ion current with the anode at +23 cm where the flutes obviously dominate the transport. The lower photo shows the same currents for the anode at −23 cm where the radial diffusion appears to be the predominant transport mechanism. The amplitude of the few remaining fluctuations is reduced an additional order of magnitude by operation in the single-coil field.

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DETERMINATION OF THE STATISTICAL PROPERTIES OF LIGHT FROM PHOTOELECTRIC MEASUREMENTS*

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Several publications have appeared in recent months, concerning the determination of some of the statistical properties of fluctuating light beams from photoelectric measurements.[1-6] This problem is of particular interest at present in connection with attempts to understand the behavior of laser light. Some of these publications deal with questions of stability of the oscillations and with some of the statistical moments of the associated probability distributions. It is the purpose of this note to show that it is possible, in principle, to determine the complete probability density of the light intensity from the knowledge of the statistical distribution of photoelectrons released from a photosensitive surface on which the light is incident.

Consider a plane, quasimonochromatic stationary light wave incident normally on a photodetector. If $I(t)$ represents the light intensity at time t (measured in photon numbers per second), then under usual experimental conditions, the probability $p(n)$ that n photoelectrons will be released in a time interval of duration T is given by Mandel's formula[5-9]

$$p(n) \equiv p(n, T) = \int_0^\infty \frac{(\alpha W)^n}{n!} e^{-\alpha W} P(W) dW, \tag{1}$$

where

$$W \equiv W(T) = \int_0^T I(t')dt'. \tag{2}$$

In Eq. (1), $P(W)dW$ represents the probability that the time-integrated intensity W will take on a value in the range between W and $W+dW$, and α represents the photoefficiency of the detector. The physical meaning of Eq. (1) is clear: It shows that the probability $p(n)$ is the average of a Poisson distribution with parameter αW over the ensemble of the incident light wave. Formula (1) is a "generalized analog" of a formula found by Bothe[10] already in 1927, in connection with his derivation of the Bose-Einstein distribution. Very recently Ghielmetti[11] showed that Formula (1) has an exact counterpart in the quantum statistics of bosons. Since Eq. (1) expresses $p(n)$ as a linear transform of $P(W)$ with a "Poisson kernel," we may say that $p(n)$ is the Poisson transform of $P(W)$.

We will now show that the Poisson transform may be inverted, i.e., that for any given $p(n)$ one can find a unique $P(W)$ which satisfies Eq. (1). This assertion implies that the probability $P(W)$ which governs the fluctuations of the (time-integrated) light intensity may be determined from the knowledge of the distribution of the photoelectric counts.

Let

$$F(x) = \int_0^\infty e^{ixW} P(W) e^{-\alpha W} dW. \tag{3}$$

Then by the Fourier inversion formula,

$$P(W) = \frac{e^{\alpha W}}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-ixW} dx. \tag{4}$$

Now from (3) we have, at least formally,

$$F(x)=\int_0^\infty \sum_{n=0}^\infty \frac{(ixW)^n}{n!}P(W)e^{-\alpha W}dW$$

$$=\sum_{n=0} \frac{(ix)^n}{n!}\int_0^\infty W^n P(W)e^{-\alpha W}dW,$$

or, using (1),

$$F(x)=\sum_{n=0}^\infty \left(\frac{ix}{\alpha}\right)^n p(n). \qquad (5)$$

Thus the required probability density $P(W)$ may be obtained from the knowledge of $p(n)$ ($n=0, 1, 2, \cdots$), by first evaluating $F(x)$ from (5) and then evaluating (4).

The function $F(x)$ is evidently closly related to the characteristic function

$$C(x)=\int_0^\infty e^{ixW}P(W)dW \qquad (6)$$

of $P(W)$. Since $C(x)$ contains Fourier components of non-negative frequencies only, it is by a well-known theorem[12] the limit on the real axis of a function $C(z)=C(x+iy)$ which is analytic and regular in the upper half ($y>0$) of the complex z plane. The analytic continuation of $C(x)$ into the upper half-plane may be obtained by simply replacing x by z in (6). On comparing the resulting expression with (3) we see that

$$F(x)=C(x+i\alpha). \qquad (7)$$

Several comments seem appropriate: In the first place, if the basic time interval T is short compared with the coherence time τ_c of the light (a condition that may readily be satisfied with laser light), $I(t)$ may be taken as constant under the integral in Eq. (2), so that one then has

$$W=IT \quad (T\ll\tau_c). \qquad (8)$$

Equation (4) then yields the probability density of the instantaneous light intensity I. Further, if $V^{(r)}(t)$ represents the fluctuating real field (for simplicity assumed to be linearly polarized) and $V(t)=V^{(r)}(t)+iV^{(i)}(t)$ is the associated analytic signal,[13,5] one may readily derive expressions for the probability densities of V and $V^{(r)}$, respectively: For a stationary quasi-monochromatic field, such as is being here considered, the probability density of V may be assumed to be independent of the phase of V and one then finds, by using the fact that $I(t)=V^*(t)V(t)$ and changing variables, that

$$P(V^{(r)},V^{(i)})=\pi^{-1}P(I). \qquad (9)$$

The probability density of $V^{(r)}$ alone is obtained by integrating (9) with respect to $V^{(i)}$ ($-\infty<V^{(i)}<\infty$). Since $I=V^{(r)2}+V^{(i)2}$, this leads to the expression[14]

$$P(V^{(r)})=\frac{1}{\pi}\int\frac{P(I)dI}{(I-V^{(r)2})^{1/2}}, \qquad (10)$$

where the integration on the right extends from $V^{(r)2}$ to ∞.

To illustrate our results, let us consider two simple examples[15]:

(1) Suppose that $p(n)$ is given by the Bose-Einstein distribution

$$p(n)=\bar{n}^n/(\bar{n}+1)^{n+1}. \qquad (11)$$

Then (5) becomes

$$F(x)=(\bar{n}+1-i\bar{n}x/\alpha)^{-1}, \qquad (12)$$

and (4) gives

$$P(W)=\bar{W}^{-1}e^{-W/\bar{W}}, \qquad (13)$$

where $\bar{W}=\bar{n}/\alpha$. Hence $P(W)$ is now an exponential distribution. Further, if the distribution (11) corresponds to measurements for which $T\ll\tau_c$, one has from (8) and (13)

$$P(I)=\bar{I}^{-1}e^{-I/\bar{I}}, \qquad (14)$$

where $\bar{I}=\bar{W}/T=\bar{n}/T\alpha$. Moreover, if the light is linearly polarized, one obtains from (10) and (14)

$$P(V^{(r)})=(\pi\bar{I})^{-1/2}\exp(-V^{(r)2}/\bar{I}). \qquad (15)$$

Equation (15) shows that the probability density of $V^{(r)}$ is now a Gaussian distribution with zero mean and variance $\frac{1}{2}\bar{I}$.

(2) As a second example, suppose that the photoelectric counts are governed by the Poisson distribution

$$p(n)=\bar{n}^n e^{-\bar{n}}/n!. \qquad (16)$$

Equation (5) now gives

$$F(x)=\exp\{\bar{n}[(ix/\alpha)-1]\}, \qquad (17)$$

and (4) shows that in this case

$$P(W)=\delta(W-\bar{W}), \qquad (18)$$

where $\bar{W}=\bar{n}/\alpha$ and δ is the Dirac delta function. This result implies that the distribution of the photoelectrons is strictly Poissonian (pure shot noise), with parameter $\bar{n}$, if and only if the incident light is perfectly stabilized in the sense that W does not fluctuate at all, but has a constant value $\bar{W}=\bar{n}/\alpha$. It should be noted that perfect stability in this sense does not imply a complete absence of fluctuations in the incident light. For W depends only on the behavior of the intensity $I=VV^*$, and so stabilization of W does not preclude fluctuations of the phase of V. If the Poisson distribution (16) refers to results of measurements for which $T \ll \tau_c$, one now also has

$$P(I)=\delta(I-\bar{I}), \tag{19}$$

where again $\bar{I}=\bar{W}/T=\bar{n}/T\alpha$. Further, if the field is also linearly polarized, one obtains from (10) and (19)

$$P(V^{(r)})=\pi^{-1}(\bar{I}-V^{(r)2})^{-1/2} \quad \text{if } |V^{(r)}|<\bar{I}^{1/2}$$
$$=0 \quad \text{if } |V^{(r)}|>\bar{I}^{1/2}. \tag{20}$$

It is evident from our analysis that experimental determination of the statistical distribution of photoelectrons in photoelectric light-detection experiments may be used to derive the probability densities that govern the fluctuations of such light. This result should prove of particular interest in connection with laser light as it could be used to decide which of the proposed descriptions of its statistical behavior is in best agreement with results of photoelectric measurements.[16]

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