S (for more details, we refer the reader to our companion paper [58]).

Under the assumption that there are no operators commuting with every S^α except the multiples of identity, one can show [59] that the Gibbs state of H_S at inverse temperature β, namely, $\sigma_\beta = Z_\beta{}^{-1} \exp(-\beta H_S)$ is the unique fixed point of the evolution generated by $\mathcal{L}$, and moreover

$$\forall\, \rho, \quad \exp(t\mathcal{L})(\rho) \overset{t\to\infty}{\to} \sigma_\beta. \tag{1}$$

An important problem concerns the speed at which the convergence (1) occurs. This is quantified by the *mixing time* of the dynamics: for $\epsilon > 0$,

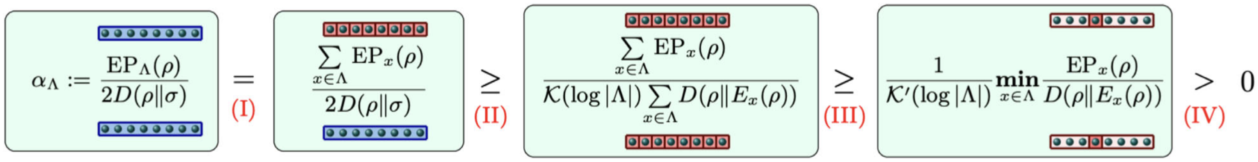

FIG. 1. Sketch of the proof of Theorem 1, which can be summarized in this chain of inequalities, further explained in the main text. The segments of spins placed above and below represent the regions where the numerator, resp. denominator, is acting. More specifically, the full blue segment is for the whole Λ, whereas the cutted-red segment means that the term is defined as the sum of local terms acting on each $x \in \Lambda$ (and their boundaries), which are individually represented by highlighting x and fainting the color in the other sites.

$$t_{\mathrm{mix}}(\epsilon) \coloneqq \inf\{t \geq 0 | \|e^{t\mathcal{L}}(\rho) - \sigma_\beta\|_1 \leq \epsilon\}, \tag{2}$$

where $\|X\|_1 \coloneqq \mathrm{tr}|X|$ denotes the trace norm. One way of controlling the mixing time is via the analysis of the spectral gap of $\mathcal{L}$. It is well known [18] that, whenever the gap can be lower bounded by a constant independent of system size $|\Lambda| = n$, $t_{\mathrm{mix}}(\epsilon) = \mathcal{O}(\sqrt{n})$. This is the case for Davies generators over spin chains with commuting interactions at any positive temperature [13]. Moreover, Glauber dynamics, which can be interpreted as the classical analogues of Davies generators, are known to thermalize logarithmically faster in 1D [10,12] with $t_{\mathrm{mix}}(\epsilon) = \mathcal{O}(\mathrm{polylog}(n))$. This property of a local (quantum) Markovian evolution is known as *rapid mixing*.

One way to prove rapid mixing is to consider the exponential decay of the relative entropy between the evolved state at time t and the invariant state σ_β:

$$D(e^{t\mathcal{L}}(\rho)\|\sigma_\beta) \leq e^{-4\alpha t} D(\rho\|\sigma_\beta). \tag{3}$$

The constant α appearing in (3) is known as the *modified logarithmic Sobolev constant* (MLSI constant) of the semigroup. By Pinsker's inequality together with the bound $D(\rho\|\sigma_\beta) = \mathcal{O}(\log(n))$, one can easily show that $\alpha = \Omega(\mathrm{polylog}(n)^{-1})$ implies the rapid mixing property. This is precisely what we achieve in this Letter.

Main result.—We now state the main result of our Letter, namely, an exponential decay for the entropy in the form of Eq. (3). We consider a finite chain Λ with n sites and the Davies generator $\mathcal{L}_\Lambda$ of a quantum Markov semigroup with unique invariant state $\sigma \equiv \sigma_\Lambda^\beta \coloneqq (e^{-\beta H_\Lambda}/\mathrm{tr}[e^{-\beta H_\Lambda}])$, the Gibbs state of a finite-range, translation-invariant, commuting Hamiltonian at inverse temperature $\beta < \infty$.

Theorem 1.—In the setting introduced above, there exists $\alpha_\Lambda = \Omega((\ln|\Lambda|)^{-1})$ such that, for all $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$ and all $t \geq 0$,

$$D(e^{t\mathcal{L}_\Lambda}(\rho)\|\sigma) \leq e^{-\alpha_\Lambda t} D(\rho\|\sigma). \tag{4}$$

A sketch of the proof of Theorem 1 is shown in Figure 1. The essential feature of the previous result is the scaling of α_Λ with $|\Lambda|$, which we show to be logarithmic, thus implying rapid mixing of the thermal evolution. To prove this, our approach is based on the idea of reducing the MLSI constant in Λ to the MLSI constants in smaller regions $A_i, B_i \subset \Lambda$, in particular taken to be composed of fixed-size (growing logarithmically with $|\Lambda|$) separated segments. By doing so, we reduce the expected scaling of the inverse MLSI constant in Λ, which would be $\mathcal{O}(\mathrm{poly}|\Lambda|)$, to that of the inverse MLSI constants in A_i and B_i, which are actually $\mathcal{O}(\log|\Lambda|)$.

As we will show in Supplemental Material [66], (4) is equivalent to the following inequality:

$$\alpha_\Lambda D(\rho\|\sigma) \leq -\mathrm{tr}[\mathcal{L}_\Lambda(\rho)(\log\rho - \log\sigma)]. \tag{5}$$

The right-hand side of (5) is called the *entropy production* in Λ and denoted by $\mathrm{EP}_\Lambda(\rho)$. Note that it is additive in the region where the Lindbladian is considered, as $\mathcal{L}_\Lambda = \sum_{x\in\Lambda}\mathcal{L}_x$. Therefore, for $A \cup B = \Lambda$ with $A \cap B = \emptyset$, this implies $\mathcal{L}_\Lambda = \mathcal{L}_A + \mathcal{L}_B$ and thus $\mathrm{EP}_\Lambda(\rho) = \mathrm{EP}_A(\rho) + \mathrm{EP}_B(\rho)$. This is (I) in Figure 1. The left-hand side, however, is much more subtle, as no such property is valid for the relative entropy. We are able to prove, though, some form of subadditivity for the relative entropy, in terms of some so-called *conditional relative entropies* (in subregions of Λ), up to a multiplicative factor which encodes how correlations decay on the thermal equilibrium of the evolution: This result is named *quasifactorization of the relative entropy* and all the combined steps listed below are represented as (II) in Figure 1.

Quasifactorization.—This part of the proof is sketched in Fig. 2. We follow the next steps to reduce the global relative entropy in Λ to on-site conditional relative entropies of each site.

Step 1: We consider the relative entropy between an arbitrary state ρ and the equilibrium state σ in Λ.

Steps 2 and 3: We reduce it to some conditional relative entropies in smaller regions $\{A_i, B_i\}$, in the spirit of the results of [20–22], and a multiplicative error term depending on how correlations decay on σ between $(\cup_i A_i)^c$ and $(\cup_i B_i)^c$, which can be interpreted as a mixing condition and is controlled using Araki's estimates [60] as in the recent [61].

Step 4: We use operator spaces to lift the results of [62] to nontracial conditional expectations to further reduce the

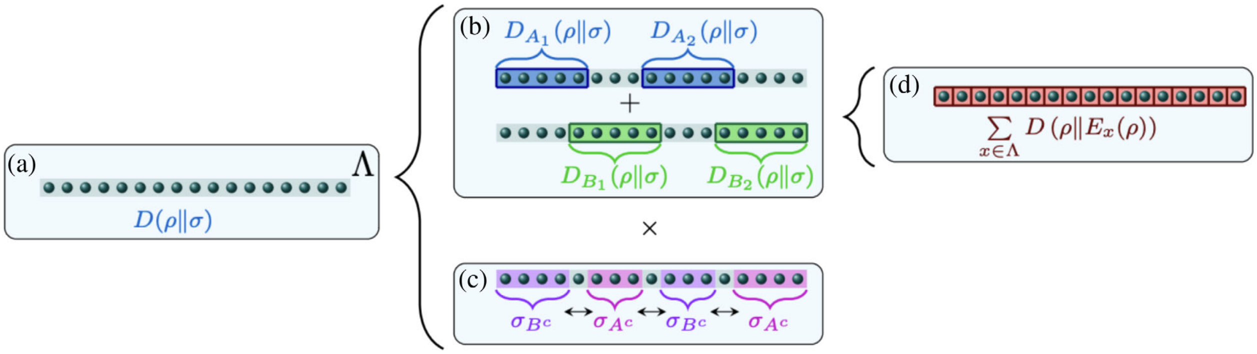

FIG. 2. Quasi-factorizations of the relative entropy used in the proof of Theorem 1: First, we consider in (a) the relative entropy between the evolved state and the equilibrium in Λ and reduce it to some conditional relative entropies in smaller regions $\{A_i, B_i\}$ as in (b), in the spirit of the results of [20–22], and a multiplicative error term in (c), depending on how correlations decay on σ between $(\cup_i A_i)^c$ and $(\cup_i B_i)^c$, which can be interpreted as a mixing condition and is controlled using Araki's estimates [60] as in the recent [61]. In (d), we use operator spaces to lift the results of [62] to nontracial conditional expectations to further reduce the latter conditional relative entropies to another form of on-site conditional relative entropies.

conditional relative entropies on each smaller region A_i and B_i to the sum of on-site conditional relative entropies.

Local control of the MLSI constant.—It is only left to show that the latter on-site conditional relative entropies can be bounded by the entropy production on each site, i.e., there is $\alpha_x > 0$ such that

$$\alpha_x D[\rho\|E_x(\rho)] \leq \mathrm{EP}_x(\rho), \tag{6}$$

for every $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$. This is obtained as a consequence of the findings of [62] and represents (IV) in Figure 1. Note that (III) follows from choosing a universal α_0 for every $x \in \Lambda$ in (6).

More details for the proof of Theorem 1 are provided in Appendix I. For a complete proof, we refer the reader to our companion paper [58].

SPT phases.—In this section, we discuss how our result applies to the understanding of the question whether nontrivial SPT phases are robust against temperature. Let us consider then an on-site symmetry u_g for some group G and a finite range frustration-free commuting system Hamiltonian H_S which also commutes with u_g,

$$[H_S, u_g] = 0 \quad \forall \ g \in G,$$

and belongs to a nontrivial SPT phase protected by the on-site symmetry u_g. The paradigmatic example is the 1D cluster state, where the group G is $\mathbb{Z}_2 \times \mathbb{Z}_2$. As a nontrivial SPT system, when considered with open boundary conditions, it has a degenerate ground space—the edge states—which is protected by the symmetry against symmetric perturbations of H_S, very much like the case of ordinary topological order [63]. We review the cluster state example in Supplemental Material [66]. and refer to [64] for a detailed introduction to SPT order.

In this setup, one needs to ask for the Davies thermalization process to also respect the symmetry G. We will do this by requiring that the Davies generator is *covariant* with respect to the symmetry u_g: for every state ρ and every $g \in G$, it holds that

$$\mathcal{L}(u_g^\dagger \rho u_g) = u_g^\dagger \mathcal{L}(\rho) u_g \quad \forall \, \rho, \, \forall \, g \in G. \tag{7}$$

We remark that a sufficient condition for this to happen is that the jump operators S^α commute with u_g up to a phase:

$$S^\alpha u_g = \omega_g^\alpha u_g S^\alpha, \qquad \omega_g^\alpha \in U(1). \tag{8}$$

It is easy to construct many examples of covariant Davies generators. In fact, this is always possible when the symmetry u_g is made of Pauli terms (tensor products of Pauli matrices), by choosing S^α to be also Pauli operators. This covers the case of the 1D cluster state.

We also remark that any such covariant generator $\mathcal{L}$ can be obtained as the weak-coupling limit of the interaction with a thermal bath that is *weakly symmetric*, in the sense that there exists a representation U_g of G acting on the Hilbert space of the environment such that, for each $g \in G$ and all α, $[H_E, U_g] = 0$ and $[S^\alpha \otimes B^\alpha, u_g \otimes U_g] = 0$. In fact, if this is not the case, one can extend the original environment by a conjugate copy of the system:

$$\tilde{B}^\alpha = B^\alpha \otimes \bar{S}^\alpha, \qquad U_g = \mathbb{1} \otimes \bar{u}_g,$$

obtaining a weakly symmetric thermal bath.

Our main result applied to the 1D cluster state implies the following.

Corollary 1.—There exist nontrivial 1D SPT phases which thermalize in logarithmic time in the system size for every inverse temperature $\beta < \infty$, even when the thermal bath is chosen to be weakly symmetric.

Since in the thermal Gibbs state all the information initially encoded in the ground space is washed out, this implies that SPT protection is in general not robust against temperature, at least in the weakly symmetric case.

Discussion.—In this Letter, we have shown that the Davies dynamics associated with any 1D spin chain translation-invariant commuting Hamiltonian at finite temperature satisfies a log-Sobolev inequality, and therefore the corresponding thermalization process converges logarithmically fast in terms of the system size (the *rapid mixing* property). This also holds under the assumption that the evolution is weakly symmetric with respect to a given symmetry, for example, in the case of SPT phases.

We expect our two-step proof strategy to be relevant in higher dimensions. We leave the study of log-Sobolev constants for Davies generators of 2D quantum double models, whose gap was recently investigated in [16], to future work.

Finally, one could ask whether our result for SPT phases would apply to the setting where the thermal bath is chosen to be *strongly symmetric*, in the sense that the representation U_g acting on the Hilbert space of the environment is the trivial one. This is not the case, given that this condition prevents the thermal evolution from being ergodic, and in particular for it to have a unique invariant state. This can be seen by noticing that strong symmetry would imply that all u_g are invariant, in the sense that $\mathrm{tr}[u_g \mathcal{L}(\rho)] = 0$ for any ρ. A sufficient condition for this to happen is that $[S^\alpha, u_g] = 0$ for each α and $g \in G$. In the presence of a full rank invariant state, this condition is also necessary [65]. When u_g is not irreducible (which is the case for local on-site symmetries), this implies that $\mathcal{L}$ has multiple invariant states and therefore it is not ergodic. This issue was solved in [43] by restricting the initial state only to the subspace of u_g-symmetric states, and studying the thermalization of the *symmetric Gibbs ensemble* (a nonfull rank state). We leave open the question of whether our techniques could be adapted to cover this case.

I. B. is supported by French ANR Grant: ANR-20-CE47-0014-01 “ESQuisses.” A. C. was partially supported by MCQST Distinguished Postdoc and the Seed Funding Program of the MCQST (EXC-2111/Projekt-ID: 390814868). L. G. was partially supported by NSF Grant No. DMS-2154903. A. L. acknowledges support from the BBVA Foundation and the Spanish “Ramón y Cajal” Programme (Grant No. RYC2019-026475-I funded by MCIN/AEI/10.13039/501100011033 and “ESF Investing in your future”). D. P. G. acknowledges support from Comunidad de Madrid (Grant QUITEMAD-CM, Ref. P2018/TCS-4342) and from the Spanish Ministry of Science and Innovation (“Severo Ochoa Programme for Centres of Excellence in R&D” CEX2019-000904-S and Grant No. PID2020–113523 GB-I00) as well as from CSIC Quantum Technologies Platform PTI-001. C. R. acknowledges financial support from a Junior Researcher START Fellowship from the MCQST, and A. C. and C. R. also acknowledge financial support by the DFG cluster of excellence 2111 (Munich Center for Quantum Science and Technology). This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant Agreement No. 648913), from the Ministry of Economic Affairs and Digital Transformation of the Spanish Government through the QUANTUM ENIA project call—QUANTUM SPAIN project, and by the European Union through the Recovery, Transformation and Resilience Plan—NextGenerationEU within the framework of the Digital Spain 2025 Agenda.

[*]ivan.bardet@inria.fr
[†]angela.capel@uni-tuebingen.de
[‡]lgao20@central.uh.edu
[§]anglucia@ucm.es
[‖]dperezga@ucm.es
[¶]cambyse.rouze@tum.de

[1] K. Temme, T. J. Osborne, K. G. Vollbrecht, D. Poulin, and F. Verstraete, Quantum metropolis sampling, Nature (London) **471**, 87 (2011).

[2] K. Huang, *Statistical Mechanics* (John Wiley & Sons, New York, 1987).

[3] R. D. Somma, S. Boixo, H. Barnum, and K. Emanuel, Quantum Simulations of Classical Annealing Processes, Phys. Rev. Lett. **101**, 130504 (2008).

[4] F. G. S. L. Brandão and K. M. Svore, Quantum speed-ups for solving semidefinite programs, in *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, New York, 2017), pp. 415–426.

[5] M. Kieferová and N. Wiebe, Tomography and generative training with quantum Boltzmann machines, Phys. Rev. A **96**, 062327 (2017).

[6] J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, Quantum machine learning, Nature (London) **549**, 195 (2017).

[7] T. S. Cubitt, A. Lucia, S. Michalakis, and D. Pérez-García, Stability of local quantum dissipative systems, Commun. Math. Phys. **337**, 1275 (2015).

[8] A. Lucia, T. S. Cubitt, S. Michalakis, and D. Pérez-García, Rapid mixing and stability of quantum dissipative systems, Phys. Rev. A **91**, 040302 (2015).

[9] R. J. Glauber, Time-dependent statistics of the Ising model, J. Math. Phys. (N.Y.) **4**, 294 (1963).

[10] R. A. Holley and D. W. Stroock, Uniform and L2 convergence in one dimensional stochastic Ising models, Commun. Math. Phys. **123**, 85 (1989).

[11] R. Holley, Rapid convergence to equilibrium in one dimensional stochastic Ising models, Ann. Prob. **13**, 72 (1985).

[12] B. Zegarlinski, Log-Sobolev inequalities for infinite one-dimensional lattice systems, Commun. Math. Phys. **133**, 147 (1990).

[13] M. J. Kastoryano and F. G. S. L. Brandao, Quantum Gibbs samplers: The commuting case, Commun. Math. Phys. **344**, 915 (2016).

[14] R. Alicki, M. Fannes, and M. Horodecki, On thermalization in Kitaev's 2D model, J. Phys. A **42**, 065303 (2009).

[15] A. Kómár, O. Landon-Cardinal, and K. Temme, Necessity of an energy barrier for self-correction of abelian quantum doubles, Phys. Rev. A **93**, 052337 (2016).

[16] A. Lucia, D. Pérez-García, and A. Pérez-Hernández, Thermalization in Kitaev's quantum double models via Tensor Network techniques, arXiv:2107.01628.

[17] L. Gross, Hypercontractivity and logarithmic sobolev inequalities for the clifford-dirichlet form, Duke Math. J. **42**, 383 (1975).

[18] M. J. Kastoryano and K. Temme, Quantum logarithmic Sobolev inequalities and rapid mixing, J. Math. Phys. (N.Y.) **54**, 052202 (2013).

[19] K. Temme, F. Pastawski, and M. J. Kastoryano, Hypercontractivity of quasi-free quantum semigroups, J. Phys. A **47**, 405303 (2014).

[20] I. Bardet, Á. Capel, A. Lucia, and D. Pérez-García, and C. Rouzé, On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems, J. Math. Phys. (N.Y.) **62**, 061901 (2021).

[21] Á Capel, C. Rouzé, and D. Stilck França, The modified logarithmic Sobolev inequality for quantum spin systems: Classical and commuting nearest neighbour interactions, arXiv:2009.11817.

[22] Á. Capel, A. Lucia, and D. Pérez-Garćida, Quantum conditional relative entropy and quasi-factorization of the relative entropy, J. Phys. A **51**, 484001 (2018).

[23] S. Beigi, N. Datta, and C. Rouzé, Quantum reverse hypercontractivity: Its tensorization and application to strong converses, Commun. Math. Phys. **376**, 753 (2020).

[24] P. Werner, K. Völker, M. Troyer, and S. Chakravarty, Phase Diagram and Critical Exponents of a Dissipative Ising Spin Chain in a Transverse Magnetic Field, Phys. Rev. Lett. **94**, 047201 (2005).

[25] L. Capriotti, A. Cuccoli, A. Fubini, V. Tognetti, and R. Vaia, Dissipation-Driven Phase Transition in Two-Dimensional Josephson Arrays, Phys. Rev. Lett. **94**, 157001 (2005).

[26] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Büchler, and P. Zoller, Quantum states and phases in driven open quantum systems with cold atoms, Nat. Phys. **4**, 878 (2008).

[27] S. Morrison and A. S. Parkins, Dissipation-driven quantum phase transitions in collective spin systems, J. Phys. B **41**, 195502 (2008).

[28] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin, and J. I, Cirac, Dissipative phase transitions in a central spin system, Phys. Rev. A **86**, 012116 (2012).

[29] B. Horstmann, J. I. Cirac, and G. Giedke, Noise-driven dynamics and phase transitions in fermionic systems, Phys. Rev. A **87**, 012108 (2013).

[30] F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum computation and quantum-state engineering driven by dissipation, Nat. Phys. **5**, 633 (2009).

[31] F. Minganti, A. Biella, N. Bartolo, and C. Ciuti, Spectral theory of Liouvillians for dissipative phase transitions, Phys. Rev. A **98**, 042118 (2018).

[32] T. Barthel and Y. Zhang, Superoperator structures and no-go theorems for dissipative quantum phase transitions, Phys. Rev. A **105**, 052224 (2022).

[33] F. Martinelli, *Lectures on Glauber Dynamics for Discrete Spin Models* (Springer Berlin Heidelberg, Berlin, Heidelberg, 1999), pp. 93–191.

[34] F. Martinelli, *Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2004), pp. 175–262.

[35] X.-G. Wen, Quantum orders and symmetric spin liquids, Phys. Rev. B **65**, 165113 (2002).

[36] Z.-C. Gu and X.-G. Wen, Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order, Phys. Rev. B **80**, 155131 (2009).

[37] X. Chen, Z.-C. Gu, and X.-G. Wen, Classification of gapped symmetric phases in one-dimensional spin systems, Phys. Rev. B **83**, 035107 (2011).

[38] S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, Topology by dissipation in atomic quantum wires, Nat. Phys. **7**, 971 (2011).

[39] D. Rainis and D. Loss, Majorana qubit decoherence by quasiparticle poisoning, Phys. Rev. B **85**, 174533 (2012).

[40] O. Viyuela, A. Rivas, and M. A. Martín-Delgado, Thermal instability of protected end states in a one-dimensional topological insulator, Phys. Rev. B **86**, 155140 (2012).

[41] C.-E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. İmamoğlu, P. Zoller, and S. Diehl, Topology by dissipation, New J. Phys. **15**, 085001 (2013).

[42] O. Viyuela, A. Rivas, and M. A. Martín-Delgado, Symmetry-protected topological phases at finite temperature, 2D Mater. **2**, 034006 (2015).

[43] S. Roberts, B. Yoshida, A. Kubica, and S. D. Bartlett, Symmetry-protected topological order at nonzero temperature, Phys. Rev. A **96**, 022306 (2017).

[44] M. McGinley and N. R. Cooper, Interacting symmetry-protected topological phases out of equilibrium, Phys. Rev. Res. **1**, 033204 (2019).

[45] A. Coser and D. Pérez-García, Classification of phases for mixed states via fast dissipative evolution, Quantum **3**, 174 (2019).

[46] M. McGinley and N. R. Cooper, Fragility of time-reversal symmetry protected topological phases, Nat. Phys. **16**, 1181 (2020).

[47] C. de Groot, A. Turzillo, and N. Schuch, Symmetry protected topological order in open quantum systems, Quantum **6**, 856 (2022).

[48] A. Altland, M. Fleischhauer, and S. Diehl, Symmetry Classes of Open Fermionic Quantum Matter, Phys. Rev. X **11**, 021037 (2021).

[49] O. Viyuela, A. Rivas, S. Gasparinetti, A. Wallraff, S. Filipp, and M. A. Martídn-Delgado, Observation of topological uhlmann phases with superconducting qubits, npj Quantum Inf. **4**, 1 (2018).

[50] H. J. Briegel and R. Raussendorf, Persistent Entanglement in Arrays of Interacting Particles, Phys. Rev. Lett. **86**, 910 (2001).

[51] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. **86**, 5188 (2001).

[52] R. Raussendorf and H. J. Briegel, Computational model underlying the one-way quantum computer, Quantum Inf. Comput. **2**, 443 (2002).

[53] W. Son, L. Amico, R. Fazio, A. Hamma, S. Pascazio, and V. Vedral, Quantum phase transition between cluster and antiferromagnetic states, Europhys. Lett. **95**, 50001 (2011).

[54] B. Buča and T. Prosen, A note on symmetry reductions of the lindblad equation: Transport in constrained open spin chains, New J. Phys. **14**, 073007 (2012).

[55] V. V. Albert and L. Jiang, Symmetries and conserved quantities in Lindblad master equations, Phys. Rev. A **89**, 022118 (2014).

[56] C. Palazuelos and T. Vidick, Survey on nonlocal games and operator space theory, J. Math. Phys. (N.Y.) **57**, 015220 (2016).

[57] E. B. Davies, Markovian master equations, Commun. Math. Phys. **39**, 91 (1974).

[58] I. Bardet, Á. Capel, L. Gao, A. Lucia, D. Pérez-García, and C. Rouzé, Entropy decay for Davies semigroups of 1D quantum spin chains, arXiv:2112.00601.

[59] M. M. Wolf, Quantum channels & operations: Guided tour, Lecture notes available at http://www-m5.ma.tum, de/foswiki/pub M, 5, 2012.

[60] H. Araki, Gibbs states of a one dimensional quantum lattice, Commun. Math. Phys. **14**, 120 (1969).

[61] A. Bluhm, Á. Capel, and A. Pérez-Hernández, Exponential decay of mutual information for Gibbs states of local Hamiltonians, Quantum **6**, 650 (2022).

[62] L. Gao and C. Rouzé, Complete entropic inequalities for quantum Markov Chains, Arch. Ration. Mech. Anal. **245**, 183 (2022).

[63] S. Bravyi, M. B. Hastings, and S. Michalakis, Topological quantum order: Stability under local perturbations, J. Math. Phys. (N.Y.) **51**, 093512 (2010).

[64] Bei Zeng, Xie Chen, Duan-Lu Zhou, and Xiao-Gang Wen, *Quantum Information Meets Quantum Matter* (Springer, New York, 2019).

[65] R. Carbone, E. Sasso, and V. Umanità, Decoherence for quantum Markov semi-groups on matrix algebras, Ann. Inst. Henri Poincaré **14**, 681 (2012).

[66] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.060401 for more details on the techniques used in the proof of the main result, as well as an overview on the cluster state. This Material includes Refs. [67–89].

[67] H. Amann, *Ordinary Differential Equations: An Introduction to Nonlinear Analysis* (Walter de gruyter, Berlin, New York 2011), Vol. 13.

[68] F. Cesi, Quasi-factorization of the entropy and logarithmic Sobolev inequalities for Gibbs random fields, Probab. Theory Relat. Fields **120**, 569 (2001).

[69] P. Dai Pra, A. M. Paganoni, and G. Posta, Entropy inequalities for unbounded spin systems, Ann. Probab. **30**, 1959 (2002).

[70] Á. Capel, Quantum logarithmic Sobolev inequalities for quantum many-body systems: An approach via quasi-factorization of the relative entropy, Ph.D thesis at Universidad Autónoma de Madrid, 2019.

[71] I. Bardet, Á. Capel, and C. Rouzé, Approximate tensorization of the relative entropy for noncommuting conditional expectations, Ann. Inst. Henri Poincaré **23**, 101 (2021).

[72] I. Bardet, Estimating the decoherence time using non-commutative functional inequalities, arxiv:1710.01039.

[73] N. LaRacuente, Quasi-factorization and multiplicative comparison of subalgebra-relative entropy, J. Math. Phys. (N.Y.) **63**, 122203 (2022).

[74] M. Junge and J. Parcet, Mixed-Norm Inequalities and Operator Space L_p Embedding Theory (American Mathematical Society, Providence, 2010).

[75] I. Bardet and C. Rouzé, Hypercontractivity and logarithmic sobolev inequality for non-primitive quantum markov semigroups and estimation of decoherence rates, Ann. Inst. Henri Poincaré **23**, 3839 (2018).

[76] L. Gao, M. Junge, and N. LaRacuente, Fisher information and logarithmic sobolev inequality for matrix-valued functions, Ann. Inst. Henri Poincaré **21**, 3409 (2020).

[77] G. Pisier, *Non-Commutative Vector Valued Lp-Spaces and Completely p-Summing Maps* (Société mathématique de France, 1998).

[78] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, The detectability lemma and quantum gap amplification, in *Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing* (2009), pp. 417–426, 10.1145/1536414.1536472.

[79] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, Quantum Hamiltonian complexity and the detectability lemma, arXiv:1011.3445.

[80] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Entanglement spectrum of a topological phase in one dimension, Phys. Rev. B **81**, 064439 (2010).

[81] J. Haegeman, D. Pérez-García, I. Cirac, and N. Schuch, Order Parameter for Symmetry-Protected Phases in One Dimension, Phys. Rev. Lett. **109**, 050402 (2012).

[82] N. Schuch, D. Pérez-García, and I. Cirac, Classifying quantum phases using matrix product states and projected entangled pair states, Phys. Rev. B **84**, 165139 (2011).

[83] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Symmetry protection of topological phases in one-dimensional quantum spin systems, Phys. Rev. B **85**, 075125 (2012).

[84] L. Fidkowski and A. Kitaev, Topological phases of fermions in one dimension, Phys. Rev. B **83**, 075103 (2011).

[85] I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, Matrix product states and projected entangled pair states: Concepts, symmetries, and theorems, Rev. Mod. Phys. **93**, 045003 (2021).

[86] M. Sanz, M. M. Wolf, D. Pérez-García, and J. I. Cirac, Matrix product states: Symmetries and two-body Hamiltonians, Phys. Rev. A **79**, 042308 (2009).

[87] J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, Matrix product density operators: Renormalization fixed points and boundary theories, Ann. Phys. (Amsterdam) **378**, 100 (2017).

[88] G. De las Cuevas, J. I. Cirac, N. Schuch, and D. Pérez-García, Irreducible forms of matrix product states: Theory and applications, J. Math. Phys. (N.Y.) **58**, 121901 (2017).

[89] Y. Ogata, Classification of gapped ground state phases in quantum spin systems, arXiv:2110.04675.