$f_0(1370)$ Controversy from Dispersive Meson-Meson Scattering Data Analyses

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We establish the existence of the long-debated $f_0(1370)$ resonance in the dispersive analyses of meson-meson scattering data. For this, we present a novel approach using forward dispersion relations, valid for generic inelastic resonances. We find its pole at $(1245 \pm 40) - i(300^{+30}_{-70})$ MeV in $\pi\pi$ scattering. We also provide the couplings as well as further checks extrapolating partial-wave dispersion relations or with other continuation methods. A pole at $(1380^{+70}_{-60}) - i(220^{+80}_{-70})$ MeV also appears in the $\pi\pi \to K\bar{K}$ data analysis with partial-wave dispersion relations. Despite settling its existence, our model-independent dispersive and analytic methods still show a lingering tension between pole parameters from the $\pi\pi$ and $K\bar{K}$ channels that should be attributed to data.

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Introduction.—Quantum chromodynamics (QCD) became the theory of strong interactions almost 50 years ago, but its low-energy regime, particularly the lightest scalar spectrum, is still under debate—see Refs. [1,2] and the "Scalar Mesons below 2 GeV" note in the Review of Particle Physics [3] (RPP). This may be surprising, since light scalars are relevant for nucleon-nucleon interactions, final states in heavy hadron decays, CP violation, the identification of the lightest glueball, and the understanding of the QCD spontaneous chiral symmetry breaking. Moreover, a precise knowledge of this sector is not only relevant by itself and QCD, but also for the accuracy frontier of nuclear and particle physics.

This debate lingers on because light scalars do not show up as sharp peaks, since some of them are very wide and overlap with others, or are distorted by nearby two-body thresholds. Indeed, their shape changes with the dynamics of the process where they appear. Hence, they must be rigorously identified from their process-independent associated poles. These appear in the complex s plane of any amplitude $T(s)$ where resonances exist. Here, s is the total CM-energy squared Mandelstam variable. Then, the pole mass M and width Γ are defined as $\sqrt{s_{\text{pole}}} = M - i\Gamma/2$. The familiar peak shape only appears in the real axis when the resonance is narrow and isolated from other singularities. Only then, simple Breit-Wigner (BW) approximations, K matrices or isobar sums may be justified, but not for the lightest scalars and definitely not for the $f_0(1370)$.

Problems identifying light scalars are crudely of two types. The "data problem" is severe in meson-meson scattering, where scalars were first observed, since data are extracted indirectly from the virtual-pion-exchange contribution to meson-nucleon to meson-meson-nucleon scattering. Hence, the initial state is not well defined, leading to inconsistencies in the data and with fundamental principles. This is not a problem for heavy-meson decays, generically with better statistics and less systematic uncertainty. The "model problem" arises when searching for poles, since analytic continuations are mathematically delicate, particularly for resonances deep in the complex plane. Unfortunately, they are often carried out with models (BW, K-matrices, "isobar" sums,...). Dispersion theory addresses both problems by discarding inconsistent data, and avoiding model dependencies in data parametrizations and resonance identification.

The RPP [3] lists the $\sigma/f_0(500)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ scalar-isoscalar resonances below 2 GeV. The longstanding controversy on the $\sigma/f_0(500)$ existence, and the similar strange $\kappa/K_0^*(700)$, (see Refs. [1,4]), was settled by dispersive studies [2,5–9]. The $f_0(980)$, close to $K\bar{K}$ threshold, has also been rigorously determined dispersively [6,7]. This narrow state illustrates the process dependence of shapes, appearing as a dip in the $\pi\pi \to \pi\pi$ cross section but as a peak in heavy-meson decays. The $f_0(1500)$ and $f_0(1710)$ are well established. The former has less than 10 MeV uncertainties for its mass and width and five accurate branching fractions listed in the RPP. The $f_0(1710)$ has mass and width uncertainties below 20 MeV and six "seen" decay modes.

In contrast, the $f_0(1370)$ remains controversial in the hadron community: some reviews find enough evidence to consider it well established [10] whereas other reviews [11,12] or recent experiments [13], do not. Indeed, a scalar-isoscalar state between 1.2 and 1.5 GeV has been reported by several experiments [14–41], but with large disagreements on its parameters and decay channels. However, it was absent in the classic $\pi\pi$ scattering analyses [42–45], where a resonant phase motion is not seen. One of our main results here is that we do find such a pole using just $\pi\pi \to \pi\pi$ data and rigorous dispersive and analytic techniques. A pole is often found in multichannel or multiprocess data analyses, where 4π are approximated as background or quasi-two-body states (see, for instance, [46], and unitarized chiral approaches in [47–50]). In general, $f_0(1370)$ analyses suffer from some aspects of the "model problem" and, worse, its appearance depends strongly on the source [51,52]. All in all, the RPP places the $f_0(1370)$ pole within a huge range, $(1200 - 1500) - i(150 - 250)$ MeV, and lists its decay modes only as seen, remarking the elusiveness of its two-pion coupling. Although inappropriate for this resonance, the RPP lists its BW parameters, separating the "$K\bar{K}$ mode" mass, always above $\sim$1.35 GeV, from the "$\pi\pi$ mode" mass, reaching as low as $\sim$1.2 GeV.

Here we confirm the $f_0(1370)$ presence in $\pi\pi \to \pi\pi$ scattering data, absent in the original analyses, providing a rigorous determination of its position and coupling, using model-independent dispersive and analytic methods. We study two-meson scattering because it obeys the most stringent dispersive constraints. Thus, we next explain the data dispersive constraints, then the analytic methods to reach the poles, and finally discuss results and checks. Nonessential details are given in [53].

Dispersion relations for $\pi\pi \to \pi\pi$, $K\bar{K}$.—We assume the customary isospin limit. Since no bound states exist in meson-meson scattering, the fixed-t amplitude $F(s,t)$, is analytic in the first Riemann sheet of the complex-s plane except for a right-hand cut (RHC) along the real axis from $s = 4m_\pi^2$ to $+\infty$. Crossing this RHC continuously leads to the "adjacent" sheet, where resonance poles sit. In addition, there is a left-hand cut (LHC) from $-\infty$ to $s = -t$ due to crossed channel cuts. For forward scattering ($t = 0$) and partial waves it extends to $s = 0$. Using Cauchy's integral formula the amplitude in the first Riemann sheet is recast in terms of integrals over its imaginary part along the RHC and LHC.

Customarily, the pole of a resonance with isospin I and spin J is obtained from $f_J^I(s)$ partial waves. In the elastic case the adjacent sheet is the inverse of the first, and this is how the $\sigma/f_0(500)$, $f_0(980)$, and $\kappa/K_0^*(700)$ poles were determined dispersively [2,5–8]. However, the $f_0(1370)$ lies in the inelastic region and the continuation to the adjacent sheet has to be built explicitly, for which we will use analytic continuation techniques. To avoid model dependencies we will continue the dispersive output of our constrained fits to data (CFD) [56], not the fits themselves.

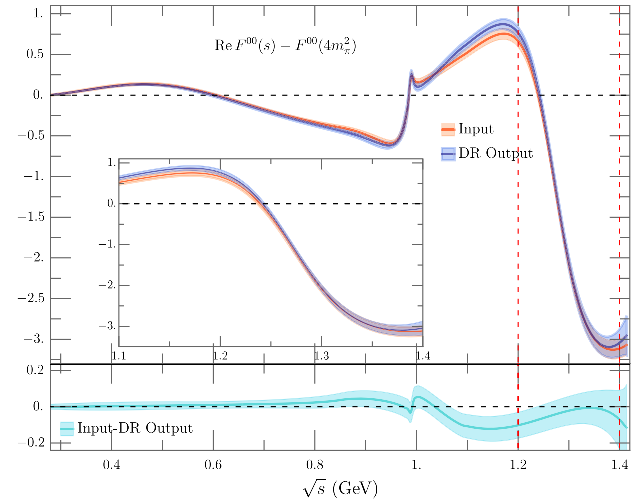

FIG. 1. F^{00} Forward dispersion relation for the CFD [56]. Note the input-output agreement between 1.2 and 1.4 GeV.

For $\pi\pi \to \bar{K}K$ we can use the output of partial-wave Roy-Steiner equations (RS), recently extended to 1.47 GeV, whose corresponding CFDs were obtained in [2,57,58].

However, the applicability of Roy and GKPY dispersion relations for $\pi\pi \to \pi\pi$ partial waves is limited to $\sim$1.1 GeV. Hence, we have implemented a novel approach, by continuing the output of $\pi\pi$ forward dispersion relations (FDR). These can rigorously reach any energy, and in [56,58,59] were used to constrain partial-wave data up to 1.42 GeV. The caveat is that FDRs alone do not determine the resonance spin. Among the different FDRs, the most precise is that for $F^{00} \equiv (F^0 + 2F^2)/3$ [56,60], where $F^I(s,t)$ are the $\pi\pi$ scattering amplitudes with isospin I. As input we will use the $\pi\pi \to \pi\pi$ CFD from [56], which describes data and satisfies three FDRs as well as Roy and GKPY equations [56,61]. Figure 1 shows that the once-subtracted F^{00} FDR is well satisfied in the 1.2–1.4 GeV region, dominated by the $f_2(1270)$. Note that FDRs have data input up to several tens of GeV, but above 1.42 GeV they were not used as constraints. Still, they should remain valid within their large and growing uncertainties well above that energy.

Analytic continuation methods.—There are several analytic continuation techniques from a real segment to the complex plane: conformal expansions [62,63], Laurent-Pietarinen expansions [64–66], sequences of Padé approximants [67–70], or continued fractions [71–74]. For the FDR, the $f_2(1270)$ pole lies between the real axis and the $f_0(1370)$ pole, as seen in Fig. 2. Determining the latter with Padé sequences thus requires very high derivatives, making them unsuitable for the FDR method, although still valid for other checks. After trying several methods, the most

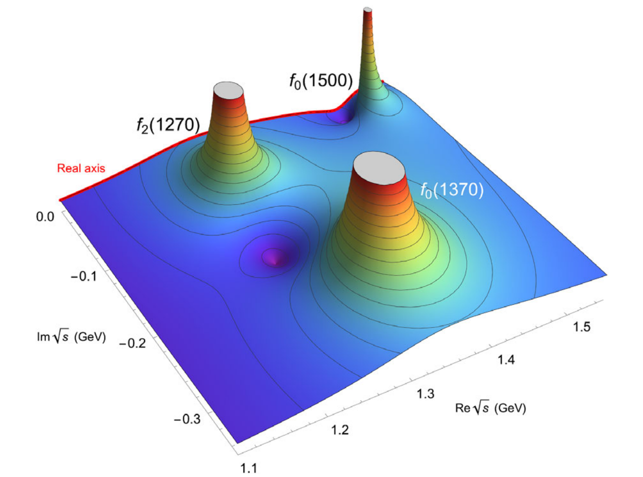

FIG. 2. $|F^{00}(s, t=0)|$ obtained from the F^{00} FDR, using as input the CFD in [56], analytically continued by means of continued fractions. Note the $f_2(1270)$ pole between the real axis and the $f_0(1370)$ pole, and the $f_0(1500)$ pole nearby.

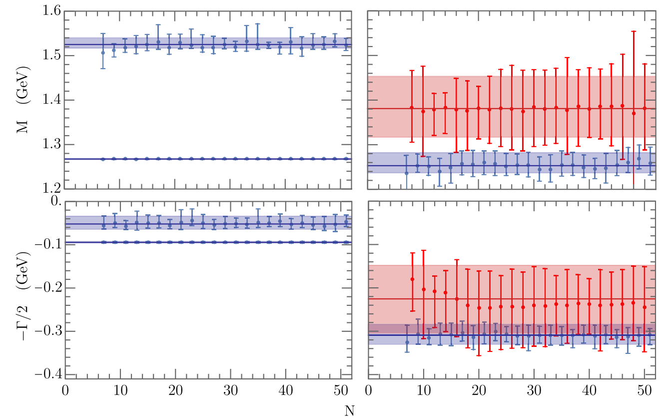

FIG. 3. Pole masses (M, top) and half-widths ($\Gamma/2$, bottom) of the $f_2(1270)$, $f_0(1500)$ (left), and $f_0(1370)$ (right). They are obtained from the output of the F^{00} $\pi\pi \to \pi\pi$ FDR (blue) or Roy-Steiner $\pi\pi \to K\bar{K}$ dispersive output (red) analytically continued to the complex plane by a continued fraction of order N (horizontal axis). CFD are used as input. Note the tension between $\pi\pi$ and $K\bar{K}$ $f_0(1370)$ determinations.

stable both for $\pi\pi$ and $K\bar{K}$ are the continued fractions, C_N. As usual, these are $N-1$ nested fractions, whose parameters are fixed by imposing $C_N(s_i) = F(s_i)$ for N real values s_i within the domain of interest.

Results.—Let us first describe the $\pi\pi \to \pi\pi$ F^{00} FDR output continuation. The C_N are calculated from $N = 7$ up to 51 equally spaced energies in the 1.2–1.4 GeV segment, which maximizes the region where the FDR is well satisfied. Still, the $f_0(1370)$ is also found with much smaller segments, even if they lie completely below 1.3 GeV. Figure 2 shows a typical case, where $f_0(1370)$, $f_2(1270)$, and $f_0(1500)$ poles are found. Finding the latter is striking since it lies above our segment and the CFD above 1.42 GeV was not fitted to partial-wave data but to total cross-section data within a Regge formalism that describes this region "on the average."

In Fig. 3 we show in blue the pole masses (top) and half widths (bottom) for each N. Statistical errors are propagated from the CFD input [56,58,59]. For each N a systematic uncertainty is added by considering several intervals up to 25 MeV lower in either segment end. Results are very stable for the three resonances, and their uncertainties are obtained from a weighted average of the values for each N. Note that the energy where the CFD tensor-isoscalar partial-wave phase reaches $\pi/2$ was fixed at 1274.5 MeV, so the $f_2(1270)$ pole appears at $1267.5 - i94$ MeV, with negligible error. The $f_0(1500)$ pole is found at $1523^{+16}_{-10} - i(52^{+16}_{-11})$ MeV. As these two resonances are fairly narrow, their pole parameters are similar to their RPP BW values [3].

Hence, in Table I we provide the $f_0(1370)$ pole parameters, obtained from the $\pi\pi \to \pi\pi$ FDR method. We assign isospin zero to this pole since a consistent but less accurate pole is also found in the $F^{I_t=1} = F^0/3 + F^1/2 - 5F^2/6$ FDR, but not in the $F^{0+} = (F^1 + F^2)/3$. FDRs alone do not fix the spin, but a consistent pole in the $\pi\pi \to \pi\pi$ scalar wave is found below with somewhat less rigorous methods.

Concerning systematic errors, since the $\pi\pi$ CFD is a piecewise function, we provided later three simple "global" analytic parametrizations [75], almost identical among themselves and to the CFD up to 1.42 GeV. Indeed, they fit Roy and GKPY equations output in the real axis and complex plane validity domains, as well as the FDRs up to 1.42 GeV. By construction, they contain $\sigma/f_0(500)$ and $f_0(980)$ poles consistent with their dispersive values. From 1.42 up to 2 GeV they describe three widely different datasets, covering radically different $f_0(1500)$ scenarios. Still, Table I shows their very similar $f_0(1370)$ poles using the same FDR + C_N method.

For our final $\pi\pi$ FDR result, in Table I, we first obtain a range covering all global fits, which we combine with the

TABLE I. $f_0(1370)$ pole parameters. First lines, from continued fractions on the F^{00} FDR output using as input $\pi\pi \to \pi\pi$ CFD or global fits. Fifth line: final FDR + C_N method result. Last line, from partial-wave RS dispersion relations, using as input $\pi\pi \to K\bar{K}$ constrained fits. Note that $G = |g_{\pi\pi}|$ for $\pi\pi \to \pi\pi$, whereas $G = |\sqrt{g_{\pi\pi} g_{K\bar{K}}}|$ for $\pi\pi \to K\bar{K}$.

Method	$\sqrt{s_{f_0(1370)}}$ (MeV)	G (GeV)
FDR + CFD + C_N	$(1253^{+29}_{-16}) - i(309^{+21}_{-25})$	6.0 ± 0.3
FDR + Global1 + C_N	$(1232^{+29}_{-31}) - i(270^{+47}_{-32})$	4.9 ± 0.4
FDR + Global2 + C_N	$(1227^{+27}_{-22}) - i(276^{+36}_{-48})$	$4.9^{+0.4}_{-0.3}$
FDR + Global3 + C_N	$(1230^{+26}_{-21}) - i(274^{+36}_{-24})$	$4.9^{+0.4}_{-0.5}$
$\mathbf{FDR_{\pi\pi\to\pi\pi} + C_N}$	$\mathbf{(1245 \pm 40) - i(300^{+30}_{-70})}$	$\mathbf{5.6^{+0.7}_{-1.2}}$
$\mathbf{RS_{\pi\pi\to K\bar{K}} + C_N}$	$\mathbf{(1380^{+70}_{-60}) - i(220^{+80}_{-70})}$	$\mathbf{3.2^{+1.3}_{-1.1}}$

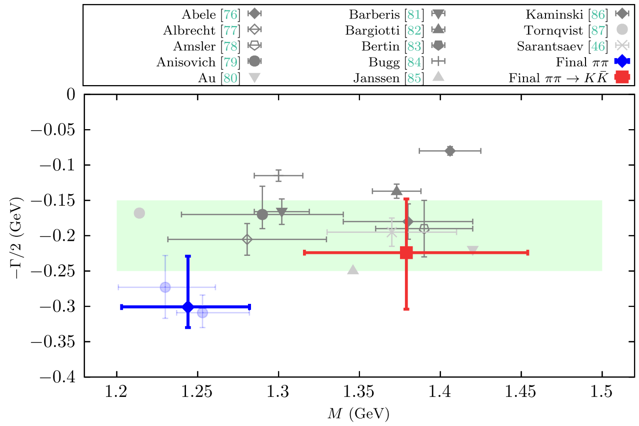

FIG. 4. The $f_0(1370)$ poles obtained from the analytic continuation of $\pi\pi \to \pi\pi$ FDR (blue) or $\pi\pi \to K\bar{K}$ RS partial-wave hyperbolic dispersion relations (red). For comparison, we provide the RPP t-matrix $f_0(1370)$ pole estimate (light green area) and the poles listed there [46,76–87] (only the latest of each group, in gray). Also in light blue the poles from the FDR + CFD and averaged of FDR + Globals in Table I.

CFD value. In Fig. 4 its position is shown (in blue) in the complex $\sqrt{s}$ plane. It overlaps within uncertainties with the RPP estimate (green area), although our central half-width is ∼50 MeV larger.

The last row of Table I is our $f_0(1370)$ result obtained from the $\pi\pi \to K\bar{K}$ scalar-isoscalar partial-wave Roy-Steiner equation. Its output in the 1.04 to 1.46 GeV segment is continued analytically by continued fractions. In Fig. 3 we show, now in red, the resulting pole parameters for $N = 8$ up to 50. Statistical uncertainties are propagated from the $\pi\pi \to K\bar{K}$ CFD parametrization used as input in the integrals. For each N, systematic uncertainties cover the existence of two CFD solutions, the different matching points needed to describe the "unphysical" region between $\pi\pi$ and $K\bar{K}$ thresholds, and a variation of +30 (−30) MeV in the lower (upper) end of the segment. Results are very stable for different N and our final value is obtained by combining the (mass or width) distributions for each N, weighted by their uncertainties. Note that, while systematic effects produce a sizable contribution for $\pi\pi \to \pi\pi$ scattering, statistical uncertainties clearly dominate the total error for $\pi\pi \to K\bar{K}$. Furthermore, even though the $f_2(1270)$ is not present in this wave, the uncertainties are much larger than from $\pi\pi$. Nevertheless, this confirms in full rigor the $f_0(1370)$ pole existence in meson-meson scattering data and its scalar-isoscalar assignment.

The pole position from the $\pi\pi \to K\bar{K}$ analysis is shown in red in Fig. 4, fully consistent with the RPP estimate. However, the central mass is two deviations away from our $\pi\pi \to \pi\pi$ value, and the width is about one deviation away. Given the negligible model dependence of our approaches, this tension should be attributed to an inconsistency between $\pi\pi \to \pi\pi$ and $\pi\pi \to K\bar{K}$ data. Recall that this tension is also hinted in the RPP between the BW $\pi\pi$ and $K\bar{K}$ modes.

TABLE II. Approximated methods yield poles compatible with the rigorous $\pi\pi$ results in Table I. We compare CFD and global parametrization (param.) inputs as well as different continuation methods: Padé sequences (P_M^N), continued fractions (C_N), or directly from the global parametrization. GKPY equations are extrapolated beyond their strict validity range.

Method	$\sqrt{s_{f_0(1370)}}$ (MeV)	$g_{\pi\pi}$ (GeV)
GKPY + CFD + C_N	$(1277^{+49}_{-42}) - i(287^{+49}_{-64})$	$5.6^{+2.1}_{-2.2}$
GKPY + CFD + P_2^N	$(1285^{+32}_{-36}) - i(219^{+40}_{-44})$	4.2 ± 0.4
GKPY + Global1 + C_N	$(1218^{+26}_{-21}) - i(218^{+34}_{-32})$	4.1 ± 1.3
GKPY + Global1 + P_1^N	$(1224^{+31}_{-22}) - i(219^{+23}_{-31})$	4.1 ± 0.4
GKPY + Global1 + P_2^N	$(1222^{+28}_{-17}) - i(214^{+26}_{-21})$	4.2 ± 0.4
Global1 param. + C_N	$(1220^{+27}_{-22}) - i(218^{+41}_{-36})$	4.2 ± 0.4
Global1 param. + P_1^N	$(1222^{+39}_{-33}) - i(220^{+42}_{-40})$	$4.2^{+0.9}_{-0.8}$
Global1 param. + P_2^N	$(1219^{+29}_{-27}) - i(213^{+43}_{-41})$	3.9 ± 0.5
Global1 param.	$(1219 \pm 29) - i(214 \pm 44)$	4.16 ± 0.08

Further checks.—Roy-like partial-wave equations still hold approximately somewhat beyond their strict validity domain [88–90]. Indeed, our f_0^0 partial-wave Roy and GKPY dispersive output, strictly valid below 1.1 GeV, still agrees within uncertainties with the CFD input up to 1.4 GeV. The continued fraction method then yields a pole compatible with our FDR result that is listed in Table II. Being more accurate than Roy equations in that region, we provide GKPY results. Despite the unknown uncertainty due to the use of GKPY equations beyond their applicability limit, this is a remarkable consistency check, particularly of the resonance spin assignment.

Moreover, since in the f_0^0 partial wave there is no $f_2(1270)$ pole hindering the $f_0(1370)$ determination, Padé sequences provide a check with a different continuation method. Recall that a Padé approximant of $f(s)$ is $P_M^N(s, s_0) = Q_N(s, s_0)/R_M(s, s_0)$, with Q_N and R_M polynomials of Nth and Mth degree, respectively, matching the $f(s)$ Taylor series to order $N + M + 1$. Namely, $P_M^N(s, s_0) = f(s) + \mathcal{O}[(s - s_0)^{N+M+1}]$. The polynomial coefficients are related to the $f(s)$ derivatives of different orders. It has been proved [91] that if $f(s)$ is regular inside a domain D, except for poles at s_{p_i}, of total multiplicity M, the sequence $P_M^N(s)$ converges uniformly to $f(s)$ in any compact subset of D, excluding the s_{p_i}. The Padé sequence choice depends on the partial-wave analytic structure. In our case, at least it must have a pole for the resonance, although we also considered sequences with more poles. We follow previous works [70], now using as input the GKPY output. We have propagated the data uncertainties and added systematic errors from the sequence truncation and s_0 choice. As seen in Table II, the pole from the GKPY output continued with the P_2^N sequences is consistent with that from continued fractions. Similar consistency is found for other Padé sequences.

In addition, we have checked the consistency and accuracy of the dispersive plus continuation methods versus the global parametrizations since, being analytic, they can be directly extended to the complex plane without continuation methods. Although not built for that, these parametrizations possess an $f_0(1370)$ pole, identical up to a few MeV, even if they differ widely among themselves above 1.42 GeV. For illustration, we list the "Global1 param." pole in Table II. This is a simple but parametrization-dependent extraction, remarkably close to our dispersive result, although somewhat narrower. In Table II we also list poles obtained from its GKPY dispersive output continued to the complex plane, either with continued fractions or different Padé sequences. All of them come very close to the direct result, although with larger uncertainties, which also happens for the other global parametrizations. Interestingly, when using Padé sequences, there is also a $f_0(1500)$ pole, with large uncertainties. Note that the three global parametrizations cover generously the $f_0(1500)$ scenarios without a significant $f_0(1370)$ change. Finally, the dispersive integral provides rigorously the FDR in the upper half complex plane. Our continued fraction method agrees, within less than half its uncertainty, with this dispersive result in the region $1.2 \le \mathrm{Re}\sqrt{s} \le 1.5$ GeV and $\mathrm{Im}\sqrt{s} \le 0.5$ GeV, even before adding systematic uncertainties. All these checks confirm the robustness of our approach.

Summary.—We have presented a method, combining analytic continuation techniques with forward dispersion relations, to find poles and determine accurately their parameters avoiding model dependencies, even in the inelastic regime. This provides rigorous dispersive results beyond the validity range of conventional partial-wave equations. Applied to $\pi\pi$ scattering, this method reproduces the $f_2(1270)$ resonance and settles the long debate about the existence of an $f_0(1370)$ pole in the $\pi\pi \to \pi\pi$ amplitude, absent in the original experimental analyses. It is found at $(1245 \pm 40) - i(300^{+30}_{-70})$ MeV. The method also provides its elusive $\pi\pi$ coupling $|g_{\pi\pi}| = 5.6^{+0.7}_{-1.2}$ GeV. Remarkably, it also displays an $f_0(1500)$ pole, although no partial-wave data are used in that region, just total cross section data. Consistent results are obtained in the extrapolation of usual Roy-like dispersion relations. Finally, a $f_0(1370)$ pole at $(1380^{+70}_{-60}) - i(220^{+80}_{-70})$ MeV is also found in the continuation of hyperbolic partial-wave dispersion relations for $\pi\pi \to K\bar{K}$ scattering, showing a slightly smaller than two-sigma tension in the mass that can only be attributed to data.

The method presented here can be easily applied to other processes to avoid the pervasive model dependence in hadron spectroscopy and opens the possibility to use total cross-section data avoiding partial-wave analyses.

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