Entanglement Entropy of Non-Hermitian Eigenstates and the Ginibre Ensemble

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(Received 18 July 2022; revised 22 October 2022; accepted 13 December 2022; published 3 January 2023)

Entanglement entropy is a powerful tool in characterizing universal features in quantum many-body systems. In quantum chaotic Hermitian systems, typical eigenstates have near maximal entanglement with very small fluctuations. Here, we show that for Hamiltonians displaying non-Hermitian many-body quantum chaos, modeled by the Ginibre ensemble, the entanglement entropy of typical eigenstates is greatly suppressed. The entropy does not grow with the Hilbert space dimension for sufficiently large systems, and the fluctuations are of equal order. We derive the novel entanglement spectrum that has infinite support in the complex plane and strong energy dependence. We provide evidence of universality, and similar behavior is found in the non-Hermitian Sachdev-Ye-Kitaev model, indicating the general applicability of the Ginibre ensemble to dissipative many-body quantum chaos.

DOI: 10.1103/PhysRevLett.130.010401

Introduction.—In quantum mechanical systems, it is standard to take the Hamiltonian of the system to be Hermitian, ensuring the reality of the energy spectrum. However, relaxing this Hermiticity has proven to lead to many novel and unexpected phenomena [1]. These phenomena are not mere theoretical curiosities, but physically relevant, describing the physics of open quantum systems.

Given the widespread applicability of non-Hermitian physics, it is natural to ask what features are universal. This is our impetus for combining two unifying subjects in the context of non-Hermitian many-body physics, entanglement and random matrix theory. Entanglement entropy has been an indispensable tool characterizing many-body physics, with milestone results in gapped [2], critical [3], topological [4,5], holographic [6], and dynamical [7] systems. Entanglement theory has only recently been applied to non-Hermitian physics, with some of the main achievements coming from characterizations of nonunitary conformal field theories [8–14].

The goal of this Letter is to move away from these ground state studies to the generic properties of typical eigenstates. This is of particular interest in the context of the emerging field of dissipative quantum many-body chaos [15–23]. Our strategy is to analyze the eigenstates of the complex Ginibre ensemble, whose matrix elements are independent and identically distributed (IID) complex Gaussian random variables [24]. This is our proposed analogy to the typical eigenstates frequently used in Hermitian systems that are eigenstates of the Gaussian unitary ensemble (equivalently "Haar random" states) [25]. The Ginibre ensemble has been demonstrated to universally emerge in non-Hermitian many-body quantum chaotic systems [15–18,26,27]. This is anticipated by the dissipative analog of the Berry-Tabor and Bohigas-Giannoni-Schmit conjectures [28,29].

Non-Hermitian Hamiltonians, H, have distinct left and right eigenvectors, $|L_i\rangle$ and $|R_i\rangle$, residing in an N-dimensional Hilbert space, that form a biorthonormal basis $\langle L_i|R_j\rangle = \delta_{ij}$. Following the biorthogonal formulation of quantum mechanics [30], we choose the density matrix of an eigenstate to inherit the non-Hermiticity of the Hamiltonian

$$\rho^{(i)} \coloneqq |R_i\rangle\langle L_i|. \tag{1}$$

With this choice, the Heisenberg evolution of general density matrices remains $i\partial_t\rho = [H,\rho]$.

We consider a bipartition of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with sub-Hilbert space dimensions N_A and N_B. Performing the partial trace on $\mathcal{H}_B$, we arrive at the reduced density matrix

$$\rho_A^{(i)} \coloneqq \mathrm{Tr}_B \rho^{(i)}. \tag{2}$$

This describes the state localized to subsystem A because the expectation values of all observables are captured by the reduced state $\langle \mathcal{O}_A\rangle = \mathrm{Tr}(\rho_A^{(i)}\mathcal{O}_A)$.

While (2) still has unit trace, its eigenvalues are generally complex. To accommodate, we use a generalized definition of the entanglement entropy [14,31]

$$S_{vN}(\rho_A) = -\mathrm{Tr}\rho_A \log|\rho_A|, \tag{3}$$

which reduces to the standard entanglement entropy for the Hermitian case. While its quantum information theoretic interpretation is not yet entirely understood, it obeys desirable properties such as $S_{vN} \neq 0$ only if A and B are entangled, it is amenable to path integral constructions, and has been useful in characterizing nonunitary conformal field theories [11–14].

The entropy as a function of $\log N_A$ is referred to as the "Page curve" [25] and has been the topic of intense study in both many-body and quantum gravitational physics [32–35]. In this Letter, we compute the Page curve for non-Hermitian systems by first identifying the structure of typical reduced density matrices, evaluating their eigenvalues (called the entanglement spectrum), then computing the expectation of the entanglement entropy and its variance. We numerically demonstrate that our results exhibit universality by studying other random matrix ensembles as well as the non-Hermitian Sachdev-Ye-Kitaev (NSYK) model.

Structure of reduced density matrix.—To get oriented, we review the Hermitian case where one considers eigenvectors of the Gaussian unitary ensemble (GUE). The eigenvectors on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\Psi\rangle = \sum_{i=1}^{N_A} \sum_{\alpha=1}^{N_B} X_{i\alpha} |i\rangle_A \otimes |\alpha\rangle_B, \tag{4}$$

where the states in the sum are orthonormal bases for the sub-Hilbert spaces and the $X_{i\alpha}$'s (matrix elements of X) are IID complex Gaussian random variables with variance N^{-1}. The random induced states on $\mathcal{H}_A$ are $\rho_A = XX^\dagger$, defining the celebrated Wishart ensemble [36]. The spectrum of Wishart matrices is given by the Marchenko-Pastur distribution. The entropy is consequently evaluated to

$$\mathbb{E}_X[S_{vN}] = \log N_A - \frac{N_A}{2N_B}, \qquad N_A \leq N_B. \tag{5}$$

The entropy is extremely close to the upper bound of $\log N_A$. This scales extensively with the system size and is independent of the eigenvalue location, which we will soon see is not the case for non-Hermitian systems.

We seek the non-Hermitian analog of $XX^\dagger$. To describe the structure of $\rho_A^{(i)}$, we use the so-called "Hermitization trick" [37]. Define

$$H^z := \begin{pmatrix} 0 & H - z \\ (H - z)^\dagger & 0 \end{pmatrix}, \tag{6}$$

with $z \in \mathbb{C}$ and H is drawn from the Ginibre ensemble. We point out that

$$\lambda \in \mathrm{Spec}(H) \Leftrightarrow 0 \in \mathrm{Spec}(H^\lambda). \tag{7}$$

This is the key observation that will enable us to compute the spectrum of $\rho_A^{(i)}$ and, consequently, its entanglement entropy. We denote the eigenvalues of H^z by $E^z_{\pm i}$ and by $|\mathbf{w}^z_{\pm i}\rangle$ the corresponding orthogonal eigenvectors. The chiral symmetry of H^z induces a symmetric spectrum around zero, i.e., $E^z_i \geq 0$ and $E^z_{-i} = -E^z_i$; accordingly, the eigenvectors $|\mathbf{w}^z_{\pm i}\rangle$ are of the form $|\mathbf{w}^z_{\pm i}\rangle = (|\mathbf{u}^z_i\rangle, \pm|\mathbf{v}^z_i\rangle)$, with $|\mathbf{u}^z_i\rangle, |\mathbf{v}^z_i\rangle \in \mathbb{C}^N$. E^z_i deterministically coincides with the singular values of $H - z$, and $|\mathbf{u}^z_i\rangle, |\mathbf{v}^z_i\rangle$ denote the corresponding left and right singular vectors, i.e.,

$$(H - z)|\mathbf{v}^z_i\rangle = E^z_i|\mathbf{u}^z_i\rangle, \qquad (H - z)^\dagger|\mathbf{u}^z_i\rangle = E^z_i|\mathbf{v}^z_i\rangle. \tag{8}$$

The representations (6) and (8) are completely equivalent.

Let λ_i be the eigenvalue with corresponding right and left eigenvectors $|R_i\rangle$, $|L_i\rangle$ from (1). Then, by (7)–(8), it follows that

$$|\mathbf{v}^{\lambda_i}_1\rangle = \frac{|L_i\rangle}{\|L_i\|}, \qquad |\mathbf{u}^{\lambda_i}_1\rangle = \frac{|R_i\rangle}{\|R_i\|}. \tag{9}$$

We introduce the notations $|\mathbf{v}\rangle := |\mathbf{v}^{\lambda_i}_1\rangle$, $|\mathbf{u}\rangle := |\mathbf{u}^{\lambda_i}_1\rangle$ and, thus, find that

$$\rho_A^{(i)} = \frac{\mathrm{Tr}_B[|\mathbf{u}\rangle\langle\mathbf{v}|]}{\langle\mathbf{u}|\mathbf{v}\rangle}. \tag{10}$$

The key point is that, for fixed deterministic z, we can compute the distribution of $|\mathbf{u}^z_i\rangle, |\mathbf{v}^z_i\rangle$ using Hermitian techniques such as the Dyson Brownian motion (DBM) for eigenvectors introduced in [38] (see, also, [39–43]). We explain this in more detail in the Supplemental Material [44]. This would not have been possible analyzing the non-Hermitian eigenvectors R_i and L_i directly since there is no known non-Hermitian analog for eigenvector DBM. By (9), we need to study the case $z = \lambda_i$, i.e., when z is random, however, we expect (and numerically confirm) the same distribution as $|\mathbf{u}^z_i\rangle, |\mathbf{v}^z_i\rangle$ for fixed z. We remark that for H Ginibre, a similar result can be obtained via Weingarten calculus [45], however, we decided to rely on DBM since this approach applies to more general ensembles for H (see Universality below).

Now, we will study the spectrum of $\rho_A^{(i)}$ conditioned on the event that $\lambda_i = z$, for some $|z| < 1$. In order to keep the argument simple and concise, we neglect the case $|z| \approx 1$; the analysis in this regime would be analogous except for the fact that the distribution of $(\|L_i\|\|R_i\|)^{-1}$ would be more complicated compared to what we have below (12) (see Ref. [46]).

Since H is a Ginibre matrix, the singular vectors $|\mathbf{v}_i\rangle$ of $H - z$ are Haar unitary distributed (here, $|\mathbf{v}_1\rangle = |\mathbf{v}\rangle$) [47]. Now, we write $|\mathbf{u}\rangle$ in the $|\mathbf{v}_i\rangle$ basis

$$|\mathbf{u}\rangle = \sum_i c_i|\mathbf{v}_i\rangle, \qquad c_i := \langle\mathbf{v}_i|\mathbf{u}\rangle. \tag{11}$$

The coefficient c_1 is distributed as

$$c_1 = \langle \mathbf{u}|\mathbf{v}_1\rangle = \sqrt{\frac{\gamma_2}{N(1-|z|^2)}}, \tag{12}$$

as computed in [46,51]. In particular, for the distribution of c_1 no DBM is required. γ_2 is a random variable drawn from the Gamma distribution with shape parameter 2, i.e., its density is given by xe^{-x}. The rest of the c_i's, for $i \geq 2$, are IID standard complex Gaussian random variables and independent of c_1. More precisely, using DBM [38–43], we can show that any finite (independent of N) collection of c_i's, for $i \geq 2$, converge to IID standard complex Gaussians. For Ginibre, one can expect, for instance, using Weingarten calculus, that this convergence holds for all the c_i's with $i \geq 2$. This is also confirmed numerically below. We point out that, to use DBM, we write

$$c_i \coloneqq \langle \mathbf{v}_i|\mathbf{u}\rangle = \langle \mathbf{w}_1^z|F|\mathbf{w}_i^z\rangle, \qquad F \coloneqq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{13}$$

with $|\mathbf{w}_i^z\rangle$ being defined below (7), since eigenvector overlaps of this form are well understood for Hermitian matrices using DBM [38–43]. See the Supplemental Material XX for a gentle explanation of this approach.

Thus, we find that

$$\rho_{AB} = \sum_i \frac{c_i}{c_1}|\mathbf{v}_i\rangle\langle \mathbf{v}_1|, \tag{14}$$

which has unit trace but is non-Hermitian, with the c_i's distributed as described above. In the $\mathcal{H}_A \otimes \mathcal{H}_B$ basis, we have

$$|\mathbf{v}_i\rangle = \sum_{j\alpha} X_{j\alpha}^{(i)}|j\rangle_A|\alpha\rangle_B, \tag{15}$$

where the $X^{(i)}$'s are $N_A \times N_B$ rectangular matrices with IID Gaussian variables with variance N^{-1} because the singular vectors, $|\mathbf{v}_i\rangle$, are Haar distributed. We neglect the normalization because it concentrates around one at large N. The singular vectors are correlated with each other only in that they are orthonormal, a subtlety that we may safely ignore in the limits we consider. Thus, the reduced density matrix is given by

$$\rho_A = \sum_i \frac{c_i}{c_1} X^{(i)} X^{(1)\dagger}, \tag{16}$$

defining a non-Hermitian analog of the Wishart ensemble. Distinct non-Hermitian analogs of the Wishart ensemble have been studied in the math literature [52–54], though these ensembles have no clear interpretation as density matrices for quantum systems. The most striking difference is that we will see that (16) has noncompact support with a heavy tail, unlike the compactly supported eigenvalue spectra previously studied. This is a consequence of the correlation of the left and right eigenvectors encoded in c_1, which the models in [52–54] are not able to capture. This makes the analysis much more delicate in the current case. As shown in [53], the product of rectangular IID matrices gives a matrix with the spectrum of a Ginibre matrix for $1 \ll N_A \ll N_B$ rescaled by $N^{-1/2}$. When $i = 1$, this limit leads to the normalized identity matrix plus a matrix with the spectrum of a GUE matrix suppressed by $N^{-1/2}$ and, hence, is irrelevant. In total, the density matrix takes the form

$$\rho_A = \frac{\mathbb{1}_A}{N_A} + \frac{M_{\text{GUE}}}{\sqrt{N}} + \sqrt{\frac{1-|z|^2}{\gamma_2 N}} \sum_{i=2}^{N} c_i M_{\text{Ginibre}}^{(i)}. \tag{17}$$

By the central limit theorem (CLT), we can add all the random matrices at large N to find

$$\rho_A = \frac{\mathbb{1}_A}{N_A} + \sqrt{\frac{1-|z|^2}{\gamma_2}} M, \tag{18}$$

where M is a Ginibre matrix independent of γ_2. Note that, to go from (16) to (18), we did not need that $X^{(i)}X^{(1)\dagger}$ is approximately Ginibre and that the c_i's for $i \geq 2$ are IID; we only needed to ensure that the CLT for the entries of ρ_A held. This remark will be relevant for the Universality discussion below when matrices H with not necessarily Gaussian entries are considered. Thus, we find that reduced density matrices of the Ginibre ensemble are Ginibre themselves, with a random, eigenvalue dependent, scaling and deterministic shift.

Entanglement spectrum.—Now that we understand the structure of the non-Hermitian ensemble defining the reduced density matrix, we compute the entanglement spectrum. Famously, the spectrum of Ginibre matrices is uniformly distributed on the unit circle [24]. In the $1 \ll N_A \ll N_B$ regime, the entanglement spectrum, conditioned on γ_2, is, therefore, given by a shifted circular law due to the Ginibre matrix in (18)

$$\mu(x, \gamma_2) = \begin{cases} \frac{\gamma_2}{\pi}, & x < \gamma_2^{-1/2} \\ 0, & x > \gamma_2^{-1/2} \end{cases}, \tag{19}$$

where $x \coloneqq (|N_A^{-1} - \lambda|/\sqrt{1-|z|^2})$. Integrating this distribution over γ_2,

$$\mu(x) = \frac{1}{\pi}\int_0^{x^{-2}} d\gamma_2 \gamma_2^2 e^{-\gamma_2}, \tag{20}$$

we then find

$$\mu(x) = \frac{1}{\pi}\left[2 - e^{-\frac{1}{x^2}}\left(\frac{1+2x^2+2x^4}{x^4}\right)\right]. \tag{21}$$

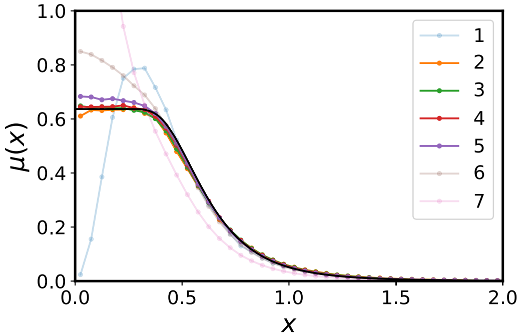

FIG. 1. The entanglement spectrum is shown for various values of $\log_2 N_A$ (labeled in the legend) and $N = 2^{14}$. The data is averaged over all eigenstates and compared with the large-N formula, (21), displayed as the solid black line. There are clear deviations from (21) outside of the $1 \ll N_A \ll N_B$ regime.

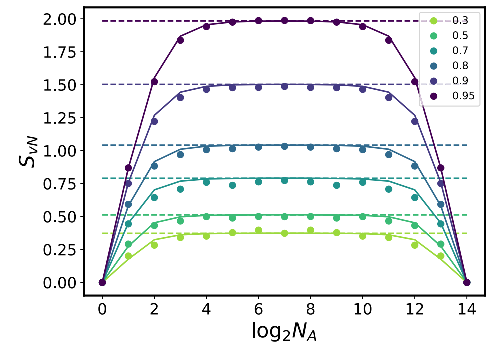

FIG. 2. The von Neumann entropy (base 2) for eigenvectors of Ginibre matrices for various values of $|z|$. The circles are numerical data, the solid line is evaluated from the finite N spectrum (21), and the dashed line is the asymptotic result (25). We have taken $N = 2^{14}$ and 10^3 disorder realizations. Only the real part is plotted because the imaginary part averages to zero.

Note that $\mu(x)$ is rotationally invariant. For $N_A > N_B$, the spectrum is identical with $N_A \leftrightarrow N_B$ and the addition of $N_A - N_B$ eigenvalues at $\lambda = 0$. This spectrum has infinite support with a very heavy tail, decaying only as $|\lambda|^{-6}$ at large $|\lambda|$, in stark contrast with the compactly supported eigenvalue spectra of [52–54]. In Fig. 1, we show the very good agreement between (21) and numerical data for small matrices. We lack an analytical expression for the spectrum at $N_A = N_B$, though we numerically show the accuracy of (14) in all regimes in the Supplemental Material XX.

Page curve.—The average entropy is given by

$$\mathbb{E}_{M,\gamma_2}[S_{vN}(|z|)] = -N_A \int d\lambda \mu(\lambda)\lambda \log|\lambda|. \tag{22}$$

Conditioned on $R \coloneqq \sqrt{[(1-|z|^2)/\gamma_2]}$, the entropy is

$$\mathbb{E}_M[S_{vN}(R)] = -\frac{N_A}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R drr(N_A^{-1} + re^{i\theta}) \times \log\left(N_A^{-2} + \frac{2r\cos(\theta)}{N_A} + r^2\right). \tag{23}$$

Expanding in N_A, only the $O(N_A^0)$ term contributes due to the integral over θ, leading to

$$\mathbb{E}_M[S_{vN}(R)] = -\log R. \tag{24}$$

Integrating over γ_2, we arrive at

$$\mathbb{E}_{M,\gamma_2}[S_{vN}(|z|)] = \frac{1 - \gamma - \log(1-|z|^2)}{2}, \tag{25}$$

where γ is the Euler-Mascheroni constant. Surprisingly, there is no scaling with N_A, in stark contrast to the Hermitian case (5). We show remarkable agreement between (25) and small matrices, even at $N_A = N_B$, in Fig. 2.

As a consequence of the large fluctuations in the structure of the density matrix, there are large fluctuations in the von Neumann entropy. Therefore, we would like to understand its variance. To do so, we split the variance into three terms

$$\begin{aligned}\mathbb{E}_{M,\gamma_2}&[|S_{vN} - \mathbb{E}_{M,\gamma_2}[S_{vN}]|^2] \\ &= \mathbb{E}_{\gamma_2}[|\mathbb{E}_M[S_{vN}]|^2] - |\mathbb{E}_{M,\gamma_2}[S_{vN}]|^2 \\ &\quad + \mathbb{E}_{M,\gamma_2}[|S_{vN} - \mathbb{E}_M[S_{vN}]|^2].\end{aligned} \tag{26}$$

The first term on the second line is simply the square of (25). The term in the first line may be analogously computed at large N_A by Taylor expanding the logarithm

$$\mathbb{E}_{\gamma_2}[|\mathbb{E}_M[S_{vN}]|^2] = \frac{\pi^2 - 6}{24} + |\mathbb{E}_{M,\gamma_2}[S_{vN}]|^2. \tag{27}$$

The most involved term is the final one. Fortunately, the variance of functions of eigenvalues, $f(\lambda_i)$, of Giniak/ciabre matrices was analyzed in [55]. There, it was found that

$$\begin{aligned}\mathbb{E}_M&\left[\left|\sum_i f(\lambda_i) - \mathbb{E}_M\left[\sum_i f(\lambda_i)\right]\right|^2\right] \\ &= \frac{1}{4\pi}\int_{\mathbf{D}} d^2z|\nabla f|^2 + \frac{1}{2}\sum_{k\in\mathbb{Z}}|k||\hat{f}(k)|^2,\end{aligned} \tag{28}$$

where $\mathbf{D}$ is the unit disk and $\hat{f}(k)$ is the kth Fourier mode of f on the perimeter of the disk. For the entropy, we must take

$$f(z) = -(N_A^{-1} + Rz)\log|N_A^{-1} + Rz|. \tag{29}$$

After averaging over γ_2, the three terms are N_A independent at large N_A, thus, the same order as the mean (25).

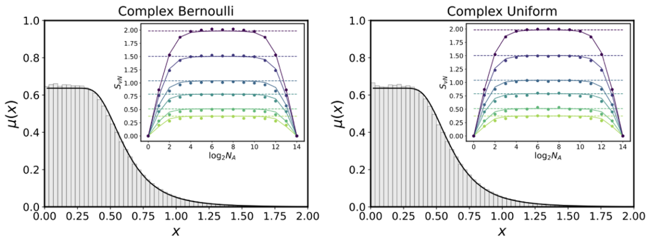

FIG. 3. The entanglement spectrum (main plot) and Page curve (upper right) for the eigenvectors of Left: random complex Bernoulli matrices (entries are independent $\pm$ real and imaginary parts) and Right: random complex uniform matrices (entries are uniformly drawn from the complex unit circle). For the spectrum, we take $N_A = 2^4$ and $N = 2^{14}$ to ensure $1 \ll N_A \ll N_B$. The black line is the analytic result for Ginibre matrices (21). The Page curves precisely agree with the Ginibre Page curve from Fig. 2.

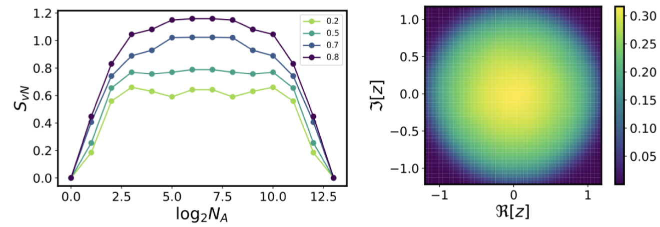

FIG. 4. Left: The von Neumann entropy (base 2) of the non-Hermitian SYK model averaged over various values of $|z|$ with $q = 4$ and $N = 26$. Right: The corresponding energy eigenvalue spectrum. The entropy shows a plateau, increasing with $|z|$, akin to the Ginibre ensemble.

Interestingly, the variance is monotonically decreasing with $|z|$, ranging between ~ 0.906 at $z = 0$ and ~ 0.161 at $z = 1$.

Universality.—It is clearly important to understand whether our results exhibit universality. A similar analysis to the one performed above holds for left and right eigenvectors of more general non-Hermitian matrix ensembles, i.e., for matrices H with IID entries but not necessarily with Gaussian distribution, matrices with independent (but not necessarily identically distributed) entries, and even for matrices with some correlation structure; however, the precise limiting constant may differ from (25). The common feature in all these models is that we expect to get an N_A-independent entropy. This is motivated from the fact that the DBM arguments in [38–43] and reviewed in the Supplemental Material XX hold true for fairly general Hermitian ensembles. In these cases, the vectors $|\mathbf{v}_i^z\rangle$ will not be Haar distributed, but the CLT still holds, so the approximation in (18) will be valid. We expect the scaling constant $(1 - |z|^2)^{-1/2}$ to be dependent only on the shape of the eigenvalue distribution, but the γ_2 distribution to be universal. For matrices obeying the circular law, we demonstrate the $(1 - |z|^2)^{-1/2}$ scaling to be the correct one in Fig. 3 for two random matrix ensembles that are very different than Ginibre, suggesting universality.

It is additionally important to consider bona fide Hamiltonian systems such as the NSYK model of N Majorana fermions

$$H_{\text{NSYK}} = \sum_{i_1 < i_2 < \ldots < i_q}^{N} (J_{i_1 i_2, \ldots, i_q} + i M_{i_1 i_2, \ldots, i_q}) \psi_{i_1} \psi_{i_2}, \ldots, \psi_{i_q}, \tag{30}$$

where $J_{i_1 i_2, \ldots, i_q}$ and $M_{i_1 i_2, \ldots, i_q}$ are IID real Gaussian random variables with zero mean, variance $(2/N^{q-1})$, and $\{\psi_i, \psi_j\} = 2\delta_{ij}$. For even q and N mod 8 is 2 or 6, the Hamiltonian is in the complex Ginibre symmetry class [26,56]. We show the rotationally symmetric but nonuniform eigenvalue distribution and average entropy in Fig. 4. "Unfolding" the eigenvalue nonuniformity warrants further attention to compare quantitatively with the Ginibre ensemble.

Discussion.—In this Letter, we have presented the entanglement spectrum and entanglement entropy of eigenvectors of Ginibre matrices, relying on the determination of the novel structure of correlations in the density matrix. We found that the support of the entanglement spectrum is not compact, with eigenvalue densities decaying at infinity with a heavy $|\lambda|^{-6}$ tail in the complex plane. This led us to find a Page curve that did not scale with the system size, vastly suppressed as compared to the Hermitian Page curve. Moreover, we found the Page curve to not be self-averaging with fluctuations of the same order as the mean, in stark contrast with the Hermitian case.

There are many interesting research directions motivated from this Letter. An important characterization of many-body chaos beyond the entanglement entropy is the eigenstate thermalization hypothesis (ETH) which, motivated by random matrix theory, describes the universal behavior of expectation values of simple observables and their fluctuations [57,58] (see, also, [40,43,59]). In [60], we generalize the ETH to non-Hermitian systems by employing the Ginibre ensemble. Along with providing the compelling prediction that observables have large intereigenstate fluctuations (hence, no thermalization), this leads to an alternate derivation of (18).

Furthermore, it may be interesting to explore the entanglement entropy in different classes of non-Hermitian many-body systems, such as those with symmetries [26,56,61], those with localization transitions [15], noninteracting fermions [26], Liouvillians [21,22,62–66], nonequilibrium systems [67,68], or many-body scars [69]. We hope to report on some of these directions in the near future.

We would like to thank Amos Chan and Shinsei Ryu for useful discussions and comments. We especially thank Kohei Kawabata for his comments that significantly improved the manuscript. J. K. F. is supported through a

Simons Investigator Award to Shinsei Ryu from the Simons Foundation (Award No. 566166) and by the Institute for Advanced Study and the National Science Foundation under Grant No. PHY-2207584. We use QUSPIN for simulating the NSYK model [70,71] and thank Laimei Nie for her help in implementation.

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