THREE-FIELD MODEL AS A PROBE OF HIGHER GROUP SYMMETRIES*

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As the number of hadrons has increased, attempts have multiplied to enlarge the underlying group structure governing the strong interactions of these particles. In view of the complications involved in mixing the proper orthochronous Lorentz group and/or the discrete Lorentz transformations (parity P and time reversal T) with the internal symmetry groups (isospin I, hypercharge Y, and baryon number B), this question was initially approached by enlarging the internal group structure while keeping the Lorentz groups inviolate. Such an approach implied that one could hope to establish correlations among hadrons with different values of I, Y, and B, but with identical values of spin and parity. The success of broken SU(3) symmetry seemed to justify this more conservative approach and some authors are now searching for an underlying group structure higher than SU(3) for the internal symmetries. It seems probable, however, that a truly unified theory of hadrons, on the group-theoretic level, will relate hadron multiplets with different spins and/or parities as well as with different values of their internal quantum numbers. Such relations can only be predicted if one finds the larger mixed Lorentz-internal symmetry group which is broken in a well-defined way to yield as a subgroup the direct product of the Lorentz and internal symmetry groups which we now use to classify particles.

Assuming that the more ambitious program just sketched may succeed (this may be the case not at all or only in a limited sense), there still remains the very difficult problem of finding the correct large mixed group. In order to restrict the possible choices, it would be helpful to utilize some sort of model in the same way in which the symmetrical Sakata model was employed to identify the SU(3) group and to select the proper symmetry-breaking terms. We believe that a very useful probe of mixed group symmetries is the model based on a triplet of "basic" fields interacting via a four-fermion interaction. (We shall reserve the term "basic" field for a massless[1] Dirac four-component field and refer to the model as a whole as the three-field model.) The choice of three "basic" fields is not new; indeed, some years ago Thirring[2] showed that at least three Weyl (two-component massless) fields are required to take account of the internal quantum numbers, I, Y, and B for the hadrons and that six Weyl fields (equivalent to three "basic" fields) are needed in order to obtain the multiplicative constants associated with the discrete transformations P, T, and C (C is charge conjugation). Moreover, shortly thereafter, we extended Thirring's argument and demonstrated[3] that there is an internal group structure underlying the three-field model which is of an even higher symmetry than contemplated by Thirring, the particular higher symmetry group depending on the choice of the four-fermion interaction

$$H' = g \sum_{\mu,\nu=1}^{3} (\bar{\psi}_\mu Q \psi_\mu)(\bar{\psi}_\nu Q \psi_\nu). \tag{1}$$

Specifically, we proved in our 1961 paper that if ψ_μ ($\mu = 1, 2, 3$) are three "basic" fields and ψ_μ is decomposed into the positive and negative (two-component) chirality projections $(1 \pm \gamma_5)\psi_\mu$, respectively, then one finds the higher symmetry groups listed in Table I. Here, R(6) is the rotation group of six dimensions, U(n) is the unitary group of n dimensions and USp(6) is the unitary symplectic group of six dimensions; the $\bar{W}_3$ group is the direct product of two U(3) groups with $U(3)^{(+)}$ and $U(3)^{(-)}$ re-

Table I. γ_5 diagonalization.

Four-fermion interaction	Higher symmetry group
Scalar (S) or pseudoscalar (P)	R(6)
Vector (V)	$\bar{W}_3 = U(3)^{(+)} \otimes U(3)^{(-)}$
Axial vector (A)	U(6)
Tensor	USp(6)

ferring to the positive and negative chiral projections of the ψ's, respectively.[4] We see from Table I that the three-field model, within the framework of γ_5 diagonalization, gives us a choice of four possible higher symmetry groups: R(6), $\overline{W}_3$, U(6), and USp(6).

While the properties of all four of these groups have been discussed, attention has been focused on the $\overline{W}_3$ group.[4] This is the smallest group in which the parity (a discrete Lorentz transformation) and the internal symmetry groups are mixed in a well-defined way and which implies a V interaction.[5] By making explicit use of the three-field vector model, it was possible to prescribe the tensorial behavior of the interactions which successively break first $\overline{W}_3$ and then U(3) symmetry and thereby (1) derive mass relations among unitary meson multiplets of opposite parity, (2) specify the normal or abnormal CP quantum numbers associated with a given representation of mesons within the $\overline{W}_3$ group, and (3) work out a theory of baryons. A gratifying feature of this approach was the essential agreement between the results obtained with the simple three-field vector model and those derived by Gell-Mann[6] in a more indirect fashion.

Now while the predictions of the $\overline{W}_3$ group are of considerable interest and worthy of experimental check, the $\overline{W}_3$ group (as well as the other groups listed in Table I) were derived by making a chiral decomposition of the triplet of "basic" fields (i.e., by choosing γ_5 diagonal). This implies that one can expect at most a mixing of the parity and the internal symmetry groups and that the proper orthochronous Lorentz group L_P will still be a factor in the direct product with each of the four groups listed in Table I. In order to mix L_P with the internal symmetry groups, we must try another type of decomposition of the three "basic" fields (e.g., diagonalize γ_4). We report here some preliminary results and point out the connection with the recent work of Sakita[7] and Gürsey and Radicati[7] on combining unitary spin and ordinary spin within the larger group SU(6). We believe that our three-field model helps to clarify both the possibilities and the limitations of the program being pursued by the last-named authors.

If we re-examine the three-field model from the point of view of γ_4 (rather than γ_5) diagonalization, we find that Table I is replaced by Table II. In Table II, the SU(2) group listed

Table II. γ_4 diagonalization.

Four-fermion interaction	Higher symmetry group
S	$U(6)^{(1)} \otimes U(6)^{(2)}$
P	$U(6)^{(3)} \otimes U(6)^{(4)}$
V and A	$W_3 \otimes SU(2)$
T	$U(3) \otimes SU(2)$

as a factor for the V, A, and T interactions refers to the ordinary spin rotation group and these interactions will not be considered further here. The U(6) groups listed for the S and P interactions have a different meaning from the U(6) group listed in Table I (for the A interaction) and will be discussed.

Define the γ_4 projections of the three "basic" fields ψ_μ $(\mu = 1, 2, 3)$ by $\varphi_\mu = \frac{1}{2}(1+\gamma_4)\psi_\mu$ and $\xi_\mu = \frac{1}{2}(1-\gamma_4)\psi_\mu$, and use the representation in which $\gamma_4 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The S interaction becomes

$$H'(S) = g_S \sum_{\mu,\nu=1}^{3} \int d^3x (\bar\psi_\mu \psi_\mu)(\bar\psi_\nu \psi_\nu)$$

$$= g_S \sum_{\mu,\nu=1}^{3} \int d^3x (\varphi_\mu{}^* \varphi_\mu - \xi_\mu{}^* \xi_\mu) \times (\varphi_\nu{}^* \varphi_\nu - \xi_\nu{}^* \xi_\nu). \tag{1a}$$

If we now write

$$\varphi_\mu = \begin{pmatrix}\varphi_{\mu 1}\\ \varphi_{\mu 2}\end{pmatrix}, \quad \xi_\mu = \begin{pmatrix}\xi_{\mu 1}\\ \xi_{\mu 2}\end{pmatrix},$$

and define

$$A_{\nu j}{}^{\mu i} = \int d^3x\, \varphi_{\mu i}{}^*(x) \varphi_{\nu j}(x),$$

$$B_{\nu j}{}^{\mu i} = \int d^3x\, \xi_{\mu i}{}^*(x) \xi_{\nu j}(x) \quad (i, j = 1, 2),$$

then it follows from the commutation relations for spinor fields that

$$[A_{\nu j}{}^{\mu i}, A_{\beta l}{}^{\alpha k}] = \delta_\nu{}^\alpha \delta_j{}^k A_{\beta l}{}^{\mu i} - \delta_\beta{}^\mu \delta_l{}^i A_{\nu j}{}^{\alpha k}, \tag{2a}$$

$$[B_{\nu j}{}^{\mu i}, B_{\beta l}{}^{\alpha k}] = \delta_\nu{}^\alpha \delta_j{}^k B_{\beta l}{}^{\mu i} - \delta_\beta{}^\mu \delta_l{}^i B_{\nu j}{}^{\alpha k}, \tag{2b}$$

$$[A_{\nu j}{}^{\mu i}, B_{\beta l}{}^{\alpha k}] = 0. \tag{2c}$$

The relations (2a)-(2c) imply that $A_{\nu j}{}^{\mu i}$ and $B_{\nu j}{}^{\mu i}$ are the generators of a U(6) group. The next step is to define two six-component spinors by

$$\Phi = \begin{pmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{21} \\ \varphi_{22} \\ \varphi_{31} \\ \varphi_{32} \end{pmatrix}; \quad \Xi = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \\ \xi_{31} \\ \xi_{32} \end{pmatrix}. \tag{3}$$

Inserting (3) into (1a) we obtain

$$H'(S) = g_S \int d^3x (\Phi^* \cdot \Phi - \Xi^* \cdot \Xi)(\Phi^* \cdot \Phi - \Xi^* \cdot \Xi). \tag{4}$$

Evidently, $H'(S)$ is invariant under two independent six-dimensional unitary transformations

$$\Phi \to U_1 \Phi, \quad U_1^\dagger U_1 = 1; \tag{5a}$$

$$\Xi \to U_2 \Xi, \quad U_2^\dagger U_2 = 1; \tag{5b}$$

so that the underlying group is $U(6)^{(1)} \otimes U(6)^{(2)}$ (cf. Table II) with the generators $A_{\nu j}{}^{\mu i}$ and $B_{\nu j}{}^{\mu i}$ specifying the first and second U(6) groups, respectively.

It is now easy to derive the underlying group structure for the P four-fermion interaction. With γ_4 diagonal, we define γ_5 by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and hence the P interaction becomes

$$H'(P) = g_P \sum_{\mu,\nu=1}^{3} \int d^3x (\varphi_\mu{}^* \xi_\mu - \xi_\mu{}^* \varphi_\mu) \times (\varphi_\nu{}^* \xi_\nu - \xi_\nu{}^* \varphi_\nu)$$

$$= -g_P \sum_{\mu,\nu=1}^{3} \int d^3x (\varphi_\mu{}'^* \varphi_\mu{}' - \xi_\mu{}'^* \xi_\mu{}') \times (\varphi_\nu{}'^* \varphi_\nu{}' - \xi_\nu{}'^* \xi_\nu{}'), \tag{6}$$

where we have defined $\varphi_\mu{}' = 2^{-1/2}(\varphi_\mu + i\xi_\mu)$, $\xi_\mu{}' = 2^{-1/2}(\varphi_\mu - i\xi_\mu)$. Comparison of (1a) and (6) shows immediately that the underlying group structure for the P interaction is the same as for the S interaction with different definitions for the U(6) groups [this is why we have used $U(6)^{(3)}$ and $U(6)^{(4)}$ in Table II rather than $U(6)^{(1)}$ and $U(6)^{(2)}$].

If the S and P interactions are both present (an assumption which is made hereafter), then we have only a common U(6) group with the generators simply the sums of the previously defined A and B generators, namely

$$C_{\nu j}{}^{\mu i} = A_{\nu j}{}^{\mu i} + B_{\nu j}{}^{\mu i}, \tag{7}$$

where $C_{\nu j}{}^{\mu i}$ satisfy the commutation relations (2a) and (2b). Equation (7) can be rewritten in the suggestive form (see below)

$$C_{\nu j}{}^{\mu i} = \tfrac{1}{2} \sum_{a=1}^{4} (\sigma_a)_{ji} \int d^3x\, \psi_\mu{}^*(x) \sigma_a \psi_\nu(x),$$

where

$$\sigma_a = (\vec{\sigma}, 1) \quad (a = 1, 2, 3, 4). \tag{8}$$

Thus far we have only considered the group properties of the four-fermion interaction term in the total Hamiltonian on the basis of a γ_4 decomposition of a triplet of "basic" fields. If we were to allow the common mass term for the triplet of fields,

$$H(m) = m \sum_{\mu=1}^{3} \int d^3x\, \bar{\psi}_\mu(x) \psi_\mu(x), \tag{9}$$

we would have the interesting result[8] that invariance under the common U(6) group would still be maintained [since $H(m)$ is a scalar]; that is to say, invariance under U(6) is not affected by assigning an arbitrarily large (rather than zero) bare mass to the fundamental triplet. On the other hand, the kinematical part of the Hamiltonian, H_0, is not invariant under the common U(6) group. This is obvious when we write H_0 in terms of Φ and Ξ, namely

$$H_0 = i \int d^3x \left[\Phi^*(x) \vec{\sigma} \cdot \frac{\partial}{\partial \vec{x}} \Xi(x) + \Xi^*(x) \vec{\sigma} \cdot \frac{\partial}{\partial \vec{x}} \Phi(x) \right]. \tag{10}$$

H_0 is therefore a symmetry-breaking term in U(6) and if we wish to exhibit its tensorial behavior in this space, it is convenient to rewrite (10) in the form

$$H_0 = \sum_{a=1}^{3} \sum_{\mu=1}^{3} (T_a)_{\mu j}{}^{\mu i} (\sigma_a)_{ij}, \tag{11}$$

where

$$(T_a)_{\mu j}{}^{\mu i} = i \int d^3x \left[\varphi_{\mu i}{}^*(x) \frac{\partial}{\partial x_a} \xi_{\mu j}(x) + \xi_{\mu i}{}^*(x) \frac{\partial}{\partial x_a} \varphi_{\mu j}(x) \right]. \tag{12}$$

The usual situation is thus reversed[8] and it is the kinematical (rather than the interaction) part of the total Hamiltonian which reduces the symmetry of the underlying group. But this is in the spirit of strong-coupling theory where we regard H_0 as a perturbation on $[H'(S) + H'(P)]$ $(g_S, g_P \to \infty)$; if we choose m in $H(m)$ as sufficiently large, we can include the (common) mass term as a part of the unperturbed Hamiltonian.[8]

The symmetry-breaking term H_0, as given by (11), has a tensorial behavior in U(6) of the type similar to the one considered by Sakita[7] and Gürsey and Radicati,[7] and is capable of splitting the masses of the $J=\frac{1}{2}^+$ unitary octet and the $J=\frac{3}{2}^+$ unitary decuplet within the $\underline{56}$ baryon representation and the masses of the $J=0^-, 1^-$ unitary octets and the $J=1^-$ unitary singlet within the $\underline{35}$ meson representation (in second order). (Of course, rigorously speaking, it is the total spin S rather than the total angular momentum J which enters in our classification.) Furthermore, in order to obtain a mass splitting within a unitary multiplet, it is natural to assume the presence of the medium-strong symmetry-breaking perturbation

$$H(m') = m' \int d^3x\, \bar{\psi}_3(x)\psi_3(x), \tag{13}$$

where m' is of the order of a mass difference within a unitary multiplet. If one recognizes that the tensorial behavior of $H(m')$ within U(6) is $T_{3i}^{\;3i}$, and treats the effect of this term together with H_0, one obtains unitary-spin–spin mass relations of the type found by the afore-mentioned authors[7] (provided we assume the rotational invariance of the theory).

Up to this point, we have shown how the three-field model with γ_4 (rather than γ_5) diagonalization can be used to mix unitary spin and spin along the lines of the recently proposed generalization of the Wigner supermultiplet theory. We shall now employ the same three-field model to exhibit a fundamental difficulty in this method of mixing L_P with the internal symmetry groups, which we believe is intrinsic in the program of Sakita[7] and Gürsey and Radicati.[7] To see this, write down the total angular momentum for the three fields:

$$\vec{J} = \sum_{\mu=1}^{3} \int d^3x\, \psi_\mu^*(x)[\tfrac{1}{2}\vec{\sigma} + (\vec{x}\times\vec{p})]\psi_\mu(x). \tag{14}$$

It is easy to show that $\vec{J}$ does not commute with the generators $C_{\nu j}^{\;\mu i}$ of the U(6) group [cf. (7)] but that it is a part of the modified set of generators $D_{\nu j}^{\;\mu i}$ defined by

$$D_{\nu j}^{\;\mu i} = \sum_{a=1}^{3} (\sigma_a)_{ji} \int d^3x\, \psi_\mu^*(x)[\tfrac{1}{2}\sigma_a + (\vec{x}\times\vec{p})_a]\psi_\nu(x). \tag{15}$$

Indeed, we can express $\vec{J}$ in terms of $D_{\nu j}^{\;\mu i}$ as follows:

$$J_a = \tfrac{1}{2}(\sigma_a)_{ij} \sum_{\mu=1}^{3} D_{\mu j}^{\;\mu i} \quad (a=1,2,3). \tag{16}$$

Unfortunately, the commutators of $D_{\nu j}^{\;\mu i}$ do not lead to a closed system so that we are dealing with an infinite-dimensional Lie algebra. The same remark also applies to a Lie algebra which is generated from the set $C_{\nu j}^{\;\mu i}$ and the generators of the Lorentz group.

The generators $D_{\nu j}^{\;\mu i}$ defined by (15) commute with H_0 and $H(m)$ but not with the S or P interactions. It is possible, however, that a nonlocal interaction can be written down which commutes with $D_{\nu j}^{\;\mu i}$ so that rotational invariance is satisfied (and perhaps even Lorentz invariance). We would then be compelled to work with the complicated case of an infinite-dimensional Lie algebra associated with the $D_{\nu j}^{\;\mu i}$ rather than with the more tractable finite-dimensional Lie algebra associated with the generators $C_{\nu j}^{\;\mu i}$. In any case, we have shown how the three-field model (with γ_4 diagonalization) can produce a mixing of L_P with the internal symmetry groups at the expense of enlarging the underlying group to an infinite number of dimensions. Further uses of the three-field model as a probe of higher symmetries are being investigated.

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[1]We shall show below that under certain circumstances, the fundamental triplet of fields may possess a finite mass without destroying the invariance under a higher symmetry group.

[2]W. E. Thirring, Nucl. Phys. 10, 97 (1959).

[3]R. E. Marshak and S. Okubo, Nuovo Cimento 19, 1226 (1961).

[4]This $\overline{W}_3$ group is not to be confused with the W_3 group studied recently [cf. J. Schwinger, Phys. Rev. 135, B816 (1964); F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. 135, B467 (1964)], since the parity operation interchanges $U(3)^{(+)}$ and $U(3)^{(-)}$ [cf. R. E. Marshak, N. Mukunda, and S. Okubo, Phys. Rev. (to be published)].

[5]Actually, one can include an A interaction without losing $\overline{W}_3$ invariance [since $\overline{W}_3$ is a subgroup of U(6)].

[6]M. Gell-Mann [Phys. Rev. 125, 1067 (1962); Physics 1, 63 (1964)] came upon the $S\overline{W}_3$ group by looking for the group generated, under equal-time commutation, of the integrals of the time components of the vector and axial-vector weak hadron current octets.

[7]B. Sakita, Phys. Rev. 136, B1756 (1964); F. Gürsey, A. Pais, and L. Radicati, Phys. Rev. Letters 13, 299 (1964), and earlier papers quoted therein.

[8]In the case of the $\overline{W}_3$ group, a finite mass destroys the invariance whereas the kinematical term preserves it (since the kinematical term is a "vector"); this is why the mass term is treated as a (symmetry-breaking) perturbation for $\overline{W}_3$ whereas the kinematical term is included in the unperturbed Hamiltonian. The situation is now reversed (see below).

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In the first column of page 692, in line 23, "$R \rightarrow N_1$" should read "$R \rightarrow F_3{}^+$."