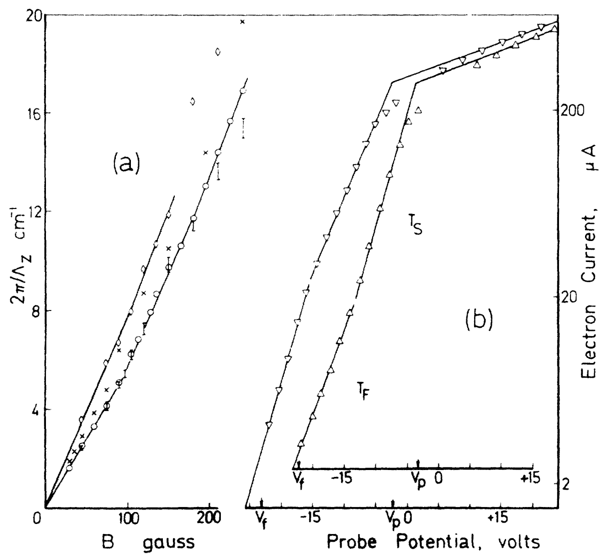

FIG. 2. (a) Dependence of spatial period Λ_z on magnetic field B in the shadow for different pressures, obstacles, and gases. Diamonds: 5 millitorr, 12-mm disc, mercury; crosses: 0.2 millitorr, 0.5-mm rod, mercury; circles: 0.1 millitorr, 12-mm disc, mercury; bars: 5 millitorr, 1-mm rod, neon. (b) Probe current-voltage curves. Inverted triangles: $z=8$ mm, $T_S=5.3$ eV, $T_F=3.5$ eV; triangles: $z=12$ mm, $T_S=3.8$ eV, $T_F=5.2$ eV.

electrons which lie within a small range of axial speed. An analysis of the particle dynamics for such a case has been carried out by Al'pert, Gurevich, and Pitaevskiĭ[3] in connection with the ion density distribution in satellite wakes in the ionosphere. Application of this analysis to the V_f measurements shows good agreement with respect to the phase difference of the curves inside and outside the shadow, the spatial period Λ_z, and the dependence of the amplitude and shape of the V_f peaks on B.

The curves of the macroscopic plasma parameters V_p, n_e, and T_S are less clearly understood. At values of z exceeding 8 mm, which is roughly the helical pitch of electrons with v_z equal to the thermal speed, the curves have a spatial period of 15 mm. They may be considered to represent a wave propagating in the negative z direction, or at on acute angle to this direction, in the frame of reference moving with the electron drift velocity. It is noted that this spatial period is equal to the helical pitch of electrons which lie in the T_F group mentioned above. Such a wave could be sustained by either the electron drift energy or by the interaction of the excess group of electrons with the remainder of the plasma after phase selection by the obstacle.

A more detailed report is to be submitted for publication.

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COLLISIONS OF THREE HARD SPHERES*

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In the deduction, from first principles, of the nonequilibrium properties of dense media (for example, the density dependence of thermal conductivity and viscosity), three-body contributions play an essential role.[1] In view of the difficulties encountered in calculating rigorously the three-body effects in general approaches,[2] the insight to be gained from a specific molecular model is needed. Therefore, the orbits of three identical, classical, rigid, elastic spheres in otherwise empty flat Euclidean three-dimensional space have been analyzed, and the phase-space domains, Γ_n, that correspond to the occurrence of n successive binary collisions have been determined. The result that Γ_4 is nonempty has been recently demonstrated by an example.[3] In this paper we state precisely, and illustrate, the form of Γ_4 and describe a rigorous proof that Γ_5 is empty. (The rather intricate demonstrations are being prepared for publication.) Hence we may state that we have established the necessary and sufficient conditions for the occurrence of four collisions and we have proved that five are impossible.

First, we describe a method of classifying collision sequences. Since a given pair of spheres cannot be involved in two successive collisions, any sequence of three collisions among three spheres belongs to one of two classes: (i) The pair involved in the third collision is the same as that involved in the first, or (ii) all three pairs are involved in the sequence. We designate the first class by the symbol r (for rebound) and the second by c (for cyclic). In any sequence of four collisions, the first three must belong to one of these classes and, similarly, the last three of the four must belong to one of them. It follows that a sequence of four can be described, insofar as ordering is concerned, by rr, rc, cr, or cc, where the first letter in each pair refers to the first three collisions and the second letter to the last three of the four. Similar reasoning leads to the description of a five-collision sequence by one of the eight symbols rrr, rrc, $\cdots$, ccc.

The previously announced four-collision sequence[3] is an example of rc. The conditions under which rc can occur are described in terms of the configuration of the spheres just at the conclusion of the second collision. With the spheres numbered such that the succession of collisions is (1, 2)-(2, 3)-(1, 2)-(1, 3), the reference frame is taken such that sphere 2 is stationary at the origin. The position and velocity of sphere 1 are denoted by $\vec{x}$ and $\vec{u}_1$; those of sphere 3 by $\vec{k}$ and $\vec{u}_3$ (Fig. 1). The diameter of the spheres is taken as the unit of length so $|\vec{k}| = 1$ and $|\vec{x}| \geq 1$.

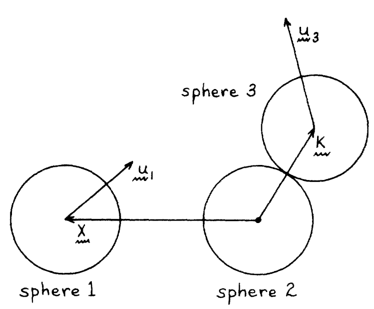

FIG. 1. Configuration at the conclusion of the second collision in an rc sequence.

Then, using $\vec{k}'$ to denote the position of sphere 1 relative to sphere 2 at the third collision, we have $\vec{k}' = \vec{x} + \vec{u}_1\tau$, where τ is the time interval between the second and third collisions and so is given by

$$\vec{u}_1^{\,2}\tau = -\vec{x}\cdot\vec{u}_1 - [(\vec{x}\cdot\vec{u}_1)^2 - \vec{u}_1^{\,2}(\vec{x}^2-1)]^{1/2}.$$

Evidently $\vec{x}$ and $\vec{u}_1$ must be so related that τ is real and nonnegative. We deduce that

$$\vec{x}\cdot\vec{u}_1 < -[\vec{u}_1^{\,2}(\vec{x}^2-1)]^{1/2}. \qquad (1)$$

With $\vec{x}$ fixed, (1) says that, for the third collision to occur, $\vec{u}_1$ must lie in a half-cone. The notation $\vec{v} \in C(\vec{p}, \vec{q}, a)$ is introduced to denote those vectors $\vec{v}$ whose endpoints lie in the interior of the half-cone with apex at $\vec{p}$ and generators tangent to the sphere whose center is at $\vec{q}$ and whose radius is a. Using this notation, the necessary and sufficient conditions (obtained in a straightforward way by following the dynamics of the collisions and motion) for the four collisions to occur in the order specified are

$$\vec{u}_1 - (\vec{k}\cdot\vec{u}_3)\vec{k} \in C(0, \vec{x}, 1); \qquad (2)$$

$$\vec{k}\cdot\vec{u}_3 > 0; \qquad (3)$$

$$\vec{u}_1 \in C(0, -\vec{x}, 1); \qquad (4)$$

$$\begin{aligned} u_3 &\in C(\vec{b}, \tau^{-1}\vec{a}, \tau^{-1}), \quad \tau > 0,\\ &\in C(\vec{b}, \vec{a}+\vec{b}, 1), \quad \tau = 0, \end{aligned} \qquad (5)$$

where $\vec{a} = \vec{k}' - \vec{k}$ and $\vec{b} = \vec{u}_1 - (\vec{k}'\cdot\vec{u}_1)\vec{k}$. Algebraically, conditions (2)-(5) constitute a system of coupled nonlinear inequalities [exemplified by (1) of which (4) is a restatement] defining that part of Γ_4, characterized by rc, which we designate $\Gamma(rc)$. But (2)-(5) cannot be called a description of $\Gamma(rc)$; as they stand, it is not even clear that the domain is nonempty.

To clarify the situation, we must be able to decouple the conditions so that the independent variables can be chosen in such an order that the restrictions on each depend only on those already chosen. This can be done as follows: Given (4), conditions (2) and (3) can be shown to be equivalent to the pair

$$\vec{k} \in C(0, -\vec{x}, 1), \qquad (6)$$

$$\vec{k}\cdot\vec{u}_3 > \alpha; \qquad (7)$$

where α is the larger root of

$$[(\vec{x}\cdot\vec{k})^2 - \vec{x}^2 + 1]\alpha^2 - 2[(\vec{x}\cdot\vec{k})(\vec{x}\cdot\vec{u}_1) - \vec{k}\cdot\vec{u}_1(\vec{x}^2-1)]\alpha + (\vec{x}\cdot\vec{u}_1)^2 - \vec{u}_1^{\,2}(\vec{x}^2-1) = 0.$$

[Conditions (4) and (6) guarantee that α is real and positive.] It is now possible to pick any $\vec{x}$ (with $|\vec{x}| \geq 1$), then $\vec{u}_1$ and $\vec{k}$ satisfying (4) and (6), respectively, and then $\vec{u}_3$ satisfying (7) to establish the first three collisions. But for the fourth collision, $\vec{u}_3$ must also satisfy (5), and, in general, the regions in $\vec{u}_3$ space defined by (5) and (7) do not intersect. It can be shown that (5) and (7) are compatible if and only if $\vec{x}$, $\vec{u}_1$, and $\vec{k}$ are such that

$$\vec{k}\cdot\vec{b} > \alpha, \tag{8}$$

in addition to satisfying (4) and (6).

But now we must consider the compatibility of (4), (6), and (8). It is found that for $|\vec{x}| > \sqrt{2}$, they are incompatible, but for any $\vec{x}$ such that $1 \leq |\vec{x}| < \sqrt{2}$, there exist choices of $\vec{u}_1$ and $\vec{k}$ that satisfy all three conditions. We have no simple description of the domain in the $(\vec{u}_1, \vec{k})$ space thus defined for each such $\vec{x}$, but we can exhibit a sample of it.

Let β and γ be the angles that $\vec{k}$ and $\vec{u}_1$, respectively, make with $-\vec{x}$, and φ the angle between the two planes generated by $\vec{x}$ and $\vec{u}_1$ and by $\vec{x}$ and $\vec{k}$; and let θ be the half-angle of $C(0, -\vec{x}, 1)$ so that $\sin\theta = |\vec{x}|^{-1}$. Then conditions (4), (6), and (8) can be expressed in terms of these four angles (since $|\vec{u}_1|$ drops out). Three cross sections (obtained by setting $\varphi = 0$ and $\theta = 90°$, 75°, and 60°) calculated from the $(\beta, \gamma, \varphi, \theta)$ domain so defined are shown in Fig. 2.

To recapitulate, a point in the domain whose

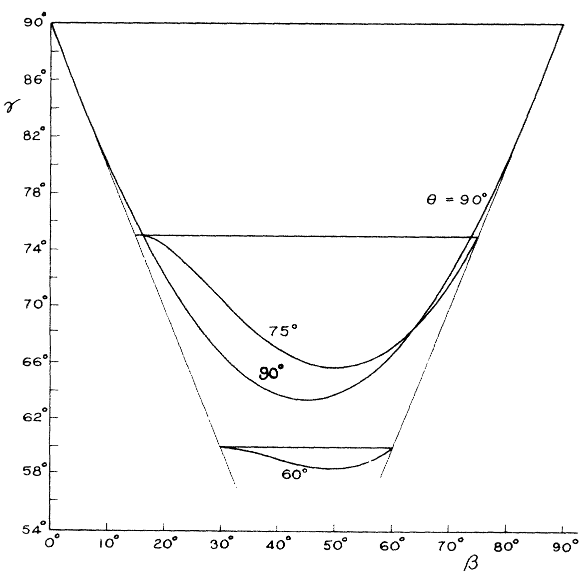

FIG. 2. Cross sections of the $(\beta, \gamma, \varphi, \theta)$ space ($\varphi = 0$).

cross section is illustrated in Fig. 2 defines, apart from orientation, values of $\vec{x}$, $\vec{u}_1$, and $\vec{k}$ such that the compatibility of (5) and (7) is assured; and choosing a value of $\vec{u}_3$ in the truncated half-cone defined by (5) and (7) finishes specification of a point in $\Gamma(rc)$.

Turning now to the case rr, we find a completely analogous set of equations but the domain defined by the analogs of (4), (6), and (8) turns out to be empty. In other words, rr is impossible. A somewhat different formulation leads to a proof that cc is impossible. By time-reversal considerations, each instance of rc gives an instance of cr and vice versa. Thus we have reached full knowledge of Γ_4.

In two special cases the maximum number of collisions is three: (i) for one-dimensional motion,[4] and (ii) for point particles.[5]

With rr and cc ruled out, the only possibilities remaining for five collisions are rcr and crc. We have found proofs that they are both impossible. Hence Γ_5 is empty. Detailed demonstrations of the foregoing assertions will be published soon. Numerical calculations were made which helped guide the construction of the proofs and corroborated the results.

We are deeply indebted to W. Thurston for his timely discovery of a four-collision sequence and to G. E. Uhlenbeck and A. Lenard for stimulating discussions.

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[2]See, for example, G. Sandri, Ann. Phys. (N.Y.) 24, 332, 380 (1963); and G. Sandri, Nuovo Cimento 31, 1131 (1964).

[3]W. Thurston and G. Sandri, Bull. Am. Phys. Soc. 9, 386 (1964). Ternary collisions are included in the theory only insofar as they can be regarded as limits, as the time interval goes to zero, of successive binary collisions. Encounters wherein spheres touch but do not exchange momentum are not counted as collisions.

[4]It is obvious that c is impossible in one dimension; thus the impossibility of rr prevents four collisions in this case. G. E. Uhlenbeck, private communication.

[5]From the condition $|\vec{x}| < \sqrt{2}$, we see that Γ_4 shrinks to the empty set as the diameter of the spheres tends to zero. This result has been obtained independently, for point particles, by A. Lenard (private communication).