Exactly Solvable Model for a Deconfined Quantum Critical Point in 1D

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We construct an exactly solvable lattice model for a deconfined quantum critical point (DQCP) in $(1+1)$ dimensions. This DQCP occurs in an unusual setting, namely, at the edge of a $(2+1)$ dimensional bosonic symmetry protected topological (SPT) phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The DQCP describes a transition between two gapped edges that break different $\mathbb{Z}_2$ subgroups of the full $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Our construction is based on an exact mapping between the SPT edge theory and a $\mathbb{Z}_4$ spin chain. This mapping reveals that DQCPs in this system are directly related to ordinary $\mathbb{Z}_4$ symmetry breaking critical points.

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Introduction.—Deconfined quantum critical points (DQCPs) describe unusual "Landau forbidden" phase transitions in which the unbroken symmetry group of one phase is not a subgroup of the unbroken symmetry group of the other phase [1,2]. The paradigm of this kind of critical point is the hypothesized $(2+1)$ dimensional DQCP between the valence bond solid (VBS) phase and the Néel phase on a square lattice. The VBS phase has internal SO(3) rotation symmetry but spontaneously breaks C_4 lattice rotation symmetry, while the Néel phase has C_4 symmetry but breaks SO(3) symmetry. Crucially, the two symmetries are intertwined: vortices of the C_4 symmetry carry uncompensated spin-$1/2$ moments [3]. As a result, disordering with respect to the C_4 symmetry can cause ordering under the SO(3) symmetry, resulting in a hypothesized direct transition between the two phases.

Thus far, DQCPs have been studied primarily using field theory and numerical methods [4,5]. One reason for this is the lack of analytically tractable lattice models for DQCPs. In this Letter, we take a step towards a more analytical microscopic approach, by constructing an *exactly solvable* lattice model for a $(1+1)$ dimensional DQCP. The exact solvability of our model makes explicit the mechanism for the DQCP, which lies in the unusual structure of the domain walls. This DQCP has a similar field theory description to the $(1+1)$ dimensional DQCP that was analyzed in Refs. [6–8] using bosonization (see also Refs. [9,10]). However, our DQCP involves a different lattice realization with different (nonspatial) symmetries.

The key idea behind our solvable lattice model is to consider a DQCP in an unusual setting, namely, at the *edge* of a $(2+1)$ dimensional symmetry protected topological (SPT) phase. SPT edge theories provide a natural setting for DQCPs because they also have intertwined symmetries [11,12]. In particular, a SPT phase with a "mixed anomaly" between two symmetries has an edge theory where domain walls of one symmetry carry fractional charge of the other symmetry [13–15]. Like in the system with the VBS and Néel phases, disordering with respect to one symmetry, by proliferating domain walls of that symmetry, may cause ordering with respect to the other symmetry, thereby realizing a DQCP.

We consider the simplest example of such a SPT edge theory: the edge theory of a 2D $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric bosonic SPT phase with a mixed anomaly between the two $\mathbb{Z}_2$ symmetries. Using an exact mapping between the SPT edge theory and a $\mathbb{Z}_4$ spin chain, we rigorously establish the existence of a DQCP and derive the full critical theory.

$\mathbb{Z}_{2a} \times \mathbb{Z}_{2b}$ SPT edge theory.—Our model for the SPT edge theory consists of a chain of spin-$1/2$'s with two spins σ_j and $\tau_{j+1/2}$ in each unit cell, labeled by j. The two $\mathbb{Z}_2$ symmetries, denoted by $\mathbb{Z}_{2a}$ and $\mathbb{Z}_{2b}$, are generated by unitary operators U_a and U_b with

$$U_a = \prod_j \sigma_j^x \qquad U_b = \prod_j \tau_{j+1/2}^x i^{\frac{1-\sigma_j^z \sigma_{j+1}^z}{2}}. \tag{1}$$

Note that U_b does not act "on site" in this representation: this is allowed since (1) describes the effective action of the symmetries on the *edge* degrees of freedom; in the original 2D spin system that describes the bulk SPT phase, both symmetries act on site.

The above symmetry action (1) carries a mixed anomaly between the two symmetries. One manifestation of this mixed anomaly is that a pair of $\mathbb{Z}_{2a}$ domain walls is charged under $\mathbb{Z}_{2b}$. To see this, consider the Hamiltonian

$$H = -\sum_j \sigma_j^z \sigma_{j+1}^z - \sum_j \tau_{j+1/2}^x. \tag{2}$$

The two degenerate ground states of this Hamiltonian, which are illustrated in Figs. 1(a) and 1(b), spontaneously break $\mathbb{Z}_{2a}$. Now consider a state with two domain walls $|\psi_{2\mathrm{DW}}\rangle$, as shown in Fig. 1(c). From (1), we can see that such a state is actually charged under $\mathbb{Z}_{2b}$: $U_b|\psi_{2\mathrm{DW}}\rangle = -|\psi_{2\mathrm{DW}}\rangle$.

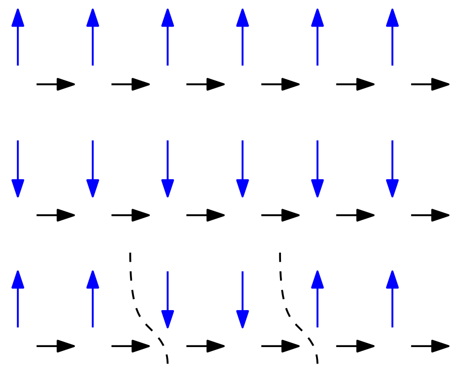

FIG. 1. (a) and (b) The two degenerate ground states of the Hamiltonian (2) that spontaneously breaks $\mathbb{Z}_{2a}$. The blue arrows represent the σ_j spins and the black arrows represent the $\tau_{j+1/2}$ spins. Both states are eigenstates of U_b with eigenvalue $+1$. (c) Domain walls occur at the boundaries between these states. A state with two $\mathbb{Z}_{2a}$ domain walls (indicated by the dashed lines) has eigenvalue -1 under U_b, meaning two $\mathbb{Z}_{2a}$ domain walls fuse to a $\mathbb{Z}_{2b}$ charge.

Evidently, two $\mathbb{Z}_{2a}$ domain walls carry a $\mathbb{Z}_{2b}$ charge, so each domain wall can be associated with "half" a $\mathbb{Z}_{2b}$ charge.

Another way to think about this anomaly is in terms of the fusion rules for domain walls. There are actually four kinds of domain walls for this system if we distinguish between states that carry different quantum numbers under the unbroken $\mathbb{Z}_{2b}$ symmetry. These four kinds of domain walls are shown in Fig. 2: (1) a "no-domain wall" state; (2) a (bare) $\mathbb{Z}_{2a}$ domain wall; (3) a $\mathbb{Z}_{2b}$ charge; (4) a composite of a $\mathbb{Z}_{2a}$ domain wall and a $\mathbb{Z}_{2b}$ charge. The fact that two $\mathbb{Z}_{2a}$ domain walls fuse to a $\mathbb{Z}_{2b}$ charge means that the fusion rules for the domain walls have a $\mathbb{Z}_4$ group structure rather than the usual $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure. This $\mathbb{Z}_4$ fusion structure points to a connection between our edge theory with an anomalous $\mathbb{Z}_{2a} \times \mathbb{Z}_{2b}$ symmetry given by (1) and an ordinary (nonanomalous) $\mathbb{Z}_4$ spin chain (this was also noted in Ref. [16]).

$\mathbb{Z}_4$ spin chain.—The $\mathbb{Z}_4$ spin chain is a spin chain where each spin can be in four different states. The two basic operators acting on the jth spin are the "clock" operator C_j and the "shift" operator S_j. These operators take the following form (in the clock eigenstate basis):

$C_j^\dagger C_{j+1}$	1	i	-1	$-i$
domain wall	↑ ↑ →	↑ ↓ →	↑ ↑ ←	↑ ↓ ←

FIG. 2. A mapping between the four kinds of domain walls in the SPT edge theory and the four kinds of domain walls in the $\mathbb{Z}_4$ spin chain, which are labeled by their eigenvalues $\{1, i, -1, -i\}$ under $C_j^\dagger C_{j+1}$. As discussed in the main text, two $\mathbb{Z}_{2a}$ domain walls (second configuration) fuse to a U_b charge, which is equivalent to a $\tau_{j+1/2}$ spin flip (third configuration).

$$C_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \qquad S_j = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Note that C_j and S_j satisfy the algebra

$$C_j^4 = S_j^4 = \mathbb{1} \qquad C_j S_j = i S_j C_j. \tag{4}$$

In this Letter, we will be interested in $\mathbb{Z}_4$ spin chains with a global $\mathbb{Z}_4$ symmetry given by $U_{\mathbb{Z}_4} = \prod_j S_j$. Such spin chains are closely related to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT edge theory described above. To see the relation, consider a symmetry breaking Hamiltonian of the form $H = -\sum_j \frac{1}{2}(C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j)$. This system has four degenerate ground states, and likewise four different species of domain walls. The different types of domain walls can be conveniently labeled by fourth roots of unity, $\{1, i, -1, -i\}$; the label associated with each domain wall is given by $C_j^\dagger C_{j+1}$ (assuming the domain wall is located between spins at sites j and $j+1$). The crucial point is that these domain walls obey $\mathbb{Z}_4$ fusion rules just like the domain walls for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT edge theory, suggesting that there may be a way to map one system onto the other.

Mapping between the models.—We will now map the Hilbert space of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT edge theory onto the Hilbert space of the $\mathbb{Z}_4$ spin chain.

As we mentioned earlier, the basic idea is to map the four kinds of domain walls in the $\mathbb{Z}_4$ spin chain onto the four kinds of domain walls in the SPT edge theory. To do this, we need to map the spin chain operator $C_j^\dagger C_{j+1}$ (which measures $\mathbb{Z}_4$ domain walls) onto a corresponding domain wall operator in the SPT edge theory. The latter operator should have the four domain wall configurations in Fig. 2 as eigenstates, with eigenvalues $1, i, -1$, and $-i$. It should also be invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, since we want our mapping to map $\mathbb{Z}_4$ symmetric operators in the spin chain (like $C_j^\dagger C_{j+1}$) onto $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric operators in the SPT edge theory. These requirements are satisfied by the operator $\tau_{j+1/2}^x i^{(1-\sigma_j^z \sigma_{j+1}^z)/2}$, so we map

$$C_j^\dagger C_{j+1} \leftrightarrow \tau_{j+1/2}^x i^{(1-\sigma_j^z \sigma_{j+1}^z)/2}. \tag{5}$$

In addition to $C_j^\dagger C_{j+1}$, we also need to work out how our mapping acts on the shift operator S_j. To do this, notice that S_j shifts the domain wall measured by $C_{j-1}^\dagger C_j$ by i and the domain wall measured by $C_j^\dagger C_{j+1}$ by $-i$. This means that S_j should map to an operator whose action on SPT domain wall states is of the form shown in Fig. 3. Another requirement is that S_j should map onto an operator that is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. These two requirements lead us to the mapping

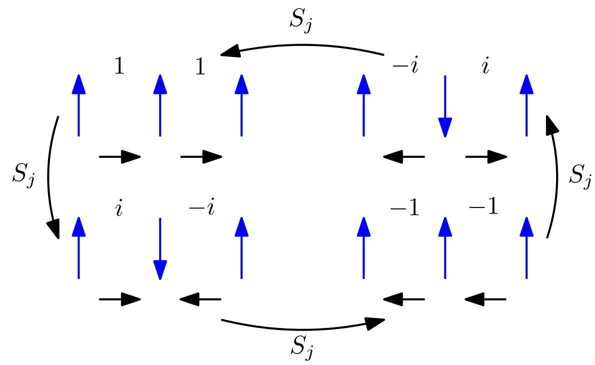

FIG. 3. The action of S_j on domain wall states in the SPT edge theory: S_j shifts the domain wall measured by $C_{j-1}^\dagger C_j$ by i and the domain wall measured by $C_j^\dagger C_{j+1}$ by $-i$. Here, j labels the spin in the middle of each five-spin configuration.

$$S_j \leftrightarrow \sigma_j^x \left[\left(\frac{1+\sigma_{j-1}^z\sigma_j^z}{2}\right) + \left(\frac{1-\sigma_{j-1}^z\sigma_j^z}{2}\right)\tau_{j-1/2}^z\right] \times \left[\left(\frac{1+\sigma_j^z\sigma_{j+1}^z}{2}\right)\tau_{j+1/2}^z + \left(\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}\right)\right]. \tag{6}$$

Equations (5) and (6) define a mapping between local $\mathbb{Z}_4$ symmetric operators in the spin chain and local $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric operators in the SPT edge theory. To understand the *global* properties of this mapping, we note that straightforward algebra shows that

$$\prod_j S_j \leftrightarrow U_a \qquad \prod_j C_j^\dagger C_{j+1} \leftrightarrow U_b. \tag{7}$$

These equations tell us how the various symmetry sectors map onto one another. In particular, we see that the two sectors of the SPT edge theory with $U_a = \pm 1$ and $U_b = 1$ map onto the two spin chain sectors with $\prod_j S_j = \pm 1$, and with *periodic* boundary conditions. On the other hand, the two SPT sectors with $U_a = \pm 1$ and $U_b = -1$ map onto spin chain sectors with $\prod_j S_j = \pm 1$ and *antiperiodic* boundary conditions. Here, the antiperiodic boundary condition can be implemented, on a closed loop of N sites, by using $C_{N+1} = -C_1$ instead of $C_{N+1} = C_1$ (which corresponds to the periodic case). Putting this all together, we see that the Hilbert space of the SPT edge theory maps onto the Hilbert space of the $\mathbb{Z}_4$ spin chain with a particular combination of sectors, namely, the two symmetry sectors $\prod_j S_j = \pm 1$, with either periodic or antiperiodic boundary conditions.

Alternatively, one can think of this particular combination of sectors as describing a $\mathbb{Z}_4$ spin chain coupled to a $\mathbb{Z}_2$ gauge field $\{\nu_{j+1/2}\}$ with the gauge constraint $\nu_{j-1/2}^x \nu_{j+1/2}^x = S_j^2$. In the gauged spin chain, the two boundary conditions correspond to sectors with even and odd $\mathbb{Z}_2$ gauge flux, while the global constraint $\prod_j S_j = \pm 1$ is imposed by gauge invariance. In this Letter, we will mostly work with the explicit sector description rather than the gauged spin chain language, but the latter is a completely equivalent way to think about our mapping.

Using the mapping.—We will now use the mapping to understand the phases and phase transitions of the SPT edge theory. We start with the $\mathbb{Z}_4$ spin chain, which is expected to support three different gapped phases: an ordered phase where the $\mathbb{Z}_4$ symmetry is spontaneously broken, a disordered phase where the symmetry is unbroken, and a partially ordered phase where the $\mathbb{Z}_4$ symmetry is broken down to $\mathbb{Z}_2$ [17]. We can diagnose each of these phases in terms of an order parameter O, and a disorder parameter D defined as follows [18]:

$$O = \lim_{|i-j|\to\infty} \langle C_i^\dagger C_j \rangle \qquad D = \lim_{|i-j|\to\infty} \left\langle \prod_{k=i}^{j} S_k \right\rangle. \tag{8}$$

Each phase has a different pattern of order and disorder parameters:

$$\begin{aligned} \text{Ordered phase:}\ & O \neq 0, \qquad D = 0 \\ \text{Disordered phase:}\ & O = 0, \qquad D \neq 0 \\ \text{Partially ordered phase:}\ & O = 0, \qquad D = 0. \end{aligned} \tag{9}$$

Now, according to (7), our mapping takes the order parameter O for the $\mathbb{Z}_4$ spin chain onto the symmetry transformation U_b restricted to an interval, which is, by definition, a disorder parameter for $\mathbb{Z}_{2b}$. Likewise, our mapping takes the $\mathbb{Z}_4$ disorder parameter D to a $\mathbb{Z}_{2a}$ disorder parameter. It follows that the ordered phase of the spin chain corresponds to a phase of the SPT edge theory with a vanishing $\mathbb{Z}_{2a}$ disorder parameter and a nonvanishing $\mathbb{Z}_{2b}$ disorder parameter—i.e., a phase with broken $\mathbb{Z}_{2a}$ symmetry and unbroken $\mathbb{Z}_{2b}$ symmetry. By the same reasoning, the disordered phase of the spin chain maps onto a phase with unbroken $\mathbb{Z}_{2a}$ symmetry and broken $\mathbb{Z}_{2b}$ symmetry. Finally, the partially ordered phase of the spin chain maps onto a phase where both $\mathbb{Z}_{2a}$ and $\mathbb{Z}_{2b}$ are broken.

The most important application of these results, for our purposes, involves phase *transitions*. In particular, consider a hypothetical critical point between the $\mathbb{Z}_{2a}$ broken ($\mathbb{Z}_{2b}$ unbroken) phase, and its partner, the $\mathbb{Z}_{2b}$ broken ($\mathbb{Z}_{2a}$ unbroken) phase. Applying our mapping, such critical points correspond to critical points between the ordered and disordered phase of the $\mathbb{Z}_4$ spin chain. This means that the problem of understanding DQCPs in the context of the SPT edge theory maps onto the problem of understanding ordinary symmetry breaking critical points for the $\mathbb{Z}_4$ spin chain. Since the latter critical points are known to exist and are well understood, this proves the existence of DQCPs and also allows us deduce their structure.

Exactly solvable model.—More concretely, we can use our mapping to construct an exactly solvable Hamiltonian that describes a continuous phase transition between the $\mathbb{Z}_{2a}$ broken ($\mathbb{Z}_{2b}$ unbroken) phase, and the $\mathbb{Z}_{2b}$ broken ($\mathbb{Z}_{2a}$ unbroken) phase of the SPT edge theory and therefore describes a DQCP. To build such a Hamiltonian, we start with an exactly solvable spin chain Hamiltonian that describes a $\mathbb{Z}_4$ symmetry breaking transition. In particular, we use the $\mathbb{Z}_4$ clock model:

$$H_{\text{clock}}(\alpha) = -(1-\alpha)\sum_j \frac{1}{2}(C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j) - \alpha \sum_j \frac{1}{2}(S_j + S_j^\dagger). \tag{10}$$

Later, we will review how to solve H_{clock} exactly; for now, the only property we need is that H_{clock} belongs to the $\mathbb{Z}_4$ ordered phase for $\alpha < \frac{1}{2}$ and the disordered phase for $\alpha > \frac{1}{2}$, with a direct transition at $\alpha = \frac{1}{2}$.

To apply our mapping, we write

$$H_{\text{clock}}(\alpha) = (1-\alpha)H_{a,\text{clock}} + \alpha H_{b,\text{clock}}, \tag{11}$$

where $H_{a,\text{clock}}$ and $H_{b,\text{clock}}$ describe the two sets of terms in (10). Notice that $H_{a,\text{clock}}$ and $H_{b,\text{clock}}$ are both sums of commuting terms. Furthermore, one can see that $H_{a,\text{clock}}$ and $H_{b,\text{clock}}$ belong to the ordered and disordered phases, respectively. Hence, applying our mapping to $H_{a,\text{clock}}$ gives a commuting Hamiltonian describing the $\mathbb{Z}_{2a}$ broken ($\mathbb{Z}_{2b}$ unbroken) phase of the SPT edge theory [19]:

$$H_a = -\sum_j \left(\frac{1+\sigma_j^z \sigma_{j+1}^z}{2}\right)\tau_{j+1/2}^x. \tag{12}$$

Similarly, applying our mapping to $H_{b,\text{clock}}$, gives a commuting Hamiltonian for the $\mathbb{Z}_{2b}$ broken ($\mathbb{Z}_{2a}$ unbroken) phase:

$$H_b = -\sum_j \left[\sigma_j^x \left(\frac{1+\sigma_{j-1}^z\sigma_{j+1}^z}{2}\right)\left(\frac{\tau_{j-1/2}^z + \tau_{j+1/2}^z}{2}\right) + \sigma_j^x\left(\frac{1-\sigma_{j-1}^z\sigma_{j+1}^z}{2}\right)\left(\frac{1+\tau_{j-1/2}^z\tau_{j+1/2}^z}{2}\right)\right]. \tag{13}$$

Our exactly solvable model that tunes between these two symmetry breaking phases is given by

$$H(\alpha) = (1-\alpha)H_a + \alpha H_b. \tag{14}$$

Like H_{clock}, this Hamiltonian describes a direct transition between the two phases (and hence a DQCP) at $\alpha = \frac{1}{2}$.

Exactly solvable critical point.—We now review how to solve the $\mathbb{Z}_4$ clock model $H_{\text{clock}}(\alpha)$ (10) and hence also $H(\alpha)$. The basic idea is to map H_{clock} onto two decoupled transverse field Ising models which undergo simultaneous symmetry breaking transitions. To do this, we map each four-dimensional spin onto two spin-1/2 degrees of freedom, denoted by μ_j and ρ_j (note that μ_j and ρ_j should not be confused with σ_j and $\tau_{j+1/2}$). We then write

$$C_j = \frac{e^{-i\pi/4}}{\sqrt{2}}(\mu_j^z + i\rho_j^z) \tag{15}$$

and

$$S_j = \mu_j^x\left(\frac{1+\mu_j^z\rho_j^z}{2}\right) + \rho_j^x\left(\frac{1-\mu_j^z\rho_j^z}{2}\right). \tag{16}$$

Using (15) and (16), we compute

$$C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j = \mu_j^z\mu_{j+1}^z + \rho_j^z\rho_{j+1}^z$$
$$S_j + S_j^\dagger = \mu_j^x + \rho_j^x. \tag{17}$$

Applying this map to the $\mathbb{Z}_4$ clock model in Eq. (10) gives

$$H_{\text{clock}} = -(1-\alpha)\sum_j \frac{1}{2}(\mu_j^z\mu_{j+1}^z + \rho_j^z\rho_{j+1}^z) - \alpha\sum_j \frac{1}{2}(\mu_j^x + \rho_j^x), \tag{18}$$

which recovers the well-known fact that the $\mathbb{Z}_4$ clock model is unitarily equivalent to two decoupled transverse field Ising models.

This mapping implies that the DQCP that occurs at $\alpha = 1/2$ is equivalent to two copies of the critical Ising theory [20]. More precisely, the DQCP is equivalent to a particular combination of *sectors* of the Ising theory: translating the sectors $\prod_j S_j = \pm 1$ and $C_{N+1} = \pm C_1$ into the Ising language, we see that $H(\alpha)$ is described by the symmetry sector $\prod_j \mu_j^x \rho_j^x = 1$, with the same (periodic or antiperiodic) boundary conditions in both μ, ρ, i.e., $\mu_{N+1}^z = \pm\mu_1^z$ and $\rho_{N+1}^z = \pm\rho_1^z$ [21].

Using this mapping we can obtain all the critical exponents of the DQCP. For example, the correlation length ξ near the critical point diverges as $\xi \sim (1/|\alpha - \frac{1}{2}|^\nu)$ with $\nu = 1$. Also, the two-point correlators for the $\mathbb{Z}_{2a}$ and $\mathbb{Z}_{2b}$ order parameters σ^z and τ^z are given by

$$\langle \sigma_i^z \sigma_j^z \rangle \sim \frac{1}{|i-j|^{1/2}} \qquad \langle \tau_{i+1/2}^z \tau_{j+1/2}^z \rangle \sim \frac{1}{|i-j|^{1/2}}. \tag{19}$$

Is the above DQCP stable to perturbations? The answer to this question depends on what additional symmetries we impose beyond $\mathbb{Z}_{2a} \times \mathbb{Z}_{2b}$. For example, suppose we impose both time-reversal and parity symmetry. In this case, it is well known that the critical point of the $\mathbb{Z}_4$ clock model does not have any relevant symmetric operators beyond the tuning parameter α, but it does have a marginal operator corresponding to $\lambda \sum_j (C_j^2 C_{j+1}^2 + S_j^2)$. Adding this operator moves the system along a critical line [17,22,23]. Therefore the DQCP that we described above is actually part of a deconfined quantum critical *line* with continuously varying exponents (see the Supplemental Material [24] for more details). On the other hand, if we don't impose additional symmetries, then the critical point has other relevant symmetric operators that drive the system into a gapless phase, destroying the direct transition. These are known as "chiral perturbations," and are given by $\lambda_\varphi \sum_j (C_j^\dagger C_{j+1} e^{i\varphi} + \text{H.c.})$ and $\lambda_\vartheta \sum_j (S_j e^{i\vartheta} + \text{H.c.})$ [27–30]. More generally, if we consider the whole critical line, there is a region (i.e., a range of λ) where the chiral perturbations are irrelevant [22,24,27–29]. In this region, time-reversal symmetry and parity symmetry are not required to stabilize the transition.

Self-duality at criticality.—An interesting aspect of the above DQCP is that it is *self-dual*: there is a duality transformation that maps the critical point to itself and exchanges the $\mathbb{Z}_{2a}$ and $\mathbb{Z}_{2b}$ order parameters in (19). This self-duality is reminiscent of the self-duality that occurs in other DQCPs, such as in the XY antiferromagnet to VBS transition obtained from adding easy-plane spin anisotropy to the Néel to VBS transition [1,2,31].

The duality transformation—denoted by U_c—is easiest to understand in terms of the $\mathbb{Z}_4$ spin chain variables: in this description, U_c maps $C_j^\dagger C_{j+1}$ onto S_{j+1} and maps S_j onto $C_j^\dagger C_{j+1}$. This is similar to the Kramers-Wannier duality, but unlike standard Kramers-Wannier duality, U_c is both (1) unitary and (2) locality preserving, in the sense that it maps local operators to local operators. These properties are due to the unusual sector structure in our $\mathbb{Z}_4$ spin chain, or equivalently the fact that the $\mathbb{Z}_4$ spin chain is coupled to a $\mathbb{Z}_2$ gauge field (see the Supplemental Material [24] for more details). One consequence of the unitarity and locality of U_c is that U_c can also be viewed as an ordinary symmetry, rather than a duality.

Discussion.—As emphasized above, at the core of our construction is the mapping between the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT edge theory (with a mixed anomaly) and the $\mathbb{Z}_4$ spin chain [(5) and (6)]. This mapping can be readily generalized to any $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ SPT edge theory with a primitive [32] mixed anomaly. Specifically, any edge theory of this kind can be mapped onto a $\mathbb{Z}_{N_1 N_2}$ spin chain in such a way that the Landau forbidden transition in the edge theory maps onto an ordinary symmetry breaking transition in the spin chain.

Moving forward, it would be interesting to find examples of these mappings for other kinds of anomalies, such as "type-III anomalies" [14,15], or for non-Abelian symmetry groups. Examples of this kind could give solvable DQCPs with richer structure. It would also be interesting to generalize to higher dimensional systems, though this is not straightforward since our construction relies on charges and domain walls having the same dimensionality, as shown in Fig. 2.

Another interesting generalization is to add disorder to our model, by drawing the coefficients of the terms in $H_{a,\text{clock}}$ and $H_{b,\text{clock}}$ from random distributions. It was shown in Refs. [34–36] that strongly disordered $\mathbb{Z}_N$ clock models have continuous transitions with critical properties that can be obtained exactly using a renormalization group analysis. In the corresponding SPT edge theory, this kind of model would give an example of a *disordered* DQCP.

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